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Journal of Mathematics

EDITED BY

FRANK MORLEY

WITH THE COOPERATION OF

A. COHEN, CHARLOTTE A. SCOTT

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

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Generalizations of Geodesic Curvature and a Theorem of Gauss Concerning Geodesic Triangles. *

By Gilbert Ames Bliss.

Introduction.

Many of the geometrical invariants of a curve on a surface can be defined in terms of the integral which expresses the length of the curve. Their geometrical invariance corresponds analytically to the fact that they remain invariant under a transformation of the parameters in terms of which the equations of the surface are expressed. In an earlier paper † the author has shown that there exists a function analogous to the angle between two given curves in a plane or on a surface, and related to an integral of the form

$$u = \int_{t_0}^{t_1} f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt \tag{1}$$

as angle on a surface is related to length. The integral (1) is to be thought of as taken along a curve in the form

$$x = x(t), \quad y = y(t), \qquad (t_0 \leqq t \leqq t_1);$$

x' and y' represent the derivatives of x and y with respect to t; and τ is the angle defined by the equations

$$\cos\tau = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin\tau = \frac{y'}{\sqrt{x'^2 + y'^2}}.$$

For the length integral on a surface the function f has the special form

$$f = \sqrt{E\cos^2\tau + 2F\cos\tau\sin\tau + G\sin^2\tau}. \tag{2}$$

In the present paper a similar generalization of the notion of the curvature of a curve in the plane or the geodesic curvature of a curve on a surface is exhibited. This generalization is called *extremal curvature* and is explained in § 1. The possession of invariants corresponding to angle and geodesic

* The results in § 1 of this paper were presented to the American Mathematical Society, April 28, 1906, under the title, "An Invariant of the Calculus of Variations Corresponding to Geodesic Curvature."

† "A Generalization of the Notion of Angle," *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 184.

curvature suggests at once the possibility of generalizing in a similar manner the formula of Gauss for the sum of the angles of a geodesic triangle on a surface. The situation is unfortunately not as simple as in the case of the surface theory. In § 2 a notion called the *area of a simply closed curve in a field of extremals* is explained, and in § 3 a generalization of Gauss' theorem is developed in which angle, extremal curvature and the area in a field are involved. In § 4 the invariance under point transformation of the quantities introduced in the preceding sections is discussed, and in § 5 the relation of the results of the paper to the usual formulas of surface theory is elucidated.

§ 1. *Extremal Curvature.*

The geodesic curvature of a curve on a surface may be defined in a number of different ways, to several of which there correspond by generalization invariants of the integral (1). The most convenient definition for the purposes

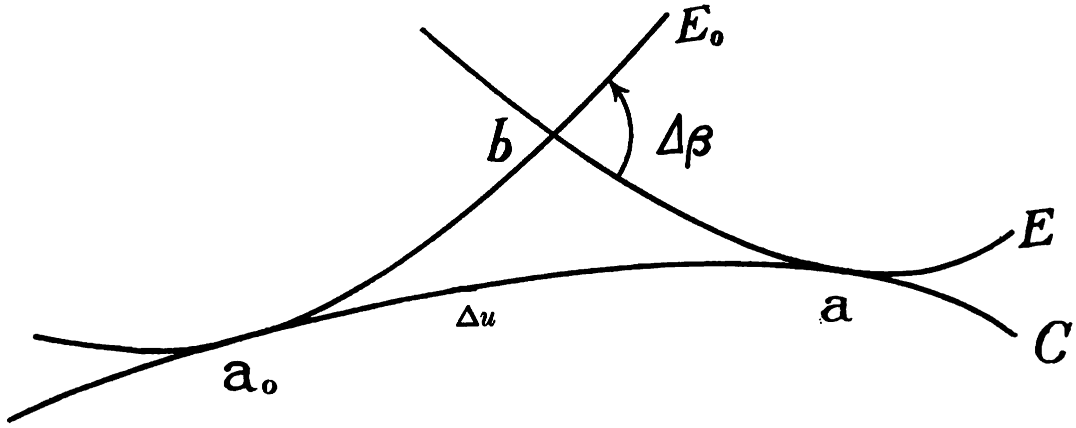

FIG. 1.

of the present paper is the following, which depends upon the notion of angle.* Suppose at a fixed point a_0 of a given curve C the geodesic E_0 tangent to C is drawn. The geodesic tangent E at a movable neighboring point a will in general meet E_0 at a point b. Let the angle between E_0 and E at b be denoted by $\Delta\beta$, and the length of the arc $a_0 a$ by Δu. Then the geodesic curvature of the curve C at the point a_0 is equal to the limit

$$\lim_{\Delta u \doteq 0} \frac{\Delta\beta}{\Delta u}. \tag{3}$$

This definition admits of a ready generalization when the function f in the integral (1) is not restricted to have the special form (2). The geodesics E_0, E

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* See Darboux, *Leçons sur la Théorie Générale des Surfaces*, Vol. III, p. 129.

can be replaced by extremals of the integral (1); the angle $\Delta\beta$ has already been generalized in the paper referred to above, and instead of the length of arc the value of the integral (1) along $a_0 a$ can be used.

The extremals for the integral (1) are the solutions of the differential equation*

$$T(x, y, \tau, \tau_s) = f_x \sin\tau - f_y \cos\tau + f_{x\tau}\cos\tau + f_{y\tau}\sin\tau + (f + f_{\tau\tau})\tau_s = 0, \quad (4)$$

which is of the second order. Through each point of the curve C there passes one of these solutions in the direction of the positive tangent to C. The equations of the one-parameter family so determined can be found in the form

$$x = \phi(s, \alpha), \quad y = \psi(s, \alpha), \quad (5)$$

where s is the length of arc of E measured from the point of tangency with C, and α is the parameter of the family. Then the curve C has the equations

$$x = \phi(0, \alpha), \quad y = \psi(0, \alpha), \quad (6)$$

with α as the variable parameter. If the function $f(x, y, \tau)$ is of class C''' and the expression $f + f_{\tau\tau}$ different from zero in a neighborhood of the values of (x, y, τ) on the curve C, and if the curve C is of class C'', then from the known properties of solutions of the differential equation (4) it follows that the functions ϕ, ψ, ϕ_s, ψ_s are of class C' for all values of α defining points on the arc C and for $|s| \leqq \delta$, if δ is properly chosen. From the results of a previous paper by the writer† it also follows, when C is not an extremal, that for negative values of s sufficiently near to zero the arcs of the extremals, such as ba in the figure, touch the curve C only at a and simply cover a region of the plane which has the curve C as a part of its boundary. An arc of the fixed extremal E_0 extending from a_0 in the direction $s > 0$ can be taken short enough so that it also lies in this region, and consequently so that through each of its points there passes one and but one of the extremals (5). In other words, the equations

$$\phi(s, \alpha) = \phi(s_0, \alpha_0), \quad \psi(s, \alpha) = \psi(s_0, \alpha_0), \quad (7)$$

where α_0 is the constant value of α, and s_0 the length of arc for the extremal E_0, have single-valued solutions for s, α $(s < 0)$ in terms of s_0. By implicit function theory these solutions $s(s_0)$, $\alpha(s_0)$ will be of class C' in an interval $0 < s_0 \leqq \epsilon$ with the value $s_0 = 0$ omitted. But at $s_0 = 0$ they are continuous and approach the values 0 and α_0 respectively.

* See Bliss, *loc. cit.*, p. 188, where the most general solutions of (5) are given and their properties stated. With the help of these properties the existence and character of the family (5) can be readily derived.

† "Sufficient Conditions for a Minimum with Respect to One-Sided Variations," *Transactions of the American Mathematical Society*, Vol. V (1904), p. 480. See also Bolza, "An Existence Proof for a Field of Extremals Tangent to a Given Curve," the same *Transactions*, Vol. VIII (1907), p. 399.

where the arguments x, y, τ in f and its derivatives are for the curve C at the point a_0, $\frac{d\tau}{d\sigma}$ is the curvature of C at a_0, and $\frac{d\tau}{ds}$ the curvature of E_0 at the point a_0. From (4) the value of $\frac{d\tau}{ds}$ can be found and substituted, giving

$$\frac{d\tau}{d\sigma} - \frac{d\tau}{ds} = \frac{T\left(x, y, \tau, \frac{d\tau}{d\sigma}\right)}{f + f_{\tau\tau}}.$$

The extremal curvature of a curve C at a point a_0 (see Fig. 1) is defined as a limit

$$\frac{1}{\rho} = \lim_{\Delta u \doteq 0} \frac{\Delta\beta}{\Delta u}.$$

Here $\Delta\beta$ is the generalized angle (8) between two extremals, E_0 and E, at their point of intersection b; E_0 is supposed to be fixed and tangent to C at a point a_0; while E is a movable extremal tangent to C at a. The quantity Δu is the value of the integral (1) taken along the curve C from a_0 to a.

According to this definition the value of $\frac{1}{\rho}$ in terms of the function f and its derivatives turns out to be

$$\frac{1}{\rho} = \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^3(f + f_{\tau\tau})} T\,(x, y, \tau, \tau_s), \tag{14}$$

where the arguments of $f, f_\tau, f_{\tau\tau}$ are the coordinates (x, y) and the direction angle τ of the curve C at a_0, while in $\bar{f} = f\,(x, y, \bar{\tau})$ the angle $\bar{\tau}$ is the angle transversal to τ defined by the equations (9). The expression $T\,(x, y, \tau, \tau_s)$ is the first member of the Euler equation (4) taken at a point a_0 of the curve C.*

§ 2. *Area in a Field.*

If a region R on any surface is simply covered by a one-parameter family of curves, the surface area of the region can be calculated by finding the set of curves orthogonal to the original family, and taking the double integral of the product of the elements of length along the curves of the two systems over the entire region. The result is the well-known integral

$$\iint \sqrt{EG - F^2}\,dx\,dy, \tag{15}$$

which is independent of the systems of curves used in the calculation because E, F and G are functions only of the parameters x and y of the surface.

* Here s is used instead of σ for the length of arc along C.

Thought of in the xy-plane the image of the region R is simply covered by two families of curves one of which is transversal to the other in the sense of the calculus of variations, transversality in the plane being equivalent to orthogonality on the surface.

In a similar manner, for any field of curves a double integral related to the integral (1) can be derived defining what will be called *area in the field.* The result is not independent of the family of curves used in its derivation, for, unlike the integrand of the integral (15), the new integrand contains not only x and y, but the function $\tau(x, y)$ defining the directions of the curves of the family. The area with respect to the integral (1) is dependent, therefore, not only upon the form of the region but upon the field of curves which covers the region.

Let the family of curves E have equations of the form (5). If τ is the direction angle at a point (x, y) of a curve E of the family, then a curve cutting E transversally at (x, y) must have a direction angle $\bar{\tau}$ which satisfies the equations (9). The parameter s can be determined as a function $s(\alpha)$ of α, so that, at any point of the curve

$$x = \phi(s(\alpha), \alpha), \quad y = \psi(s(\alpha), \alpha), \tag{16}$$

the direction angle has the value $\bar{\tau}$ transversal to τ. For, on account of the properties of ϕ, ψ and f, and with the further assumption that f does not vanish in the field, the equations

$$\left.\begin{aligned} \phi_s s_\alpha + \phi_\alpha &= k(f \sin\tau + f_\tau \cos\tau), \\ \psi_s s_\alpha + \psi_\alpha &= k(-f \cos\tau + f_\tau \sin\tau), \end{aligned}\right\} \tag{17}$$

are solvable for s_α and k as functions of class C' in s and continuous in α, the arguments x, y and τ being here supposed replaced by their values in terms of s and α in the field. From the existence theorems on differential equations it results from this that through each point of the field there passes one and but one transversal curve (16).

The integrand of the integral (1) taken along a transversal (16) has the form

$$\bar{f}\sqrt{(\phi_s s_\alpha + \phi_\alpha)^2 + (\psi_s s_\alpha + \psi_\alpha)^2}\, d\alpha,$$

where the arguments of $\bar{f}$ are ϕ, ψ and $\bar{\tau}$ for the curve (16). From (17) this expression becomes

$$\bar{f}\sqrt{f^2 + f_\tau^2}\,\frac{\Delta}{f}\, d\alpha, \tag{18}$$

where Δ is the functional determinant of the equations (5). Similarly, along

Since the extremals are the curves along which the curvature $1/\rho$ vanishes, the first part of the theorem is evident from (21) and (23). If f is different from zero for all values of τ, the same will be true of ω_τ, and the equation (23) will have a solution through an arbitrarily chosen element (x, y, τ). Hence, when the totality of extremals is the same as the totality of solutions of (23), the equation (24) must be an identity in x, y, τ. This is the case which occurs in the surface theory when the function f has the form (2), as will be shown later. It is also possible to have an identity of the form (24) for other values of f, for example when f does not contain x and y, but it is difficult to determine the most general function f for which such an identity holds.

If the equation (23) *has among its solutions a one-parameter family of extremals forming a field, the functions M and N are expressible in the form*

$$M = g' \cos\tau' - g'_\tau \sin\tau' - H \sin\tau', \quad N = g' \sin\tau' + g'_\tau \cos\tau' + H \cos\tau'. \tag{25}$$

Here H is a suitably chosen functinn of x and y, and the arguments of g' and g'_τ are x, y and the function $\tau'(x, y)$ which represents the angle at the point (x, y) between the x-axis and the tangent to an extremal of the field.

For the equation (24) must be an identity in the field when $\tau'(x, y)$ is substituted for τ, and the expressions (25) with $H = 0$ are solutions of this equation. It follows readily that any functions M and N for which the equation is satisfied are expressible in the form (25). When H is identically zero, the following theorem holds:

The sum of the variations of the angle ω along the sides of a triangle Δ whose sides are solutions of the equation

$$\frac{d\omega}{ds} = (g' \cos\tau' - g'_\tau \sin\tau') \cos\tau + (g' \sin\tau' + g'_\tau \cos\tau') \sin\tau, \tag{26}$$

where $\tau'(x, y)$ is the angle function for an arbitrarily chosen field of curves, is expressible in the form

$$\iint_\Delta T(g')\, dx\, dy.$$

The expression in the second member of (26) *is exactly $g(x, y, \tau)$ when g is linear in $\cos\tau$ and $\sin\tau$ with coefficients functions of x and y.*

For, if both sides of equation (26) are integrated around the boundary of Δ, the result on the left is the sum of the variations of the angle ω on the sides of the triangle, while on the right Green's theorem gives

$$\iint_\Delta \left\{ \frac{\partial}{\partial x}(g' \sin\tau' + g'_\tau \cos\tau') - \frac{\partial}{\partial y}(g' \cos\tau' - g'_\tau \sin\tau') \right\} dx\, dy,$$

and it is readily seen that the expression under the integral sign is exactly $T(g')$.

If the angle function $\tau(x, y)$ (used for convenience instead of the $\tau'(x, y)$ in the last theorem) *belongs to a field of extremals, the value of* $T(g)$ *is*

$$T(g) = \left(K + \frac{1}{V}\frac{d^2 V}{d u^2}\right)\bar{f}\sqrt{f^2 + f_\tau^2}, \tag{27}$$

where K and V are given by formulas (29) *and* (30) *below.*

The function K is an invariant associated with Jacobi's equation which reduces to the curvature of a surface when f is the element of length on a surface, in which case also the value of V is unity. It has been shown in the preceding section that the expression $\bar{f}\sqrt{f^2 + f_\tau^2}\,dx\,dy$ may be regarded as an element of area in the field. For the case of the surface theory it reduces to the element of area on the surface.

When the function f involves the curvature τ_s as well as the variables x, y, τ, the value of $T(f)$ will be defined to be

$$T(f) = f_x \sin\tau - f_y \cos\tau + \tau_s(f - \tau_s f_{\tau_s}) + \frac{d}{ds}\left(f_\tau - \frac{d}{ds}f_{\tau_s}\right).^*$$

The operator T has the following properties for arbitrary functions f and g:

$$\left.\begin{aligned} &T\left[\frac{d}{ds}f(x, y, \tau)\right] \equiv 0, \quad T(f+g) = T(f) + T(g), \\ &T(fg) = f\,T(g) + g\,T(f) - fg\tau_s \\ &\qquad + \left(f_\tau - \frac{d}{ds}f_{\tau_s}\right)\frac{dg}{ds} + \left(g_\tau - \frac{d}{ds}g_{\tau_s}\right)\frac{df}{ds} - \frac{d}{ds}\left(f_{\tau_s}\frac{dg}{ds} + g_{\tau_s}\frac{df}{ds}\right). \end{aligned}\right\} \tag{28}$$

The relation (21) may be written

$$g(x, y, \tau) = \frac{d\omega}{ds} - W\,T(f),$$

where

$$W = \frac{\omega_\tau}{f_1} = \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^2 f_1}.$$

On account of the relations (28) it follows, for the function g just discussed, that

$$T(g) = -W\,T[T(f)] + \frac{d}{ds}\left(f_1\frac{dW}{ds}\right) - [T(W) - W\tau_s]\,T(f) - W_\tau\frac{d\,T(f)}{ds}.$$

In a field of extremals of the function f the last two terms vanish, while

$$T[T(f)] = \frac{\partial T(f)}{\partial x}\sin\tau - \frac{\partial T(f)}{\partial y}\cos\tau - f_1\tau_s^2 = f_2.\dagger$$

* See Radon, "Über das Minimum des Integrals $\int F(x, y, \theta, \kappa)\,ds$," *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften zu Wien*, Vol. CXIX (1910), Abtheilung IIa, p. 1257.

† See Bliss, "A New Form of the Simplest Problem of the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 405.

Let K and V be defined by the formulas *

$$K = -\frac{f_2}{f_1 f^2} - \frac{1}{\sqrt{f f_1}} \frac{d^2}{d u^2} \sqrt{f f_1}, \tag{29}$$

$$V = \sqrt{\pm f f_1}\, W = \frac{\bar{f} \sqrt{f^2 + f_\tau^2}}{f \sqrt{\pm f f_1}}, \tag{30}$$

in which the sign under the radical may be so chosen that the radical is real. Furthermore, let the differentiations with respect to s be transformed into differentiations with respect to u, where

$$\frac{d u}{d s} = f(x, y, \tau).$$

Then, after some calculation, it is found that the value of $T(g)$ is exactly that given in the formula (27). For convenience in notation the prime has been dropped throughout these calculations.

In any field of extremals there exists a two-parameter family of curves, defined by the equation (26), *to which the extremals of the field themselves belong, and which have the property that in any polygon whose sides consist of these curves the sum of the variations of the generalized angle* ω *along the sides of the polygon is equal to the double integral*

$$\iint \left(K + \frac{1}{V} \frac{d^2 V}{d u^2} \right) d A$$

taken over the interior of the polygon. Here K *is a known invariant* (29) *connected with Jacobi's equation,* dA *is the element of area in the field, and the function* V *is given by the equation* (30). *The derivative* $d^2 V / d u^2$ *is taken with respect to the variable*

$$u = \int f(x, y, \tau)\, d s,$$

in which the integral is taken along an extremal of the field.

§ 5. *Invariantive Properties.*

The expressions which enter into the results of the preceding sections are either relative or absolute invariants under point transformations. Let the variables x, y, τ be transformed by the transformation

$$x = X(\xi, \eta) \qquad y = Y(\xi, \eta) \tag{31}$$

* Compare the formula given for K with that given by Underhill for K_0 in his paper, "Invariants of the Function $F(x, y, x', y')$ in the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol, IX (1908), p. 334, formula (35). For the relation of K_0 to the second variation, see p. 336. The value of the function K in the text above is the negative of Underhill's K_0.

into a new set ξ, η, θ. The relation between τ and θ is determined by the equations

$$R\cos\tau = X_\xi \cos\theta + X_\eta \sin\theta, \qquad R\sin\tau = Y_\xi\cos\theta + Y_\eta\sin\theta, \tag{32}$$

where

$$\begin{aligned} R &= \sqrt{(X_\xi\cos\theta + X_\eta\sin\theta)^2 + Y_\xi\cos\theta + Y_\eta\sin\theta)^2} \\ &= \cos\tau\,(X_\xi\cos\theta + X_\eta\sin\theta) + \sin\tau\,(Y_\xi\cos\theta + Y_\eta\sin\theta). \end{aligned} \tag{33}$$

By differentiating (32), it is found that

$$\left.\begin{aligned} R_\theta &= \cos\tau\,(-X_\xi\sin\theta + X_\eta\cos\theta) + \sin\tau\,(-Y_\xi\sin\theta + Y_\eta\cos\theta), \\ R\tau_\theta &= -\sin\tau\,(-X_\xi\sin\theta + X_\eta\cos\theta) + \cos\tau(-Y_\xi\sin\theta + Y_\eta\cos\theta) = \frac{D}{R}, \\ R_\xi &= \cos\tau\frac{dX_\xi}{d\sigma} + \sin\tau\frac{dY_\xi}{d\sigma}, \\ R\tau_\xi &= -\sin\tau\frac{dX_\xi}{d\sigma} + \cos\tau\frac{dY_\xi}{d\sigma}, \end{aligned}\right\} \tag{34}$$

with similar expressions for the derivatives with respect to η. The symbol D stands for the functional determinant of the transformation (31), and σ is the length of arc along a curve in the $\xi\eta$-plane.

The integral (1) becomes

$$\int h\,(x, y, \theta)\,\sqrt{\xi'^2 + \eta'^2}\,dt, \tag{35}$$

where

$$\left.\begin{aligned} h &= fR \\ &= (f\cos\tau - f_\tau\sin\tau)(X_\xi\cos\theta + X_\eta\sin\theta) + (f\sin\tau + f_\tau\cos\tau)(Y_\xi\cos\theta + Y_\eta\sin\theta), \\ h_\theta &= fR_\theta + f_\tau R\tau_\theta \\ &= (f\cos\tau - f_\tau\sin\tau)(-X_\xi\sin\theta + X_\eta\cos\theta) + (f\sin\tau + f_\tau\cos\tau)(-Y_\xi\sin\theta + Y_\eta\cos\theta), \\ h_\xi &= R\,(f_x X_\xi + f_y Y_\xi)\,fR_\xi + f_\tau R\tau_\xi \\ &= R\,(f_x X_\xi + f_y Y_\xi) + (f\cos\tau - f_\tau\sin\tau)\frac{dX_\xi}{d\sigma} + (f\sin\tau + f_\tau\cos\tau)\frac{dY_\xi}{d\sigma}, \end{aligned}\right\} \tag{36}$$

with a similar expression for the derivative of h with respect to η.

It follows readily that

$$\left.\begin{aligned} h\cos\theta - h_\theta\sin\theta &= X_\xi(f\cos\tau - f_\tau\sin\tau) + Y_\xi(f\sin\tau + f_\tau\cos\tau), \\ h\sin\theta + h_\theta\cos\theta &= X_\eta(f\cos\tau - f_\tau\sin\tau) + Y_\eta(f\sin\tau + f_\tau\cos\tau). \end{aligned}\right\} \tag{37}$$

From equations (9), therefore,

$$\left.\begin{aligned} X_\xi\cos\bar\theta + X_\eta\sin\bar\theta &= D\sqrt{\frac{f^2 + f_\tau^2}{h^2 + h_\theta^2}}\cos\bar\tau, \\ Y_\xi\cos\bar\theta + Y_\eta\sin\bar\theta &= D\sqrt{\frac{f^2 + f_\tau^2}{h^2 + h_\theta^2}}\sin\bar\tau, \end{aligned}\right\} \tag{38}$$

showing that *the direction $\bar{\tau}$ transversal to τ at a point in the xy-plane is transformed by equations* (32) *into the direction $\bar{\theta}$ transversal to θ at the corresponding point of the $\xi\eta$-plane.*

The relation between the Euler expressions, $T(f)$ and $T(g)$, for the integrals (1) and (35) can be calculated from the relations (36) and (37). From (31),

$$\frac{ds}{dt} = R\frac{d\sigma}{dt}, \quad \frac{d}{d\sigma} = \frac{d}{dt} \Big/ \frac{d\sigma}{dt} = R\frac{d}{ds}.$$

It follows then that

$$\sin\theta\, T(h) = h_\xi - \frac{d}{d\sigma}(h\cos\theta - h_\theta \sin\theta) = \quad D\sin\theta\, T(f),$$

$$-\cos\theta\, T(h) = h_\eta - \frac{d}{d\sigma}(h\sin\theta + h_\theta\cos\theta) = -D\cos\theta\, T(f).*$$

Hence, the Euler expression $T(f)$ for the integral (1) *and the corresponding quantity $T(h)$ for* (35) *are related by the formula*

$$T(h) = D\,T(f). \tag{39}$$

From equations (33) and (38),

$$R(\xi, \eta, \bar{\theta}) = D\sqrt{\frac{f^2 + f_\tau^2}{h^2 + h_\theta^2}}, \tag{40}$$

so that, with the help of the second of equations (34) and the first of (36),

$$\omega = \int_\theta^{\theta'} \frac{\bar{h}\sqrt{h^2 + h_\theta^2}}{h^2}\, d\theta = \int_\tau^{\tau'} \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^2}\, d\tau, \tag{41}$$

where τ' corresponds to θ', and τ to θ, by means of the transformation (32).

The generalized angle with respect to the integral (1) *between two directions τ and τ' at a point of the xy-plane, is equal to the generalized angle with respect to the integral* (35) *between the corresponding directions at the corresponding point of the $\xi\eta$-plane.*

By differentiating equations (37) for θ and making use of (32) and the value of τ_θ from the second of equations (34), it follows that

$$h + h_{\theta\theta} = \frac{D^2}{R^3}(f + f_{\tau\tau}). \tag{42}$$

From the first of the relations (36), with (40), (42) and (39), it can be shown that *the extremal curvature*

$$\frac{1}{\rho} = \frac{\bar{f}\sqrt{f^2 + f_\tau^2}}{f^3(f + f_{\tau\tau})}\, T(f)$$

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* See *Transactions*, Vol. VIII (1907), p. 407.

with respect to the integral (1), *is the same as the extremal curvature with respect to the integral* (35) *at corresponding points of curves which are equivalent under the transformation* (31).

Similarly, from the first of the relations (36), with (40),

$$\iint \bar{h}\sqrt{h^2 + h_\theta^2}\, d\xi\, d\eta = \iint \bar{f}\sqrt{f^2 + f_\tau^2}\, D\, d\xi\, d\eta = \iint \bar{f}\sqrt{f^2 + f_\tau^2}\, dx\, dy,$$

so that *the area enclosed by a curve C in a field of curves in the xy-plane, is equal to the area enclosed by the image of C in the $\xi\eta$-plane taken with respect to the image of the original field defined by the transformation* (31).

It remains to show that the two integrals which occur in the generalization of Gauss' theorem have also invariantive properties. From the behavior of ω, ρ and h it follows that

$$g(h) = R\, g(f),$$

where $g(h)$ and $g(f)$ are the valnes of the function (22) formed for h and f, respectively. From this it follows also that the integral

$$\int g(x, y, \tau)\, ds$$

is invariant under the transformation (31). Equations (37) show that the expression

$$(g \cos\tau - g_\tau \sin\tau)\, dx + (g \sin\tau + g_\tau \cos\tau)\, dy$$

is equal to the expression formed in a similar way for the integral (35). The Euler expression $T(g)$ is multiplied by the factor D when the transformation (31) is applied. Consequently, the function $T(g)/(\bar{f}\sqrt{f^2 + f_\tau^2})$ is an absolute invariant. From equation (27), the value of this fraction in a field consisting of extremals is

$$\frac{T(g)}{\bar{f}\sqrt{f^2 + f_\tau^2}} = K + \frac{1}{V}\frac{d^2 V}{du^2}, \tag{43}$$

and it follows at once that the value of the second member of the equation, taken in a field of extremals, is an absolute invariant. But from equations (30), (36), (40) and (42) it is seen that V has the same property. Furthermore, the derivative of any invariant with respect to t is always invariant, and the same is therefore true of the derivatives of V with respect to u, since u is an invariant and

$$\frac{dV}{du} = \frac{dV}{dt} \Big/ \frac{du}{dt}.$$

It follows then that *the three integrals*

$$\int g(x, y, \tau)\, ds, \qquad \iint \left(K + \frac{1}{V}\frac{d^2 V}{du^2}\right) dA,$$

$$\int [(g \cos\tau - g_\tau \sin\tau)\, dx + (g \sin\tau + g_\tau \cos\tau)\, dy],$$

where g is the function defined by equation (22), are all invariant under the transformation (31). In the integrand of the second of these the function V and its derivative d^2V/du^2 are invariants for all values of the arguments $x, y, \tau, \tau_s, \tau_{ss}$, while K is invariant at least when the arguments τ, τ_s, τ_{ss} are the functions of x and y defining the direction, curvature and derivative of curvature for an extremal of a field.

§ 6. *Application to the Case of the Surface Theory.*

It is interesting to note how the invariants found above are related to the well-known invariants of the surface theory when the function f has the value (2). With the help of the notations

$$\lambda = E\cos\tau + F\sin\tau, \qquad \mu = F\cos\tau + G\sin\tau,$$

it follows, as in a preceding paper,* that

$$f = \frac{\lambda\cos\tau + \mu\sin\tau}{f}, \qquad f_\tau = \frac{-\lambda\sin\tau + \mu\cos\tau}{f},$$

$$\bar{f} = \sqrt{EG - F^2}\,\frac{f}{\sqrt{\lambda^2 + \mu^2}}, \quad f^2 + f_\tau^2 = \frac{\lambda^2 + \mu^2}{f^2}, \quad f + f_{\tau\tau} = \frac{EG - F^2}{f^3},$$

so that the integral (41) giving the generalized angle takes the form

$$\omega = \int_\tau^{\tau'} \frac{\sqrt{EG - F^2}}{E\cos^2\tau + 2F\cos\tau\sin\tau + G\sin^2\tau}\,d\tau.$$

The Euler expression $T(f)$ has often been calculated. It has the value

$$\begin{aligned} f^3\,T(f) = {} & (EG - F^2)\,\tau_s \\ & + (E\cos\tau + F\sin\tau)\,[(F_x - \tfrac{1}{2}E_y)\cos^2\tau + G_x\cos\tau\sin\tau + \tfrac{1}{2}G_y\sin^2\tau] \\ & - (F\cos\tau + G\sin\tau)\,[\tfrac{1}{2}E_x\cos^2\tau + E_y\cos\tau\sin\tau + (F - \tfrac{1}{2}G_x)\sin^2\tau], \end{aligned}$$

which, substituted in (14), gives the well-known formula for the geodesic curvature†

$$\frac{1}{\rho} = \frac{T(f)}{\sqrt{EG - F^2}}.$$

The integral of $g(x, y, \tau)$ from equation (22), which occurs in the proof of Gauss' theorem, can be found after some calculation. It has the value

$$g(x, y, \tau) = \frac{-2EF_x + EE_y + E_xF}{2E\sqrt{EG - F^2}}\cos\tau + \frac{E_yF - EG_x}{2E\sqrt{EG - F^2}}\sin\tau. \quad (44)$$

Since g is linear in $\cos\tau$ and $\sin\tau$, the value of $T(g)$ is independent of the

* *Transactions*, Vol. VII (1906), p. 193.

† See, for example, Bolza, "Vorlesungen über Variationsrechnung," p. 210.

direction angle τ and consequently entirely independent of the particular field in which it is taken.

The value of V is found to be unity, so that the invariant K in a field of extremals is exactly the expression $T(g)/(\bar{f}\sqrt{f^2+f_\tau^2})$ in equation (43). The numerator of the latter, calculated directly from (44), is given by the equation

$$\begin{aligned} 4\,(EG-F^2)^{3/2}\,T(g) = {} & E\,(G_x^2 + E_y G_y - 2F_x G_y) \\ & + F\,(E_x G_y + 4F_x F_y - 2E_y F_y - 2F_x G_x - E_y G_x) \\ & + G\,(E_y^2 - 2E_x F_y + E_x G_x) \\ & - 2\,(EG-F^2)\,(E_{yy} - 2F_{xy} + G_{xx}), \end{aligned}$$

which, taken with the expression

$$\bar{f}\sqrt{f^2+f_\tau^2} = \sqrt{EG-F^2},$$

shows that K is the Gaussian curvature.

These results were found by direct calculations which were somewhat long. It would be still more difficult to identify K with the curvature by substituting in the expression (43), without a special choice of coordinates, the value of f from (2) and the values of τ_s and τ_{ss} derived from Euler's equation. But the calculation becomes very simple when a transformation has been made which takes the transversals of the field into the lines $x=a$, and the extremals of the field into the lines $y=b$, and which furthermore makes the new x-coordinate of any point equal to the value of the integral (1) taken along the extremal arc joining the point in question to some fixed initial transversal. When such a transformation has been made, the function $f(x, y, \tau)$ will have special values for $\tau=0$. Since

$$x = \textstyle\int_0^x f(x, y, 0)\, dx,$$

it follows that

$$f(x, y, 0) = 1. \tag{45}$$

The direction $\bar{\tau} = \pi/2$ is everywhere transversal to $\tau = 0$, and consequently, from equations (9),

$$f_\tau(x, y, 0) = 0. \tag{46}$$

In the field the values of τ and τ_s are zero, and it is easy to see from the formula just preceding (29), with (45) and (46), that $f_2=0$. Consequently, from (29),

$$K = -\frac{1}{\sqrt{ff_1}}\frac{d^2}{dx^2}\sqrt{ff_1}. \tag{47}$$

For the length integral on a surface the conditions (45) and (46) mean that

$$f(x, y, \tau) = \sqrt{\cos^2\tau + m^2 \sin^2\tau},$$

where G has been put equal to m^2. Then, in the field, $f \equiv 1$ and $f_1 \equiv m^2$; and it follows, by substituting in equation (47), that

$$K = -\frac{1}{m}\frac{d^2 m}{d x^2}.$$

This is the well-known formula for the curvature of a surface when geodesic coordinates are used.*

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* Gauss, *loc. cit.*, p. 28.

On the Medians of a Closed Convex Polygon.

By Arnold Emch.

§ 1. Introduction.

In an article which appeared in this Journal* I have proved that all *medians* of a closed convex analytic curve, defined as loci of the mid-points of chords of all systems of parallel chords, are continuous and analytic curves. With each pair of orthogonal directions σ and τ are associated two medians M_σ and M_τ which always intersect in one and only one point. It is assumed that the tangent changes continuously as the point of tangency moves continuously on the curve. With every pair (σ, τ) a rhomb $A_1 A_2 A_3 A_4$ is associated whose diagonals $A_1 A_3$ and $A_2 A_4$ are parallel to σ and τ and whose vertices lie on the given curve. This fact then leads to the conclusion:

To every closed convex analytic curve without rectilinear segments, at least one square may be inscribed.†

In what follows I shall show that at least one square may be inscribed in any convex rectilinear polygon, and generally in any closed convex curve formed by analytic arcs.

When nothing to the contrary is stated, all points of the boundary are supposed to be included by the domain bounded by the curve. Such a curve has the property that all points of the segment joining any two distinct points of the curve belong to the enclosed domain, and that no other points of the domain lie on the straight line joining the two points.‡ Excluding for the present certain cases where pairs of rectilinear segments of the curve become parallel, two medians M_σ and M_τ, as defined above, accordingly can not have more than one point of intersection. This is also true in all cases of pairs of

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* Vol. XXV (1913), No. 4, pp. 407–412.

† The problem connected with this theorem was suggested to me by Dr. Kempner. After I had found a proof of the theorem and sent it to this Journal for publication, I was told that Dr. Toeplitz (at that time in Göttingen) and his pupils had succeeded in solving the problem. A statement to this effect, without proof, may be found in the *Verhandlungen der schweizerischen Naturforschenden Gesellschaft in Solothurn*, August 1, 1911, p. 197, according to which Dr. Toeplitz succeeded in proving the theorem for convex curves only. I have not been able to find a publication of his method and results.

‡ Minkowsky, "Theorie der konvexen Körper," zweiter Band, p. 154.

parallel sides, when the distance between two parallel sides is less than the smaller of the sides. If there were more than one point of intersection, it would be possible to inscribe at least two rhombs in the curve with parallel diagonals. Among the eight vertices of two rhombs of this kind there would, however, be at least four that would form a reentrant quadrangle, which is against the assumption of a convex curve. The case of pairs of parallel sides will be considered later.

§ 2. Medians of a Convex Rectilinear Polygon.

I. *Medians Near Four Consecutive Vertices of the Polygon.*

To study the variation of the medians as the direction of the system of parallel chords changes continuously, take first a direction $\sigma_1 \parallel DB$. The median of the chords parallel to σ_1 within the triangle BCD is the line

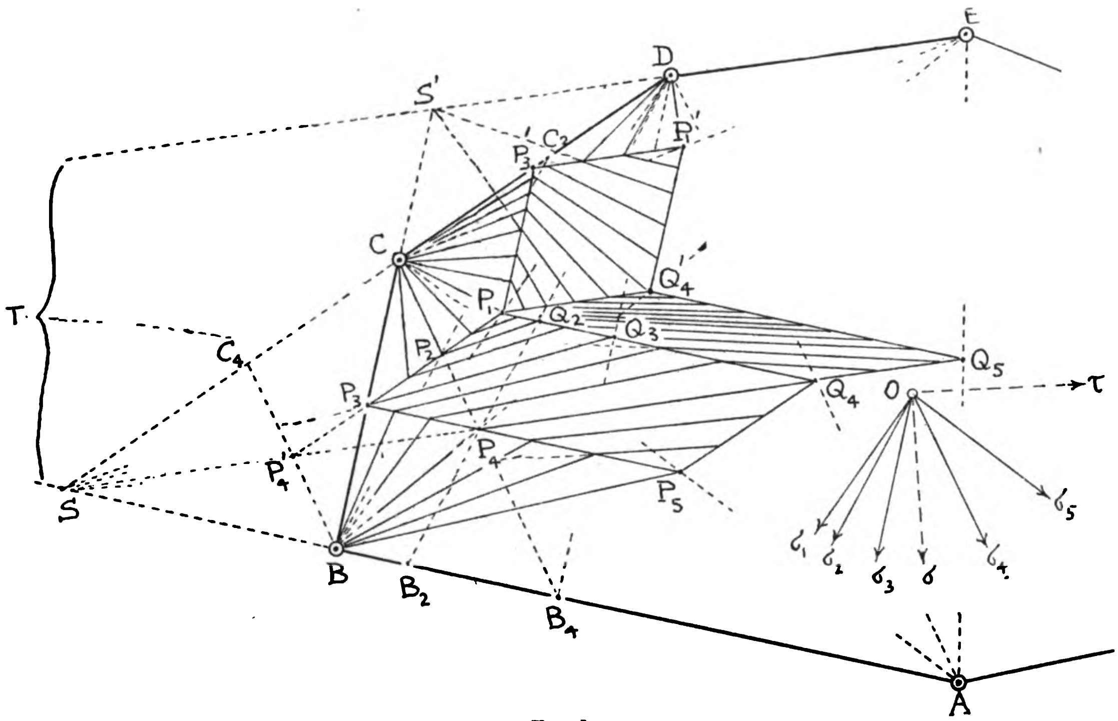

Fig. 1.

joining C with the mid-point P_1 of DB. Let σ_2 be an intermediate direction in the continuous change in the positive sense from σ_1 to $\sigma_5 \parallel CA$. Draw C_2B and $DB_2 \parallel \sigma_2$, and designate by P_2 and Q_2 their mid-points (Fig. 1). The median of all parallel chords parallel to σ_2 between C and DB_2 consists of CP_2 and P_2Q_2. For $\sigma_3 \parallel CB$, the median P_3Q_3 terminates in the mid-point P_3 of CB. As σ_2 approaches σ_3, CP_2 approaches CP_3 as a limit. For $\sigma_4 \parallel DA$,

the median consists of the lines BP_4 and P_4Q_4, where Q_4 is the mid-point of DA. Turning in the opposite direction, as σ_4 approaches σ_3, BP_4 approaches BP_3 as a limit. Finally, for $\sigma_5 \parallel CA$, the median consists of the line BP_5. It will be noticed that as σ_4 approaches σ_5, P_4Q_4 approaches $P_5(0)$ as a limit. The prolongations of P_2Q_2, P_3Q_3, P_4Q_4, ... all pass through S, the intersection of DC and AB produced. Hence, as the direction of the system of parallel chords changes continuously from σ_1 to σ_5, corresponding medians, originating from C, P_3 and B, change continuously, except for $\sigma_3 \parallel CB$, where as a limit CP_3 changes abruptly into BP_3. The medians consist of polygonal lines of two sides, whose vertices $P_1, \ldots, P_2, Q_2, \ldots, P_3, Q_3, \ldots, P_4, Q_4, \ldots, P_5$, with the exception of the initial points C and B, are located on the sides of the parallelogram formed by the mid-points of BC, AD, DB and CA. From this fact results a simple construction of the system of medians: Construct the parallelogram $P_1P_3P_5Q_4$; draw all lines bounded by this parallelogram and whose prolongations pass through S; and connect the points on P_1P_3 with C and those on P_3P_5 with B. Within the parallelogram the medians form a continuous set.

As σ_1 changes to σ_2, P_1 moves on P_1P_3 to P_2, so that BP_1 describes a pencil parallel to the pencil of directions $(O.\sigma_1 \ldots \sigma_2)$. But there is also

$$C.P_1P_2 \ldots \barwedge B.P_1P_2 \ldots \barwedge .S.P_1P_2 \ldots,$$

and consequently

$$S.P_1P_2 \ldots \barwedge O.\sigma_1\sigma_2 \ldots,$$

or

$$S.P_1Q_2 \ldots \barwedge O.\sigma_1\sigma_2 \ldots.$$

The question is whether these projectivities are confined to the points of a side of the parallelogram only or whether they hold for the whole pencil through S contained within the parallelogram. Take, for instance, P_4, which is obtained by drawing $B_4C \parallel \sigma_4$. $BC_4 \parallel B_4C$ cuts P_1P_3 produced at P'_4. As $CP_3 = BP_3$ and $P_3P'_4 \parallel CC_4$, there is also $BP'_4 = P'_4C_4$; hence, as $CP_4 = P_4B_4$, SP'_4 produced must pass through P_4. From this follows immediately the projective relation:

$$S.P_1P_2 \ldots P_3 \ldots P_4 \ldots P_5 \barwedge O.\sigma_1\sigma_2 \ldots \sigma_3 \ldots \sigma_4 \ldots \sigma_5.$$

Considering next the quadrangle $BCDE$, the system of medians associated with it may be constructed in precisely the same manner as in case of $ABCD$. The prolonged portions of the medians within the parallelogram $P'_1P'_3P_1Q'_4$ all pass through S', the intersection of ED and BC produced, while the remaining portions are obtained by joining the points of P'_3P_1 to C and those

of $P'_3 P'_1$ to D. From this it is seen that the parts of all medians originating at any vertex, like C, form a pencil whose rays completely fill the parallelogram $P'_3 C P_3 P_1$, and that consequently *no two medians of the complete system of medians associated with the given polygon can intersect within the parallelograms determined by half the sides adjacent at any vertex.*

II. *Medians Associated with a Quadrangle Determined by Any Two Non-adjacent Sides of the Polygon.*

Let $HJKL$ be such a quadrangle (Fig. 2). In perfect analogy with the discussion of the medians within the parallelogram $P_1 P_3 P_5 Q_4$ (Fig. 1), it is found that as the direction of the system of parallel chords changes continu-

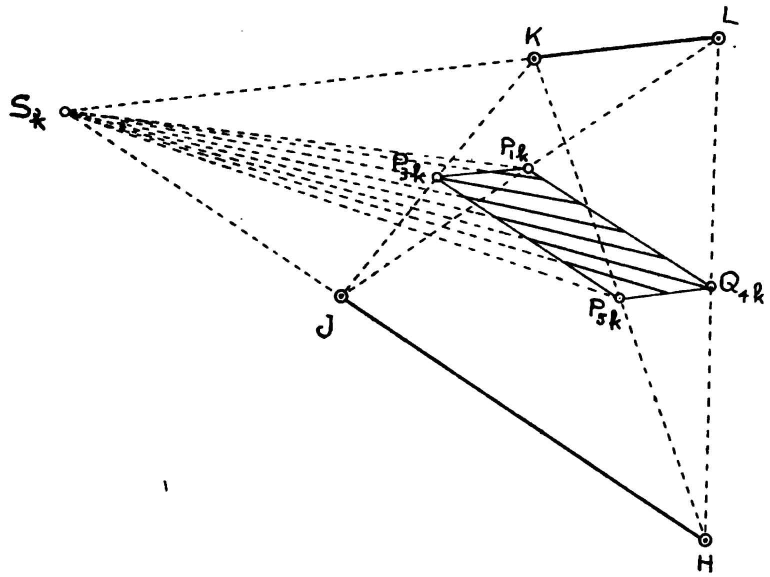

FIG. 2.

ously in the positive sense from LJ to KH, the corresponding medians form a pencil of rays through S_k, the intersection of LK and HJ, within the parallelogram $P_{1k} P_{3k} P_{5k} Q_{4k}$, such that the pencil through S_k is projective to the pencil of corresponding directions. Thus, in Fig. 1, the continuation of the medians terminating in points of $P_1 Q_4$ and $P_1 Q'_4$ are obtained by completing the parallelogram $P_1 Q_4 Q_5 Q'_4$ associated with the quadrangle $ABDE$, and connecting the extremities of the previously constructed medians on $P_1 Q_4$ and $P_1 Q'_4$ with the intersection T of AB and ED. The rays of this pencil within the parallelogram $P_1 Q_4 Q_5 Q'_4$ are the continued medians.

In a polygon of n sides we can now first form n parallelograms, like $P_1 P_3 P_5 Q_4$, having one of their vertices at the mid-points of the sides. Then,

with every side are associated $(n-5)$ parallelograms as defined under II, so that there are $\frac{n(n-5)}{2}$ parallelograms of this kind. Altogether there are $n+\frac{n(n-5)}{2}=\frac{n(n-3)}{2}$ parallelograms on whose sides the vertices of the polygonal lines forming the medians are located. The sides of these parallelograms are parallel to the sides of the given polygon. In every triangle with one side of the polygon as a base and any two diagonals from the extremities of the base to a vertex of the polygon as the other two sides, the line (parallel to the base) joining the mid-points of these two sides is common to two, but only two, parallelograms of the system. Thus, in the triangle DAB (Fig. 1), P_1Q_4 is common to the parallelograms $P_1Q_4P_5P_3$ and $P_1Q_4Q_5Q'_4$ associated with the quadrangles $ABCD$ and $ABDE$. Every point of P_1Q_4 therefore belongs to one and only one median of the entire system of medians. These results may be stated in

THEOREM 1. *In the foregoing system of parallelograms enclosing pencils of medians, every side, except those terminating in mid-points of sides of the polygon, is common to two parallelograms only. Through every point of these common sides only one median of the entire system of medians passes.*

Associated with each of these parallelograms is a point S, or T, through which the prolonged medians within the parallelogram pass. The sides of the n parallelograms terminating at the mid-points of the n sides of the given polygon form a polygon of $2n$ points bounding a closed domain (P) within which the $\frac{n(n-3)}{2}$ parallelograms are located. Within each of these parallelograms the medians associated with it, and prolonged, form a pencil which is projective with the pencil of corresponding directions. These results may be stated in

THEOREM 2. *Within the domain (P), the system of medians of a closed convex rectilinear polygon associated with the continuous set of all directions σ through a fixed point (pencil) is continuous with this pencil. The directions of the medians within each parallelogram, and associated with it, form a pencil which is projective with the pencil of corresponding directions σ. Within the domain of the given polygon (excluding the points of the boundary), and exterior to the domain (P), no medians can intersect. Within the domain (P), the system of medians is continuous.*

So far we have assumed that no pair of sides of the polygon are parallel. From the foregoing development it is obvious that with every quadrangle

determined by two distinct sides of the polygon is associated uniformly a parallelogram whose sides are parallel in pairs to the two sides considered, and which is continuously filled with the segments of a pencil of medians.

In Fig. 1, with the sides AB and DE, for example, is associated the parallelogram $P_1Q_4Q_5Q_4'$. When $AB \parallel DE$, the parallelogram degenerates into a single line P_1Q_5, and the medians of the parallelogram, as AB becomes parallel to DE, will be transformed also into a continuous set on the segment P_1Q_5, varying from 0 at P_1 to the length of P_1Q_5 and from this again to 0 at Q_5. From this it is seen that continuity of the system of medians with respect to both magnitude and position is maintained also in case of pairs of parallel sides.

Theorem 2 is therefore also valid in this case.

§ 3. INSCRIBED RHOMBS.

Consider again a closed convex polygon with no pair of parallel sides and two orthogonal directions σ and τ through O, and construct the medians M_σ and M_τ associated with them. The extremities A_σ and Z_σ of M_σ, and A_τ and Z_τ of M_τ always lie on pairs of opposite sides of a rectangle. M_σ and M_τ therefore necessarily intersect. As stated in the introduction, for a convex curve there can be only one point of intersection R, which, according to theorem 1, lies within the domain (P). As the pair $(\sigma\tau)$ of orthogonal directions changes continuously, the medians M_σ and M_τ within (P) change continuously also, and consequently their point of intersection R describes a continuous curve.

In the neighborhood of R the medians $M_\sigma M_{\sigma'} M_{\sigma''} \ldots\ldots$ and $M_\tau M_{\tau'} M_{\tau''} \ldots\ldots$ form pencils, so that

$$M_\sigma M_{\sigma'} M_{\sigma''} \ldots\ldots \barwedge O \,.\, \sigma\sigma'\sigma'' \ldots\ldots,$$

and

$$M_\tau M_{\tau'} M_{\tau''} \ldots\ldots \barwedge O \,.\, \tau\tau'\tau'' \ldots\ldots;$$

and as, obviously,

$$O \,.\, \sigma\sigma'\sigma'' \ldots\ldots \barwedge O \,.\, \tau\tau'\tau'' \ldots\ldots,$$

there is also

$$M_\sigma M_{\sigma'} M_{\sigma''} \ldots\ldots \barwedge M_\tau M_{\tau'} M_{\tau''} \ldots\ldots.$$

In the neighborhood of the point considered, the curve described by R is therefore an arc of a conic.

Now turning σ and τ continuously through a right angle, the two directions and consequently the two medians M_σ and M_τ are interchanged by a continuous process. R describes therefore a closed curve. We have, therefore:

THEOREM 3. *The locus of the points of intersection R of the medians M_σ and M_τ associated with all pairs of orthogonal directions is a closed continuous curve composed of arcs of conics.*

The case remains to be investigated where one or more pairs of sides of the polygon each consist of parallel sides. We may limit ourselves to one pair, since the effect on the medians by any other pair will be of the same nature.

As stated before, all portions of medians determined by two parallel sides lie in the mid-line between the two sides and vary continuously in their length

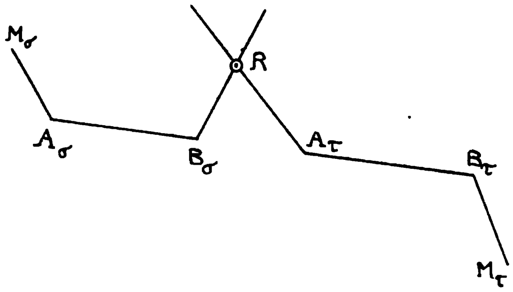

FIG. 3.

and extremities. Consequently, if we take two medians M_σ and M_τ corresponding to a pair of orthogonal directions (σ, τ) and consider the portions $A_\sigma B_\sigma$ and $A_\tau B_\tau$ of each determined by the two parallel sides, two cases may occur:

1) For all pairs (σ, τ) the segments $A_\sigma B_\sigma$ and $A_\tau B_\tau$ have no point in common (Fig. 3), so that M_σ and M_τ have in every position one and only one definite point of intersection R. As can easily be established by elementary geometry, this will, for instance, always be the case when the distance between the two parallel sides of the polygon is less than the smaller of the two sides.

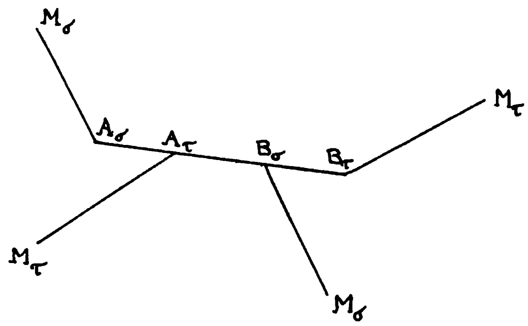

FIG. 4.

Every position of R is the center of an inscribed rhomb. As σ changes continuously, the system of chords also changes continuously. Consequently, as R describes a continuous curve, the corresponding inscribed rhombs form a continuous set as to magnitude and position.

2) For all pairs (σ, τ) of a certain continuous set S, the segments $A_\sigma B_\sigma$ and $A_\tau B_\tau$ have a portion $A_\tau B_\sigma$ in common (Fig. 4). This means that for

4

every pair (σ, τ) of S there is an infinite number of inscribed rhombs whose centers form the continuous set of points on the segment $A_\tau B_\sigma$, and whose axes are parallel to the directions σ and τ. Their vertices lie in pairs on the two parallel sides. But as σ and τ within S change continuously, also the extremities A_τ and B_σ, and consequently the segment $A_\tau B_\sigma$ and with it the infinite set of rhombs, including those with centers at A_τ and B_σ, associated with every pair (σ, τ) of the system S, change continuously. For the extreme pairs (σ, τ) of the system S the segment $A_\tau B_\sigma$ reduces to a point and is continuously connected with the points of the curve traced by the uniform intersections of all pairs M_σ and M_τ belonging to the remaining set (σ, τ) outside of S.

Evidently, when M_σ and M_τ have a common segment $A_\tau B_\sigma$, there are not other intersections of the two medians outside of the segment. Otherwise we could construct four points forming a reentrant quadrangle, which is not possible in a convex curve. Hence also in the second case the whole set of inscribed rhombs is continuous, and we have:

THEOREM 4. *The set of rhombs inscribed in a convex rectilinear polygon is continuous.*

Turning σ and τ through a right angle and designating the inscribed rhomb in the initial position by $A_1 A_2 A_3 A_4$, then, after the interchange of σ and τ, the original rhomb, in the same order, will have continuously changed into a rhomb $A_1' A_2' A_3' A_4' \equiv A_2 A_3 A_4 A_1$. The smaller diagonal, say $A_1 A_3$, changes into the larger, $A_2 A_4$, while simultaneously the larger, $A_2 A_4$, passes into the smaller, in both cases through a continuous set of diagonals. Hence, we have exactly the same situation as in case of an oval (*loc. cit.*), and we can state

THEOREM 5. *In every convex rectilinear polygon, at least one square may be inscribed.*

§ 4. CONVEX CURVE FORMED BY ANALYTIC ARCS.

The foregoing construction is evidently valid for a convex polygon of any number of rectilinear sides.

Suppose now that a convex polygon formed by analytic arcs (including rectilinear segments) be given. Replace all non-rectilinear arcs by polygonal lines of any number of sides inscribed in these arcs, so that for the complete rectilinear convex polygon obtained in this manner the foregoing constructions may be applied. They hold no matter how small the sides of the polygonal lines inscribed in the arcs finally become. Increasing the number of sides and at the same time decreasing their length indefinitely, the

inscribed polygon will approach the given convex curve formed by analytic arcs as a limit. If σ is any direction, then any line l parallel to σ cutting the curve in two points C_1 and C_2 cuts the polygon in general in two points P_1 and P_2 (Fig. 5). Designating the mid-point of C_1C_2 by C and that of P_1P_2 by P, choosing on l any point O, so that

$$OC_1 = a_1, \quad OC_2 = a_2, \quad OP_1 = b_1, \quad OP_2 = b_2,$$

then

$$OC = \tfrac{1}{2}(a_1 + a_2), \quad OP = \tfrac{1}{2}(b_1 + b_2),$$

and

$$PC = OC - OP = \tfrac{1}{2}[(a_1 - b_1) + (a_2 - b_2)].$$

The side of the inscribed polygon on which P_1 lies approaches the arc in which it is inscribed (as a chord) as a limit, so that $OC_1 - OP_1 = a_1 - b_1 = \varepsilon_1$

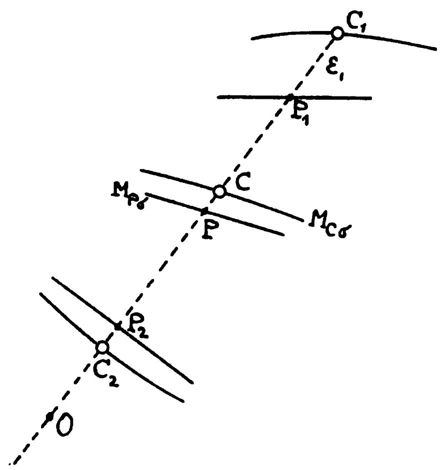

FIG. 5.

can be made as small as we please. Likewise $OC_2 - OP_2 = a_2 - b_2 = \varepsilon_2$ can be made arbitrarily small. But C is a point of the median associated with σ and the given curve. Consequently, as ε_1 and ε_2 become arbitrarily small, *i. e.*, as the inscribed polygon approaches the given curve as a limit, also

$$PC = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)$$

becomes arbitrarily small. As P is a point of the median associated with the inscribed polygon, we conclude from this that as the inscribed polygon approaches the given curve as a limit, the median $M_{P\sigma}$ associated with the inscribed polygon approaches the median $M_{C\sigma}$ associated with the given curve as a limit. Passing to the limit does not destroy any of the properties of continuity and uniformity of the intersections of $M_{P\sigma}$ and $M_{P\tau}$ while passing into $M_{C\sigma}$ and $M_{C\tau}$. The domain (P) approaches a definite domain (P') as a

limit. Outside of (P') no two medians intersect, and within (P') the set of medians is continuous. By the same argument that led to theorem 5, we conclude:

THEOREM 6. *It is always possible to inscribe at least one square in any convex curve formed by analytic arcs.*

An interesting special case is obtained when the curve is symmetric with respect to a fixed center. All medians now pass through the center; the locus of the R's coincides with the center. Hence:

THEOREM 7. *The center of a square inscribed in a convex curve formed by analytic arcs, and with central symmetry, coincides with the center of the curve.*

The Rational Sextic Curve, and the Cayley Symmetroid. * †

By J. R. Conner.

§ 1.

The rational space sextic curve, ρ_3^6, may be regarded as determined by a single form involving binary and quaternary variables,

$$(a\xi)\,(\alpha t)^6 \equiv (m\,t)^6 = 0, \tag{1}$$

where the symbolic a's and α's must be taken in combination to have an actual meaning. Evidently (1) is the equation in plane coördinates of the point of the curve to which is attached the parameter t. Fixing ξ, (1) gives the parameters of the six points in which the plane ξ meets ρ_3^6. The notation $(m\,t)^6 = 0$, where m is a form whose coefficients are linear homogeneous functions of the four ξ's, will often be convenient.

Any invariant of the form $(m\,t)^6$ equated to zero represents a contravariant surface of (1); it is precisely the locus of planes which meet ρ_3^6 in six points whose parameters are the roots of a binary sextic for which the given invariant vanishes. An important invariant of the binary sextic is the so-called *catalecticant* — the invariant whose vanishing is the condition that the sextic have an apolar cubic. For the form (1) this is the symmetrical determinant

$$\begin{vmatrix} m_0, & m_1, & m_2, & m_3 \\ m_1, & m_2, & m_3, & m_4 \\ m_2, & m_3, & m_4, & m_5 \\ m_3, & m_4, & m_5, & m_6 \end{vmatrix} = 0. \tag{2}$$

The m's being linear in the coefficients ξ_i, (2) represents a surface of class 4. Cayley ‡ studied the quartic surface defined by a symmetrical determinant of order 4 whose elements are linear in the coördinates of a point of space. He called this surface the *symmetroid* and deduced many of its properties. It has ten nodes; these nodes are a symmetrical group of ten points such that any nine are projected from the tenth into the base-points of a pencil of plane cubics. The ten minors of the symmetrical determinant represent the ten linearly independent cubic surfaces on the ten nodes. (2) may be called a

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† Presented to the American Mathematical Society, December 31, 1912.

‡ Cayley, "A Memoir on Quartic Surfaces," *Proceedings of the London Mathematical Society*, Vol. III, pp. 19–69.

Cayley symmetroid; it is the dual of the surface of Cayley—it has ten tropes each of which meets the other nine in the base-lines of a range of plane curves of class 3. We shall frequently have occasion to consider the rational sextic curve (1), and hence (2) in its dual form. We shall show in this paper that (2) is a general symmetroid; that the reduction of the general symmetroid to the form (2) is possible in two ways; and hence, when the symmetroid is given, there are two forms (1) whose catalecticant gives the surface. Hence, it is seen that rational sextic curves in space are paired, either curve of a pair determining the other uniquely and in the same manner. Our main object is the discussion of the surface (2) in its relation to the rational sextic curve. Many features of the geometry on the surface are quite readily treated from this point of view. Cayley pointed out that the symmetroid may be birationally transformed into a quartic surface which is the Jacobian of a threefold linear system of quadrics. It may be proved that a cubic Cremona transformation is sufficient to accomplish this.

Certain special cases of the symmetroid are of considerable interest, for instance the Kummer surface, and the Hessian of a general cubic surface. The detailed consideration of these is reserved for a later memoir.

It is hoped that the beauty and suggestiveness of the hyperspatial methods used are made sufficiently apparent. While the arguments could perhaps be made briefer by direct analytical presentation, many geometrical relations between apparently unconnected theorems would be lost.

§ 2. *The Norm-Sextic in S_6, and the Rational Plane and Space Sextics.*

1. We shall use the letter R throughout this paper for the rational norm-curve in a space of six dimensions. This norm-curve is supposed to be non-degenerate, and chosen once for all. $S_0, S_1, \ldots, S_5$ will be used to denote flats lying in the given six-space, and of the dimension indicated by the subscript. S_0, S_1, S_2, S_3 are a point, a line, a plane, and a space, respectively. An S_p having $(p+1)$-point contact with R will be called "an S_p of R." S_k's meeting R once, twice, three times, ..., r times, will be called S_k's unisecant, bisecant, trisecant, ..., r-secant to R, respectively.

R may be given parametrically, in S_5's by

$$\xi_i = \binom{6}{6-i} t^{6-i} \qquad (i = 0, 1, \ldots, 6), \tag{3a}$$

and in points by

$$x_i = (-)^{6-i} t^i \qquad (i = 0, 1, \ldots, 6). \tag{3b}$$

An S_5, ξ, meets R in the six points whose parameters are the roots of

$$\xi_0 - \xi_1 t + \ldots + \xi_6 t^6 \equiv (\xi t)^6 = 0, \tag{4a}$$

and S_5's of R on a point x have contact with R at points whose parameters are given by

$$x_0 t^6 + 6 x_1 t^5 + 15 x_2 t^4 + \ldots . + x_6 \equiv (x t)^6 = 0. \tag{4b}$$

(4a) and (4b) will be called "the sextic ξ" and "the sextic x," respectively, or merely "ξ" and "x" when there is no doubt as to the meaning. It follows from (4a) and (4b), and is, besides, known, that S_5's on a point x cut from R sextics apolar to the sextic x. S_{r-1}'s on x and r-secant to R meet R in the roots of the apolar r-ics of x. *In this fact lies the fundamental reason for the study of the plurisecant spaces of R to which we are about to proceed.* It will be seen for the rational sextic, and it is true in general, that the hyperspatial point of view unifies many apparently unconnected theorems relating to rational curves. As an example, there may be cited the intimate connection between the notion of osculants and that of perspective curves.*

Rational sextic curves in flats of dimension less than 6 may be derived from R by projection or section, these methods giving general rational curves of order 6 and class 6, respectively, if the given flats are in general position relative to R. We indicate the rational curve of order n in S_p by ρ_p^n, and that of class n in S_p by r_p^n. An invariant of (4a) or of (4b) equated to zero is the equation in S_6 of a contravariant or covariant five-way spread of R; similarly, the coefficients of a covariant of either of these forms equated simultaneously to zero give a spread covariantly connected with R. The operations of projection or section on these, by which a rational sextic is obtained from R, give covariant spreads of the resulting rational curve, whose relation to this curve is frequently easy to deduce.

2. If the sextic x have an apolar cubic, x is on a plane trisecant to R. By a simple enumeration, the locus of such points must be a five-way spread in S_6. The condition that x have an apolar cubic is the vanishing of its catalecticant. Hence,

(a) *The locus of planes trisecant to R has as equation the catalecticant of the sextic* (4b) *and is a five-way of order* 4.

We call this locus the spread G. Its equation is

$$|x| \equiv \begin{vmatrix} x_0, & x_1, & x_2, & x_3 \\ x_1, & x_2, & x_3, & x_4 \\ x_2, & x_3, & x_4, & x_5 \\ x_3, & x_4, & x_5, & x_6 \end{vmatrix} = 0. \tag{5}$$

Calling the three-way spread of bisecants to R, of order 10,† H, we now prove

(b) *The spread G contains H as a double spread.*

* Compare Conner: "Multiple Correspondences Associated with the Rational Plane Quintic Curve," *Transactions of the American Mathematical Society*, Vol. XIII, pp. 265–275.

† The rational plane sextic has 10 nodes.

The line joining two points x and y of S_6,

$$z_i = x_i + \lambda y_i,$$

meets (5) in four points whose parameters, λ_i, are the roots of

$$|z| = |x + \lambda y| = 0. \tag{6}$$

If $(xt)^6$ have an apolar quadratic, that is, if x is on a line bisecant to R, $(xt)^6$ may be reduced to the form

$$(xt)^6 = t^6 + \alpha$$

by a collineation on the parameter t, and λ^2 is a factor of (6), proving that the line xy meets (5) in two coincident points at x.

3. The surface (2) (p. 29) will be called the surface (of planes) S of ρ_3^6; dually, r_3^6 has a similarly defined contravariant surface (of points), Σ say. Cutting the S_5's of R by a space σ, we obtain a rational sextic of planes, r_3^6. G meets σ in the surface Σ of r_3^6. This surface may be defined as the locus of points of σ which determine, by planes of r_3^6 on them, catalectic sextics in the binary domain on r_3^6. Theorems (a) and (b) give

(c) *The surface Σ of r_3^6 is a ten-nodal quartic surface; the ten nodes are the points of intersection of σ with the three-way spread H. A node of Σ gives on r_3^6 a binary sextic reducible to the form $t^6 + \alpha$; that is, a cyclic sextic.*

These ten nodes will be called the cyclic points of r_3^6. Dually, ρ_3^6 has ten cyclic planes, the tropes of S.

4. A plane in S_6 has twelve constants. There are ∞^5 S_4's five-sècant to R, and ∞^6 planes on each; thus it is a single condition on a plane in S_6 to carry an S_4 five-secant to R. This condition is easily calculated. If x, y and z are three points on such a plane, the sextics x, y and z have a common apolar quintic. Let $(at)^5$ be this quintic; then

$$|xa|^5 (xt) \equiv 0,$$
$$|ya|^5 (yt) \equiv 0,$$
$$|za|^5 (zt) \equiv 0,$$

for all t's. Eliminating the a's from these six equations, we obtain

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{vmatrix} = 0; \tag{7}$$

(7) is of degree 2 in the determinants $|x_i y_j z_k|$, the coördinates of the plane xyz. Hence,

(d) *Planes in S_6 carrying S_4's five-secant to R are in a quadratic system (hypercomplex) of ∞^{11} planes.*

From (d) we have that the locus of planes passing through a line, p, of S_6, and carrying S_4's five-secant to R, is a quadric spread. If $p = xy$, (7) is the equation of this spread in coördinates z. A quadric spread in S_6 containing ∞^1 S_4's must have a nodal plane. This plane in the present case passes through the line p, and is common to all S_4's on p and five-secant to R. We may see this in another way. There is a pencil of binary quintics apolar to all sextics of a given pencil, but a net of sextics apolar to a given pencil of quintics. Thus, giving p determines, by S_4's five-secant to R and on p, a pencil of quintics apolar to the pencil of sextics defined on R by points of p; to this pencil of quintics, however, is apolar a net of sextics determined by points of a plane through the line p. This plane is determined in a similar way by any of its lines; it belongs to a system of ∞^8 planes in S_6 — this system is useful in the study of the (unique) cubic surface passing through ρ_3^6 (§ 5).

5. In the space σ of r_3^6, we have from (d):

(e) *Planes whose points define on r_3^6 a linear system of binary sextics having a common apolar quintic are planes of a quadric surface, K, in σ.*

The surface K was discovered by Stahl,* and will be called the Stahl quadric of r_3^6. It is the locus of planes joining three at a time the sets of four stationary points (apparent cusps) of quartic osculants of the curve.

6. A pencil of binary sextics has in general a pencil of apolar quintics. But if the sextics are the first polars of a binary septimic, the pencil has a net of apolar quintics, namely, the apolar quintics of the septimic. Conversely, if a pencil of sextics have a net of apolar quintics, they are first polars of a septimic. Lines whose points give on R first polars of a binary septimic are in a system of ∞^7 lines in S_6; since the general line in S_6 has ten constants, it is three conditions on a line to belong to this system, and there is a ruled surface of these lines in the general space σ. Let the pencil of sextics

$$(xt)^6 + \lambda\,(yt)^6 = 0$$

have a net of apolar quintics; that is, let the line xy carry a double infinity of S_4's five-secant to R. Then to any third sextic $(zt)^6 = 0$ is apolar a unique quintic of the net. In geometric language,

*Stahl, "Ueber die Fundamentalinvolutionen auf rationalen Curven," *Journal für Mathematik*, Vol. CIV, p. 56. In connection with this paper and with several others of Stahl, the author is inclined to think that many of his results were obtained by the use of hyperspace, and, for some reason—probably for greater convenience of statement—were subsequently translated into algebraic language. Compare the introductory pages of the above paper and those of the one entitled "Ueber die rationale ebene Curve vierter Ordnung," *Journal für Mathematik*, Vol. CI, p. 300. At all events, the results of Stahl are those most easily accessible to the hyperspace apparatus.

(f) *If a line xy of S_6 carry a double infinity of S_4's five-secant to R, any plane on xy carries a unique S_4 five-secant to R,*

with the immediate consequence,

(g) *The Stahl quadric K of r_3^6 is the locus of lines in σ which carry a double infinity of S_4's five-secant to R; i. e., it is the locus of lines whose points give on r_3^6 the first polars of a binary septimic.*

7. Closely connected with the subject of paragraph 5 is the system of lines in S_6 which carry S_3's four-secant to R. There are ∞^4 S_3's four-secant to R, and ∞^4 lines on each; there are thus ∞^8 lines in S_6 which carry S_3's four-secant to R, and there is hence a congruence of such lines in the general space σ. We desire to prove

(h) *Lines in a space σ and carrying S_3's four-secant to R are in a* (3, 6) *congruence;*

that is, three such lines pass through the general point, and six lie on the general plane of σ. S_3's on a point x of S_6 and four-secant to R meet R in roots of quartics apolar to x (∞^1 quartics) and lie on a four-way spread, this spread being the locus of points y such that the line xy carries an S_3 four-secant to R; *i. e.*, x and y have a common apolar quartic. Let $(k\,t)^4$ be this quartic. Then

$$|x\,k|^4\,(x\,t)^2 \equiv 0,$$
$$|y\,k|^4\,(y\,t)^2 \equiv 0;$$

eliminating k, we obtain

$$\left\|\begin{matrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 & x_5 & x_6 \\ y_0 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_2 & y_3 & y_4 & y_5 & y_6 \end{matrix}\right\| \equiv 0. \qquad (8)$$

The order of this matrix in y is easily seen to be 3. Through a point x of σ there pass three lines of our congruence, since σ meets the four-way defined by (8) in three lines through x.

Again, let π be a general plane of S_6; projecting R from π, we obtain a rational space sextic ρ_3^6; let p be a fourfold secant line of ρ_3^6. The S_4 $p\pi$ meets R in four points, and the S_3 on these four points meets π in a line of the congruence in question. The lines of the congruence which lie on π are thus in (1, 1) correspondence with the fourfold secants of ρ_3^6, and there are hence six of these lines on π.*

* Pascal, "Repertorium der höheren Mathematik," II (Geometrie) (1902), p. 270.

8. Projecting R from a plane π, we obtain a space curve ρ_3^6. Plane sections of this curve are the projections of the sections of R by S_5's on π. It follows that points of π define on R, or on the section of R by π — an r_2^6 — binary sextics apolar to the plane-sections of ρ_3^6. ρ_3^6 and r_2^6 will be called *conjugate* curves. Points of π give on r_2^6 the fundamental involution of ρ_3^6; similarly, plane-sections of ρ_3^6 are the fundamental involution of r_2^6.

Dually, projecting from a space σ carrying a curve r_3^6 cut from R by σ, we obtain the conjugate sextic, ρ_2^6, of this curve. A node of ρ_2^6 arises when a line bisecant to R meets σ. These lines mark on σ the ten nodes of a symmetroid, the surface Σ of r_3^6. Hence,

(i) *The ten nodes of a rational plane sextic curve are in one-one correspondence with the ten nodes of a projectively definite symmetroid.*

§ 3. *The Stahl Quadric is a Rational Covariant Surface of the Symmetroid.*

9. There are ∞^2 six-planes circumscribing K and inscribed in the surface Σ of a sextic of planes, r_3^6.* Each of these six-planes is associated with one set of the fundamental involution of r_3^6. Regarding r_3^6 as cut from R by a space σ, an S_5 on σ meets R in a set of six points — a set of the fundamental involution of r_3^6 — and S_4's on five out of six these meet σ in one of the six-planes in question.† These six-planes are intimately connected with the osculants of r_3^6 and the perspective curves of the conjugate ρ_2^6. The author expects to publish an account of this theory later.

10. One of the above six-planes is uniquely determined when one of its twenty vertices (a point of Σ) is given. For, let x be such a vertex; there is a unique plane on x trisecant to R, and on this plane and σ a unique S_5 which determines a unique six-plane with vertex at x. Opposite vertices of these six-planes are hence in involutory and one-to-one correspondence, thus giving a birational transformation of Σ into itself which we shall call T. The principal theorem of this section is an immediate consequence of one of the properties of the fundamental curves of T.

A singular point of T can arise when, and only when, the plane on a point x of Σ and trisecant to R is indeterminate. This happens only at the nodes of Σ. Let $n_1, \ldots, n_{10}$ be the ten nodes of Σ. There is a line, p_1, on n_1 meeting R in two points, t_1 and t_2. Choosing a plane on t_1, t_2 and t_3, any third point of R, the S_5 on σ and the plane $t_1 t_2 t_3$ meets R in three further points, t_4, t_5, t_6. The point $t_4 t_5 t_6 \sigma$ (the trace on σ of the plane $t_4 t_5 t_6$) is on the fundamental

* Stahl, "Ueber Fundamentalinvolutionen," *loc. cit.*, pp. 56, 57.

† A complete verification of these statements would require the development of the theory of osculants of R. Cf. Conner, "Multiple Correspondences," etc., *loc. cit.*

curve corresponding by T to n_1. The locus of $t_4 t_5 t_6 \sigma$ as t_3 varies is this fundamental curve, $S^{(1)}$ say. S_5's on σ and p_1 meet R in t_1, t_2 and in roots of quartics of a pencil. The points $t_3 t_4 t_5 t_6$, mentioned above, are roots of such a quartic. The space σ meets the complete six-point $t_1 \ldots . t_6$ in six planes, of which four pass through n_1 and two meet on the line $t_3 t_4 t_5 t_6 \sigma$. Call this line p_λ, λ being the parameter of $t_3 t_4 t_5 t_6$ in the pencil of quartics. On p_λ are four points of $S^{(1)}$, namely, $t_3 t_4 t_5 \sigma$, $t_3 t_4 t_6 \sigma, \ldots .$. A pencil of binary quartics is apolar to a definite binary sextic, x, the pencil being cut from R by S_3's on x and four-secant to R. (x is here not on σ.) From the matrix (8), p. 34, we see that the locus of these S_3's is a cubic four-way. This enables us to find the order of the locus of p_λ as λ varies. A cubic four-way in S_6 can not meet a space in a surface of order greater than 2 without containing it entirely. It follows that the order of the locus of p_λ is not greater than 2. But an S_5 on σ may be made to contain the lines p_1 and p_2 on the nodes n_1 and n_2 of Σ and bisecant to R; hence the locus of p_λ is on all nodes of Σ but n_1. It follows that this locus is the quadric surface, Q_1, on the nodes $n_2, \ldots ., n_{10}$. There can not be a pencil of quadric surfaces on $n_2, \ldots ., n_{10}$, for in that case the pencil taken with Q_2 would give a net of quadrics on $n_3, \ldots ., n_{10}$, and no eight nodes of Σ are base-points of a net of quadrics.* Hence,

(j) *The nodes of Σ are singular points of the correspondence T. The fundamental curve corresponding to any node, n_1, is cut out of Σ by the quadric surface on the remaining nine nodes, and is a rational curve of order* 8 *with nine actual nodes.*

Recurring to our notation above, the point $t_4 t_5 t_6 \sigma$ of $S^{(1)}$ may be named by the parameter t_3 of R. The plane $t_1 t_2 t_4 t_5 t_6 \sigma$, on the point $t_4 t_5 t_6 \sigma$, is a plane of the enveloping cone to the quadric K from n_1, and may be named, as a plane of this cone, by the same parameter, t_3. Projecting $S^{(1)}$ from n_1, we obtain a rational plane octavic curve, the enveloping cone to K from n_1 giving a perspective conic of this curve. *The enveloping cone to K from n_1 is thus unique when Σ is given,* since a rational plane octavic can have at most one perspective conic; two conics in one-one correspondence generate a rational quartic. Hence we obtain the principal theorem of this section:

(k) *Given the symmetroid Σ associated with a rational sextic curve, r_3^6, the enveloping cones to the Stahl quadric K from its ten nodes are uniquely determined. It follows that K is a rational covariant surface of Σ.*

§ 4. *The Two Sextics Determined by a Given Symmetroid.*

11. Let x be the symbol of a point in a space σ_x, and y that of a point in a second space σ_y. The equations

* Cayley, *loc. cit.*, and Pascal, "Repertorium," *loc. cit.*, p. 298.

$$\left.\begin{aligned} x_0 &= (\alpha y)^2, \\ x_1 &= (\beta y)^2, \\ x_3 &= (\gamma y)^2, \\ x_4 &= (\delta y)^2, \end{aligned}\right\} \tag{9}$$

α, β, γ and δ being four linearly independent quadric surfaces in σ_y, determine a one-to-eight correspondence, V,* between the spaces σ_x and σ_y. Planes of σ_x pass by V into quadric surfaces of a linear threefold system, C_y, in σ_y. Planes of σ_y, on the other hand, are sent by V into Steiner quartic surfaces in σ, a plane η of σ_y being mapped on its correspondent surface by means of the three-fold linear system of conics in which quadrics C_y meet η. To a point x correspond eight y's; the surface along which two y's with a common correspondent coincide is the Jacobian surface, J, of the system C_y. J is a quartic surface containing ten lines; there are ten quadrics C_y which degenerate into pairs of planes, and the line of intersection of two planes of such a pair is on J.

We use the notation $\xi\, p$, $\eta\, q$ for a plane and line of σ_x and σ_y, respectively. To a line p of σ_x corresponds the intersection of $V\xi$ and $V\xi'$, if ξ and ξ' are two planes on p; this curve is an elliptic quartic, the intersection of two quadrics C_y. VJ is a surface of order 16, since the quartic curve Vp meets J in sixteen points. VJ is of class 4, since there are four nodal quadrics in a pencil.

12. If in

$$(\xi x) = 0 \tag{10}$$

are substituted the values of x as given by (9), the discriminant as to y of (10) is the surface VJ in planes. If $(Ay)^2$ is the form (10), then VJ is

$$\begin{vmatrix} A_{00}, & A_{01}, & A_{02}, & A_{03} \\ A_{10}, & A_{11}, & A_{12}, & A_{13} \\ A_{20}, & A_{21}, & A_{22}, & A_{23} \\ A_{30}, & A_{31}, & A_{32}, & A_{33} \end{vmatrix} = 0, \tag{11}$$

* It is convenient, for the purposes of this paper, to discuss the correspondence V without reference to the norm-curve R, though it is of more than passing interest to point out how it arises from the notion of the norm-curve. The spread of S_4's of R is a five-way of order 10; it has a triple three-way spread of order 8, δ_8, any three S_4's of R meeting in a point of δ_8. Now it may be shown that δ_8 is mapped from a space σ_y by quadrics orthic to a norm-curve N in σ_y, and that N itself passes into R. Regarding the curve ρ^6_3 as obtained by projecting R from a plane π in S_6, the (8, 1) correspondence is here between the eight points in which a space on π meets δ_8 and the point which it determines on the space (σ_x) into which the projection is made. In particular, if π meets δ_8 six times, the quadrics $\alpha\,\beta\,\gamma\,\delta$ of the text become quadrics on six points, and the surface S of ρ^6_3 is a Kummer surface. Thus,

If a rational plane sextic r^6_2 has six triple points (*dually, six triple tangents*), *the conjugate curve ρ^6_3 has a Kummer surface as covariant symmetroid.*

The properties of the correspondence V which we use are so similar to those of the (1, 2) correspondence determined by (9) when $\alpha\,\beta\,\gamma\,\delta$ have six common points, that it does not seen necessary to develop them in great detail. See Sturm, "Die Lehre von den Geometrischen Verwandschaften," Vol. IV, part 12, and especially pp. 436 ff. Compare also Snyder, *Transactions of the American Mathematical Society*, Vol. XII, p. 354; Cayley, *loc. cit.*

a symmetroid (in planes) in σ_x. The surface S of a rational sextic curve

$$(a\xi)\,(\alpha t)^6 \equiv (m t)^6 = 0 \tag{1}$$

in σ_x is

$$\begin{vmatrix} m_0, & m_1, & m_2, & m_3 \\ m_1, & m_2, & m_3, & m_4 \\ m_2, & m_3, & m_4, & m_5 \\ m_3, & m_4, & m_5, & m_6 \end{vmatrix} = 0. \tag{2}$$

The form (11) involves twenty-four constants, and the rational space sextic curve has twenty-four constants. The question arises, Given the surface (11), is a reduction to the form (2) possible?

The equations

$$\left.\begin{aligned} y_0 &= t^3, \\ y_1 &= 3\,t^2, \\ y_2 &= 3\,t, \\ y_3 &= 1 \end{aligned}\right\} \tag{12}$$

represent a rational cubic curve in σ_y. The quadric orthic to the curve (12) (*i. e.*, apolar to the quadrics touching the planes of the curve) and cutting out the sextic $(a\,t)^6 = 0$ is

$$a_0 y_0^2 + a_2 y_1^2 + a_4 y_2^2 + a_6 y_3^2 + 2\,a_1 y_0 y_1 + 2\,a_2 y_0 y_2 + 2\,a_3 y_0 y_3 \\ + 2\,a_5 y_2 y_3 + 2\,a_4 y_3 y_1 + 2\,a_3 y_1 y_2 = 0,$$

having the discriminant

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix} = 0;$$

thus it appears that if all quadrics C_y are orthic to the norm-curve (12), the equation of the symmetroid VJ, with properly chosen tetrahedron of reference in σ_y, assumes the form (2). The cubic curve (12) then maps by V into the rational sextic (1) whose catalecticant gives the surface (2).

Given a symmetroid of which (11) is the general form, the surface J is unique to within collineations, and the system C_y is unique, as is obvious from the form of (11). The problem of finding a rational sextic curve having a given symmetroid as surface S is thus reduced to that of finding a cubic curve orthic to every quadric of the system C_y. Reye* has shown that there are two such cubic curves, N_1 and N_2. N_1 and N_2 have the ten lines of J as common

* "Ueber lineare Systeme und Gewebe von Flächen zweiten Grades," *Journal für Mathematik*, Vol. LXXXII, pp. 78, 79.

axes. J is conversely determined by N_1 and N_2, as is easily shown.* We may state the preceding in the following theorem:

(l) *Given a symmetroid (of planes) in a space σ_x, there are two rational sextic curves (of points) having this surface as covariant surface S. The covariant surface S of a general rational sextic curve in space is a general symmetroid.*

Comparing theorem (g), § 2, and the results of § 3 with the above, we obtain

(m) *The two sextic curves which have a given symmetroid as covariant surface S have a common Stahl quadric K. The two sextics pair with the two systems of generators on K and also with the two systems of generators on a quadric Q_i on nine out of ten of the tropes of the given symmetroid.*

It follows that *if K has a node, every quadric Q_i is a conic.*

12. An interesting result may be drawn from the correspondence which V establishes between J and $VJ \equiv S$. $V\eta$ is a Steiner quartic surface, C_x, touching S along the correspondent of the quartic curve of intersection of J and η. Any line q maps by V into a conic touching S four times.

C_x is mapped from η by the intersections of η with the system C_y—a threefold linear system of conics; that is, a system every conic of which is apolar to all conics of a range. Let the base-lines of this range be q_1, q_2, q_3, q_4. Since q_i^2 is apolar to all conics of the range, it follows that a quadric C_y, necessarily a cone, touches η along q_i; hence q_i is a line of a cone C_y, and the plane of Vq_i touches S at a point of Vq_i. The planes of Vq_i are the four tropes of C_x, while the points $V(q_i q_j)$ are the six pinch-points. It may readily be shown that J passes through the six points $q_i q_j$; in fact, J is the locus of points $y y'$ apolar to all quadrics C_y, and $q_i q_j$, $q_l q_m$ are such a pair of points. The lines q_i map into the four conics in the tropes of C_x; these conics each touch S four times, three contacts of each being three pinch-points in a trope of C_x. It follows that the edges of the tetrahedron of tropes of C_x touch S at the pinch-points and

(n) *There are ∞^6 tetrahedra circumscribed to S whose edges touch S.*

These are analogous to the Humbert tetrahedra of the Kummer surface.†

§ 5. *The Cubic Surface on ρ_3^6.*

13. It is a single condition on a rational plane sextic curve to have a perspective conic. Stahl‡ showed that this condition is the condition for the

* Compare Meyer, "Apolarität und rationale Curven," § 31, p. 319.
† Hudson, "Kummer's Quartic Surface," p. 58.
‡ Stahl, "Zur Erzeugung der ebenen rationalen Curven," *Math. Ann.*, Vol. XXXVIII, pp. 565, 566.

existence of a binary octavic apolar to all line-sections of the curve. If $(a\,t)^6$, $(b\,t)^6$, $(c\,t)^6$ are three linearly independent line-sections, the condition is

$$\begin{vmatrix} a_0, & a_1, & a_2, & \ldots, & a_6, & 0, & 0 \\ b_0, & b_1, & b_2, & \ldots, & b_6, & 0, & 0 \\ c_0, & c_1, & c_2, & \ldots, & c_6, & 0, & 0 \\ 0, & a_0, & a_1, & \ldots, & a_5, & a_6, & 0 \\ 0, & b_0, & b_1, & \ldots, & b_5, & b_6, & 0 \\ 0, & c_0, & c_1, & \ldots, & c_5, & c_6, & 0 \\ 0, & 0, & a_0, & \ldots, & a_4, & a_5, & a_6 \\ 0, & 0, & b_0, & \ldots, & b_4, & b_5, & b_6 \\ 0, & 0, & c_0, & \ldots, & c_4, & c_5, & c_6 \end{vmatrix} = 0. \tag{13}$$

This is of degree 3 in the determinants

$$p_{ijk} = |\, a_i\, b_j\, c_k \,|$$

and also of degree 3 in the complementary determinants π_{lmnr} formed from four linearly independent sets of the fundamental involution of the given plane curve. Let (13) expressed in terms of the π's be

$$\phi\,(\pi_{lmnr}) = 0.$$

If now a, b, c are the forms generating the fundamental involution of a space sextic ρ_3^6, then any added form $(d\,t)^6$ defines planes on a point of the space of ρ_3^6, plane-sections on this point being apolar to a, b, c and also d. The determinants π_{lmnr} formed from a, b, c, d are proportional to determinants linear in the coördinates of the point.* Hence,

(o) *The locus of points from which ρ_3^6 projects into a plane sextic curve with a perspective conic is a cubic surface, P.*

The equation of this surface is intrinsically contained in (13).

A rational plane quintic curve has a unique perspective conic, and ρ_3^6 projects from one of its points into a rational quintic. Again, a plane sextic with a fourfold point has a perspective conic—the fourfold point repeated. Hence,

(p) *The cubic surface P is the unique cubic surface on ρ_3^6. It contains the fourfold secants of ρ_3^6.*

14. Certain facts regarding the surface P may easily be obtained by using the apparatus in S_6 which was developed in § 2 of this paper. We saw (theorem (d)) that it is a single condition on a plane in S_6 to carry an S_4 five-secant to R. Any two S_4's five-secant to R meet in a plane, and this plane carries ∞^1 S_4's five-secant to R. Let us call such a plane a plane $\bar{\omega}$. Since

* Stahl, "Zur Erzeugung der rationalen Raumcurven," *Math. Ann.*, Vol. XL, p. 2.

a plane $\bar{\omega}$ is uniquely determined by a pencil of binary quintics on R, there are ∞^8 planes $\bar{\omega}$ in S_6. We have shown that there is a unique plane $\bar{\omega}$ on every line of S_6, and that a plane $\bar{\omega}$ is equally well determined in this way by any line on it. If a plane $x\,y\,z$ carry ∞^1 S_4's five-secant to R, x, y and z have a pencil of apolar quintics, and they are second polars of an octavic, the pencil of quintics being the apolar quintics of the octavic. Any sextic apolar to x, y and z is apolar to the octavic. Hence,

(q) *R is projected from a space carrying a plane $\bar{\omega}$ into a sextic curve with a perspective conic.*

S_5's on the given space and on the S_4's on $\bar{\omega}$ and five-secant to R give the lines of the perspective conic. Again,

(r) *Spaces carrying planes $\bar{\omega}$ are in a cubic hypercomplex of spaces of which* (13) *is the equation, if a, b and c are three linearly independent S_5's.*

In a space carrying a plane $\bar{\omega}$, the Stahl quadric of r_3^6 is a conic in the plane $\bar{\omega}$. It follows that

(s) *The invariant conditions that a curve r_3^6 have a perspective conic and that the Stahl quadric of its conjugate ρ_3^6 have a node are identical.*

We need especially the following theorems:

(t) *Any plane on an S_3 four-secant to R is a plane $\bar{\omega}$.*

(u) *In an S_3 on a point of R there is a unique plane $\bar{\omega}$ on this point.*

Theorem (t) is sufficiently obvious. Theorem (u) follows, by projection from the point of R, from the fact that for the rational norm-curve R^5 in S_5 there is a unique line on a plane of S_5 carrying ∞^1 S_3's four-secant to R^5.*

16. Projecting R from the plane π on a space σ, we obtain a rational sextic curve, ρ_3^6 in σ; spaces on π carrying planes $\bar{\omega}$ meet σ in points of the cubic surface, P on ρ_3^6. On π there are six lines carrying S_3's four-secant to R (theorem (h)). Let us call these $p_1, \ldots, p_6$. The S_4 containing π and on an S_3 four-secant to R and meeting π in a line p_i, meets σ in a line q_i which lies entirely on P, since the pencil of planes on p_i and in the four-secant space are all planes $\bar{\omega}$. The lines q_i are the fourfold secants of ρ_3^6. In like manner, it follows from theorem (u) that P contains ρ_3^6, for on the space on π and a point of R there is a unique plane $\bar{\omega}$. This space marks on σ a point of ρ_3^6 and of P. The important additional point which we can make is that *there is a natural one-to-one correspondence between the lines of π and the points of P.* For,

* Marletta, "Sulle curve razionali del quinto ordine," *Rendiconti del Circolo matematico di Palermo*, Vol. XIX (1905), p. 94.

given a general line p on π, there is a unique plane $\bar{\omega}$ on it; on this plane and π there is an S_3 marking on σ a point of P. Conversely, given a point of P, the S_3 on this point and π carries a plane $\bar{\omega}$ meeting π in the line p. The six lines p_i are simple singular lines of the correspondence, for on a line p_i there is a pencil of planes $\bar{\omega}$. Hence,

(v) *The lines of π are mapped on the points of P by means of the linear threefold system of curves of class* 3 *on the six lines p_i. The correspondent of ρ_2^6 in π is a curve of class* 10, Ψ *say, having the lines p_i as fourfold tangents.*

On a plane of a space S_5 cutting out of a norm-curve R^5 a rational curve of class 5, r_2^5, there is a unique line carrying ∞^1 S_3's four-secant to R^5. This covariant line of r_2^5 was studied by the author* in a former paper; points of this line give on r_2^5 first polars of a binary sextic — the unique sextic apolar to the fundamental involution of r_2^5. Calling this covariant line of r_2^5 Ω, we have

(w) *The rational ten-ic Ψ is the locus of covariant lines Ω of quintic osculants of the curve r_2^6 in π.*†

The preceding theorems may be summed up in the following:

(x) *Given a plane sextic curve (of points), ρ_2^6, in a plane π, there are six points p_i of π having the property that lines on them cut from ρ_2^6 a pencil of sextics with a common apolar quartic. The locus of the covariant point Ω of quintic osculants of ρ_2^6 is a rational curve of order* 10, Ψ, *having p_i as fourfold points. Ψ is in one-to-one correspondence with ρ_2^6. A set of the fundamental involution of ρ_2^6 is cut from Ψ by a curve of order* 3 *on the six points p_i, the ∞^3 such curves giving the ∞^3 sets of the fundamental involution. Mapping π by means of this system of cubic curves gives a cubic surface P, and the ten-ic curve Ψ maps into the conjugate sextic of ρ_2^6. The lines p_i map into the fourfold secants of this conjugate curve; these fourfold secants are a sixer on P.*

The author suspects most strongly that the second sextic associated with the symmetroid of the conjugate curve lies also on the cubic surface P, and has as fourfold secants the other half of the double-six. This has not been proved.

BRYN MAWR COLLEGE, *December*, 1913.

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* Conner, *Transactions of the American Mathematical Society*, Vol. XII (1912), p. 265; see in particular, §§ 3, 6, 7, 8.

† See in particular § 6, Conner, *loc. cit.*

Limited and Illimited Linear Difference Equations of the Second Order with Periodic Coefficients.

By Tomlinson Fort.

In this paper, I show how a method developed by A. Liapounoff * for the linear differential equation of the second order can be extended to the difference equation in which the independent variable is restricted to integral values. Certain portions of Liapounoff's work can be applied to the difference equation, with changes which are in no wise fundamental. I shall consequently state some results without proof, the proofs being readily supplied from the paper of Liapounoff. The fundamental theorem of Liapounoff, the proof of which, as given by him, is exceedingly difficult and long, covering some sixteen pages, when stated for the difference equation admits a proof both short and simple. This simplicity does not, however, extend throughout the theory, as the formulas to be used in the calculations are usually more difficult to obtain and somewhat more complicated for the difference than for the differential equation

§ 1. *A Necessary and Sufficient Condition that the Equation be Limited.*

Given

$$y(i+2)+M(i)\,y(i+1)+y(i)=0, \tag{1}$$

where the function $M(i)$ is real and defined for all integral values of the argument, and satisfies the relation $M(i+\omega)\equiv M(i)$.

Let y_1 and y_2 be two linearly independent solutions. Then, as $y_1(i+\omega)$ and $y_2(i+\omega)$ are also solutions,

$$\left.\begin{aligned} y_1(i+\omega)&\equiv a_{11}\,y_1(i)+a_{12}\,y_2(i),\\ y_2(i+\omega)&\equiv a_{21}\,y_1(i)+a_{22}\,y_2(i),\end{aligned}\right\} \tag{2}$$

where a_{11}, a_{12}, a_{21} and a_{22} are constants; and since the determinant

$$\begin{vmatrix} y_1(i) & y_1(i+1)\\ y_2(i) & y_2(i+1)\end{vmatrix}$$

is a constant,

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=1.$$

* *Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg,* 8e Série, Tome 13, 1902.

Consequently, the characteristic equation* of (1),

$$\begin{vmatrix} a_{11}-\rho & a_{12} \\ a_{21} & a_{22}-\rho \end{vmatrix} = 0,$$

reduces to $\rho^2-(a_{11}+a_{22})\rho+1=0$, which we write

$$\rho^2-2A\rho+1=0. \tag{3}$$

From the first of equations (2),

$$y_1(i+\omega)=2Ay_1(i)+a_{12}y_2(i)-a_{22}y_1(i). \tag{4}$$

Write equations (2) in the form

$$\left.\begin{aligned} y_1(i)&=a_{11}y_1(i-\omega)+a_{12}y_2(i-\omega), \\ y_2(i)&=a_{21}y_1(i-\omega)+a_{22}y_2(i-\omega). \end{aligned}\right\} \tag{5}$$

Solve (5) for $y_1(i-\omega)$ and substitute in (4). We get

$$y_1(i+\omega)+y_1(i-\omega)=2Ay_1(i).$$

But A, being a coefficient of the characteristic equation, is independent of the particular fundamental system of solutions chosen. Hence, when y is any solution of (1),

$$y(i+\omega)+y(i-\omega)\equiv 2Ay(i). \tag{6}$$

We call A the characteristic constant of the difference equation, (1).

From (6), proceeding exactly as is done by Liapounoff in the case of the differential equation, we obtain the following theorem:

THEOREM I. *If $A^2>1$, all solutions of (1), not identically zero, are illimited.*† *If $A^2<1$, all solutions of (1) are limited. If $A=1$, there exists at least one solution, not identically zero, having the period ω; all solutions not having the period ω are illimited. If $A=-1$, there exists at least one solution, not identically zero, satisfying the relation $y(i+\omega)\equiv -y(i)$; all solutions not having the period 2ω are illimited.*

§ 2. *The Finite Series,* $1-A_1+A_2-\ldots.+(-1)^\omega A_\omega$.

The problem of the calculation of A next presents itself.

Let $f(i)$ and $\phi(i)$ be the two solutions of (1) such that

$$f(0)=1,\quad \Delta f(0)=0;$$
$$\phi(0)=0,\quad \Delta\phi(0)=1.$$

From (6), $f(\omega)+f(-\omega)=2A$. Moreover, $\Delta\phi(\omega)f(i+\omega)-\Delta f(\omega)\phi(i+\omega)$ is a linear combination of $f(i+\omega)$ and $\phi(i+\omega)$, two solutions of (1), and hence

* The characteristic equation of (1) is the analogue of the characteristic equation of the differential equation, $\frac{d^2y}{dx^2}+m(x)y=0$, where $m(x)$ is periodic. Compare Floquet, *Ann. Sci. de l'École Normale Supérieure*, 2e Série, XII, p. 47.

† A solution of (1) is said to be limited if it remains finite as i becomes infinite, and to be illimited in the contrary case.

is itself a solution. Moreover, it and its first difference at 0 are equal respectively to $f(0)$ and $\Delta f(0)$; hence

$$f(i) \equiv \Delta\phi(\omega) f(i+\omega) - \Delta f(\omega)\phi(i+\omega). \tag{7}$$

From (7), $f(-\omega) = \Delta\phi(\omega)$, and hence

$$2A = f(\omega) + \Delta\phi(\omega). \tag{8}$$

If ω is small, the calculation of A from (8) is easy. We calculate $f(2)$, $f(3), \ldots, f(\omega)$ successively from (1), then $\phi(2), \phi(3), \ldots, \phi(\omega+1)$, and substitute in (8).

If ω is large, this process is tedious, and for very large values of ω is prohibitive. In the following pages a process, analogous to that employed by Liapounoff for the differential equation, is developed for the treatment of this case. We begin by writing the difference equation in the form

$$\Delta^2 y(i) + p(i)\, y(i+1) = 0, \tag{9}$$

where $p(i)$ replaces $M(i) + 2$.

Treat $f(i)$ and $\phi(i)$ by a method of successive approximations* similar to that employed in existence theorems for differential equations. Denote the successive terms in the two series by $f_0, -f_1, \ldots, (-1)^n f_n, \ldots$, and $\phi_0, -\phi_1, \ldots, (-1)^n \phi_n$ respectively. Adopting the convention $\sum_{i=k}^{g} F(i) = 0$, $k > g$; when $i \geq 0$,

$$\phi_0(i) = i,$$

$$\phi_1(i) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1-1} p(i_2)(i_2+1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\phi_n(i) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{2n-1}=0}^{i_{2n-2}} \sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_2)\, p(i_4) \cdots p(i_{2n})(i_{2n}+1), \tag{10}$$

$$f_0(i) = 1,$$

$$\ldots\ldots\ldots,$$

$$f_n(i) = \sum_{i_1=0}^{i-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{2n-1}=0}^{i_{2n-2}} \sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_2)\, p(i_4) \cdots p(i_{2n}). \tag{11}$$

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* Consider equation (9). Form successive approximations for a solution, y, such that $y(a) = c_0$, $y(a+1) = c_1$, where a is any integer, and c_0 and c_1 arbitrary constants. Assuming y_{n-1} as known, we determine y_n from $\Delta^2 y_n(i) = -p(i)\, y_{n-1}(i+1)$, subject to the conditions $y_n(a) = c_0$, $y_n(a+1) = c_1$. Choose $y_0 \equiv (i-a)(c_1 - c_0) + c_0$, and let $z_n(i) = y_n(i) - y_{n-1}(i)$ when $n \geq 1$, $z_0(i) \equiv y_0(i)$. Clearly, when $n \geq 1$,

$$\Delta^2 z_n(i) = -p(i)\, z_{n-1}(i+1) \text{ and } z_n(a) = z_n(a+1) = 0. \tag{j}$$

Adopting the convention $\sum_{i=k}^{g} F(i) = 0$, $k > g$, we have, when $n \geq 1$ and $i \leq a$,

$$z_n(i) = -\sum_{i=a}^{i-1} \sum_{i=a}^{i-1} p(i)\, z_{n-1}(i+1). \tag{jj}$$

From (jj) it is immediate that if $z_{n-1}(\bar{i}) = 0$ when $\bar{i} = a, a+1, \ldots, i-1$, $z_n(\bar{i}) = 0$ when $\bar{i} = a$, $a+1, \ldots, i$. But $z_1(a) = z_1(a+1) = 0$. Hence $z_2(a) = z_2(a+1) = z_2(a+2) = 0$, and in general $z_n(a) = z_n(a+1) = \ldots = z_n(a+n) = 0$. Consequently, the series $z_0(i) + z_1(i) + z_2(i) + \ldots$ has all its terms zero after the $(i-a)$-th, and hence converges. Moreover, it satisfies the difference equation and the conditions at a, and accordingly is the solution of the difference equation sought when $i \geq a$.

From (10),

$$\Delta\phi_n(i) = \sum_{i_2=0}^{i-1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3-1}\cdots\sum_{i_{2n}=0}^{i_{2n-1}-1} p(i_2)\,p(i_4)\cdots p(i_{2n})(i_{2n}+1)$$

$$= \sum_{i_1=0}^{i-1}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_{2n-1}=0}^{i_{2n-2}-1}\sum_{i_{2n}=0}^{i_{2n}-1} p(i_1)\,p(i_3)\cdots p(i_{2n-1}). \tag{12}$$

Let $2A_n = f_n(\omega) + \Delta\phi_n(\omega)$. Clearly,

$$A = 1 - A_1 + A_2 - \cdots + (-1)^\omega A_\omega. \tag{13}$$

Consider

$$f_2(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} p(i_2) \sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3-1} p(i_4).$$

Let $\sum_{i=0}^{i-1} p(i) = P(i)$, thus defining $P(i)$; and sum by parts, considering i_2 as variable of summation. We get

$$f_2(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(P(i_1) - P(i_2))\,P(i_2).$$

Now apply similar summation by parts to $f_n(\omega)$, considering successively, as variables of summation, $i_2, i_4, \ldots, i_{2n-2}$. The above result is clearly general for any single summation, and we write

$$f_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_{2n}=0}^{i_{2n-1}-1}(P(i_1)-P(i_2))\cdot(P(i_2)-P(i_3))\cdots$$
$$(P(i_{n-1}) - P(i_n))\,P(i_n). \tag{14}$$

Consider next

$$\Delta\phi_2(\omega) = \sum_{i_1=0}^{\omega-1} p(i_1)\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2-1} p(i_3)\sum_{i_4=0}^{i_3} 1.$$

Sum by parts, considering successively i_1 and i_3 as variables of summation.

$$\Delta\phi_2(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(P(\omega) - P(i_1))\,(P(i_1) - P(i_2)).$$

In general, letting $P(\omega) = \Omega$,

$$\Delta\phi_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}(\Omega - P(i_1))\,(P(i_1) - P(i_2))\cdots$$
$$(P(i_{n-1}) - P(i_n)). \tag{15}$$

Combining (14) and (15),

$$2A_n = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\cdots\sum_{i_n=0}^{i_{n-1}-1}(\Omega - P(i_1) + P(i_n))\cdot(P(i_1) - P(i_2))$$
$$\cdots(P(i_{n-1}) - P(i_n)). \tag{16}$$

Remark that when $F(i)$ is any function, $\sum_{i_1=0}^{i-1}\sum_{i_2=0}^{i_1} F(i_2) = \sum_{i_1=0}^{i}\sum_{i_2=0}^{i_1-1} F(i_2)$.

Then, from (11) and (12),

$$f_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}\sum_{i_4=0}^{i_3}\cdots\sum_{i_{2n-1}=0}^{i_{2n-2}-1}\sum_{i_{2n}=0}^{i_{2n-1}} p(i_2)\,p(i_4)\cdots p(i_{2n}), \tag{17}$$

$$\Delta\phi_n(\omega) = \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2}\sum_{i_4=0}^{i_3-1}\cdots\sum_{i_{2n-1}=0}^{i_{2n-2}}\sum_{i_{2n}=0}^{i_{2n-1}} p(i_1)\,p(i_3)\cdots p(i_{2n-1}). \tag{18}$$

Apply summation by parts to (17) and (18), considering as variables of summation $i_1, i_3, \ldots, i_{2n-1}$ and $i_2, i_4, \ldots, i_{2n-2}$ respectively. We arrive at the following formula:

$$2A_n = \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} (\omega - i_1 + i_n)(i_1 - i_2) \cdots (i_{n-1} - i_n)\, p(i_1)\, p(i_2) \cdots p(i_n), \tag{19}$$

an alternative formula to (16). One draws the conclusion from it, as easily in various other ways, that if $p(i) \geqq 0$ at all points, $A_n \geqq 0$.

§ 3. *Fundamental Theorem.*

THEOREM II. *If $p(i) \geqq 0$ at all points, then, if $A_n = 0$, $A_{n+1} = 0$; if $A_n \neq 0$, $\frac{A_{n+1}}{A_n} < \frac{n}{n+1} \frac{A_n}{A_{n-1}}$.*

To prove the first conclusion of this theorem, we refer to (19). This formula can be written

$$A_n = \Sigma\, (\omega - i_1 + i_n)(i_1 - i_2) \cdots (i_{n-1} - i_n)\, p(i_1)\, p(i_2) \cdots p(i_n),$$

where Σ denotes the sum of all possible products of the form expressed, $i_1, i_2, \ldots, i_n$ taken in every possible way from $\omega - 1, \omega - 2, \ldots, 0$, subject to the restrictions $i_1 > i_2 > \cdots > i_n$. $(\omega - i_1 + i_n)(i_1 - i_2) \cdots (i_{n-1} - i_n)$ is always positive. Hence, if $A_n = 0$, each product $p(i_1) \cdots p(i_n)$ must be zero; that is, there do not exist n numbers of the set $\omega - 1, \omega - 2, \ldots, 0$ for which $p \neq 0$. But A_{n+1} is the sum of products of the form

$$(\omega - i_1 + i_{n+1})(i_1 - i_2) \cdots (i_n - i_{n+1})\, p(i_1)\, p(i_2) \cdots p(i_{n+1});$$

$i_1, i_2, \ldots, i_{n+1}$ numbers of the set $\omega-1, \omega-2, \ldots, 0$ and $i_1 > i_2 > \cdots > i_{n+1}$; and hence is zero.

For the second conclusion of the theorem we refer to (16). This can be written

$$2A_n = \Sigma\, (\Omega - P(i_1) + P(i_n))(P(i_1) - P(i_2)) \cdots (P(i_{n-1}) - P(i_n)),$$

where Σ denotes the sum of all products of the form expressed, the letters i_1, i_2, i_n, chosen in every possible way from the numbers $\omega-1, \omega-2, \ldots, 1, 0$, subject to the restrictions $i_1 > i_2 > i_3 > \cdots > i_n$.

If we conceive of the numbers $\omega-1, \omega-2, \ldots, 1, 0$ as equally spaced points on a circle of circumference ω, in the expression

$$(\Omega - P(i_1) + P(i_n))(P(i_1) - P(i_2)) \cdots (P(i_{n-1}) - P(i_n)),$$

the first factor is in no manner different from any other, and (16) can be written

$$2A_n = \Sigma \left[\left(\sum_{i=k_0}^{k_{l_1}} p(i) \right) \left(\sum_{i=k_{l_1}+}^{k_{l_2}} p(i) \right) \cdots \left(\sum_{i=k_{l_{n-1}}+1}^{k_{l_n}} p(i) \right) \right],$$

where Σ denotes the sum of all possible products of the form expressed, $k_0, k_1, \ldots, k_{l_n} = k_\omega$ being the numbers $\omega - 1, \omega - 2, \ldots, 0$ taken always in the same cyclic order, namely $\omega - 1, \omega - 2, \ldots, 0$. For brevity we write

$$2A_n = \Sigma\, {}_0D_{l_1} \cdot {}_{l_1}D_{l_2} \ldots {}_{l_{n-1}}D_{l_n}.$$

Then

$$4A_n^2 = \{\Sigma\, {}_0D_{\lambda_1} \cdot {}_{\lambda_1}D_{\lambda_2} \ldots {}_{\lambda_{n-1}}D_{\lambda_n}\}\{\Sigma\, {}_0D_{\mu_1} \cdot {}_{\mu_1}D_{\mu_2} \ldots {}_{\mu_{n-1}}D_{\mu_n}\}, \tag{20}$$

$$4A_{n-1}A_{n+1} = \{\Sigma\, {}_0D_{\nu_1} \cdot {}_{\nu_1}D_{\nu_2} \ldots {}_{\nu_{n-2}}D_{\nu_{n-1}}\}\{\Sigma\, {}_0D_{\rho_1} \cdot {}_{\rho_1}D_{\rho_2} \ldots {}_{\rho_n}D_{\rho_{n+1}}\}, \tag{21}$$

where, instead of using only the letter l, we use distinct letters, λ, μ, ν, ρ.

We shall consider (20) and (21). Begin by supposing $a_1, a_2, \ldots, a_{2n}$ numbers of the succession $\omega - 1, \omega - 2, \ldots, 0$, and assume that among the a's there are exactly k distinct numbers, and that no number occurs more than twice among them. If $k \geqq n+1$, the product $p(a_1)p(a_2)\ldots p(a_{2n})$ will occur in the expanded right-hand members of both (20) and (21). We shall show that the ratio of its coefficient in (20) to its coefficient in (21) is greater than or equal to $\dfrac{n+1}{n}$.

Omitting coefficients, let $p(j_1)\ldots p(j_{n-1})$ be a term of A_{n-1} and $p(i_1)\ldots p(i_{n+1})$ a term of A_{n+1} such that $p(j_1)\ldots p(j_{n-1})\,p(i_1)\ldots p(i_{n+1})$ is identical with $p(a_1)\,p(a_2)\ldots p(a_{2n})$, and let $p(\bar{j}_1)\ldots p(\bar{j}_n)$ and $p(\bar{i}_1)\ldots p(\bar{i}_n)$ be terms of A_n such that $p(\bar{j}_1)\ldots p(\bar{j}_n)\,p(\bar{i}_1)\ldots p(\bar{i}_n)$ is identical with $p(a_1)\,p(a_2)\ldots p(a_{2n})$. We shall show that the ratio of the number of ways in which $\bar{j}_1, \ldots, \bar{j}_n, \bar{i}_1, \ldots, \bar{i}_n$ can be chosen to the number of ways in which $j_1, \ldots, j_{n-1}, i_1, \ldots, i_{n+1}$ can be chosen is greater than or equal to $\dfrac{n+1}{n}$. These numbers are exactly the coefficients of $p(a_1)\ldots p(a_{2n})$ in (20) and (21) respectively.

Require, first, that $p(j_1), p(j_2), \ldots, p(j_{n-1})$ be each the first term of one of the parentheses ${}_0D_{\nu_1}, {}_{\nu_1}D_{\nu_2}, \ldots, {}_{\nu_{n-2}}D_{\nu_{n-1}}$, and that $p(i_1), p(i_2), \ldots, p(i_{n+1})$ be each the first term of one of the parentheses ${}_0D_{\rho_1}, {}_{\rho_1}D_{\rho_2}, \ldots, {}_{\rho_n}D_{\rho_{n+1}}$. Under this requirement the number of ways in which $j_1, \ldots, j_n, i_1, \ldots, i_{n+1}$ can be chosen is the number of ways in which $j_1, \ldots, j_{n-1}$ can be chosen from $a_1, a_2, \ldots, a_{2n}$. The $2n - k$ numbers which occur twice among $a_1, a_2, \ldots, a_{2n}$ necessarily occur among $j_1, j_2, \ldots, j_{n-1}$. There remain $k - n - 1$ of the j's which can be chosen arbitrarily from the remaining $2k - 2n$ numbers. Letting $k - n = N_k$, this can be done in $\dfrac{2N_k(2N_k - 1)\ldots(N_k+2)}{(N_k - 1)!}$ ways. Similarly, the number of ways that $\bar{j}_1, \ldots, \bar{j}_n, \bar{i}_1, \ldots, \bar{i}_n$ can be chosen, requiring that $p(\bar{j}_1), \ldots, p(\bar{j}_n)$ be each the first term of one of the parentheses ${}_0D_{\lambda_1}, \ldots,$

${}_{\lambda_{n-1}}D_{\lambda_n}$, and $p(\bar{i}_1), \ldots, p(\bar{i}_n)$ be each the first term of one of the parentheses ${}_0D_{\mu_1}, \ldots, {}_{\mu_{n-1}}D_{\mu_n}$, is $\dfrac{2N_k(2N_k-1)\ldots(N_k+1)}{N_k!}$. The second is larger in the ratio $\dfrac{N_k+1}{N_k}$. But $N_k \leqq n$, and hence $\dfrac{N_k+1}{N_k} \geqq \dfrac{n+1}{n}$.

We generalize as follows: Instead of requiring that each p be the first term of a parenthesis, let us require that $p(a_1)$ be the λ-th, $p(a_2)$ the μ-th, $\ldots$, $p(a_{2n})$ the ζ-th. For convenience we shall refer to $j_1, \ldots, j_{n-1}$ and $\bar{j}_1, \ldots, \bar{j}_n$ as the sets J and to $i_1, \ldots, i_{n+1}$ and $\bar{i}_1, \ldots, \bar{i}_n$ as the sets I. As above, those a's occurring twice among $a_1, \ldots, a_{2n}$ necessarily occur in both the sets J and I. Consider them as fixed. We proceed as before, choosing the remainder of the sets J.

It may happen that the fact that a_m lies in the sets J (or I) requires that $a_{m+\nu}$ lie in the corresponding sets I (or J). Thus, suppose that a_m is the ρ-th term of a parenthesis and $a_{m+\nu}$ the $\bar{\rho}$-th, and suppose that $\bar{\rho} > \nu$; then, if a_m is one of the set J, in order for $p(a_{m+\nu})$ to lie in a different parenthesis from $p(a_m)$, as it must, it must necessarily be a member of the set I. Moreover, the fact that a_m lies in the set J can require that only one of the a's lie in the sets I; for, suppose that $p(a_{m+\mu})$ is the $\bar{\bar{\rho}}$-th term of a parenthesis and $\bar{\bar{\rho}} > \mu \geqq \nu$, then $p(a_{m+\nu})$ and $p(a_{m+\mu})$ belong to the sets I and lie in different parentheses. Hence, $\bar{\bar{\rho}} \leqq \mu - \nu$ but $\bar{\bar{\rho}} > \mu$, a contradiction.

In the way that we are choosing the sets J, let us suppose all $p(a)$'s that impose any restriction on others as fixed. Let this number be L. Then there are thereby fixed R in the sets I, and necessarily $R \leqq L$. The remaining a's can now be distributed in sets J and I at pleasure. This can be done in (21) and (20) in

$$\frac{(2N_k-L-R)(2N_k-L-R-1)\ldots(N_k-R+2)}{(N_k-L-1)!}$$

and

$$\frac{(2N_k-L-R)(2N_k-L-R-1)\ldots(N_k-R+1)}{(N_k-L)!}$$

ways respectively. The second is the larger in the ratio $\dfrac{N_k-R+1}{N_k-L}$, which is greater than $\dfrac{n+1}{n}$. We thus conclude that the coefficient of $p(a_1)\ldots p(a_{2n})$ in (20) is greater than its coefficient in (21) by a ratio greater than or equal to $\dfrac{n+1}{n}$.

We have considered $k \geqq n+1$, which exhausts the terms of (21). There are in addition in (20) terms of the form $p(a_1)\ldots p(a_{2n})$, where $k=n$; that

is, terms of the form $(p(a_1)\ldots\ldots p(a_n))^2$. These are not all zero as $A_n \neq 0$. All coefficients are positive, and hence we conclude

$$A_n^2 > \frac{n+1}{n} A_{n+1} A_{n-1},$$

from which we immediately draw the desired conclusion

$$\frac{A_{n+1}}{A_n} < \frac{n}{n+1} \frac{A_n}{A_{n-1}}. \tag{22}$$

From theorem II one readily proves the following:

THEOREM III. *If* $p(i) \geqq 0$ *at all points,*

when $-1 + A_2 - A_3 + \ldots\ldots + A_{2n} \leqq 0, \quad A < 1;$
when $2 - A_1 + A_2 - \ldots\ldots - A_{2n-1} \geqq 0, \quad A > -1;$
when $2 - A_1 + A_2 - \ldots\ldots + A_{2n} \leqq 0, \quad A < -1;$
when $-A_1 + A_2 - A_3 + \ldots\ldots - A_{2n-1} \geqq 0, \quad A > 1.$

§ 4. *The Calculation of* A_2 *and* A_3.

The problem now proposed is the calculation of A_2, A_3, etc., with as little labor as possible. We retain the supposition $p \geqq 0$ at all points.

From (16),

$$A_2 = \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} (\Omega - P(i_1) + P(i_2))(P(i_1) - P(i_2)) = \tfrac{1}{2}\Omega \sum_{i=0}^{\omega-1} iP(i) - \tfrac{1}{2} \sum_{i=0}^{\omega-1} i(P(i))^2$$
$$+ \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} P(i_1)P(i_2) - \tfrac{1}{2}\Omega \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} P(i_2) - \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} (P(i_2))^2.$$

This expression can be still farther reduced by summing by parts those terms in which two Σ's occur. We obtain

$$A_2 = \Omega \sum_{i=0}^{\omega-1} iP(i) - \frac{\omega}{2} \sum_{i=0}^{i-1} (P(i))^2 + \tfrac{1}{2}\Big(\sum_{i=0}^{\omega-1} P(i)\Big)^2 - \Omega \frac{\omega-1}{2} \sum_{i=0}^{\omega-1} P(i).$$

From this we easily verify the formula

$$A_2 = \Omega^2 \frac{\omega^2-1}{24} - \tfrac{1}{2} R, \tag{23}$$

where

$$R = \omega \sum_{i=0}^{\omega-1} \Big(P(i) - \Omega \frac{i}{\omega}\Big)^2 - \Big[\sum_{i=0}^{\omega-1} \Big(P(i) - \Omega \frac{i}{\omega}\Big)\Big]^2. \tag{24}$$

If $a(i)$ is any real function,

$$\omega \sum_{i=0}^{\omega-1} (a(i))^2 \geqq \Big(\sum_{i=0}^{\omega-1} a(i)\Big)^2.$$

Hence $R(i) \geqq 0$. Consequently we can use $\frac{\omega^2-1}{24}\Omega^2$ as a superior limit for A_2,

or more loosely $\frac{\omega^2\Omega^2}{24}$. But from (16), $A_1 = \frac{\omega\Omega}{2}$; hence

$$A_3 < \tfrac{1}{6} A_1^2.$$

With this last result we can proceed as is done by Liapounoff (§ 14), obtaining the same result as is obtained by him.

We have defined $P(i)$ as $\sum_{i=0}^{i-1} p(i)$, but we might equally well have defined $P(i)$ by the equation $P(i) = \Sigma p(i)$, where Σ denotes the indefinite sum, retaining the notation $\Omega = \sum_{i=0}^{\omega-1} p(i)$, since in the formula for A_n, (16), $P(i)$ occurs only in the combination $(P(i_j) - P(i_k))$. Let us particularize $P(i)$ by choosing the arbitrary constant of summation so that

$$\sum_{i=0}^{\omega-1} P(i) = \Omega \frac{\omega - 1}{2}. \tag{25}$$

It is immediate that

$$\sum_{i=0}^{\omega-1} (P(i) - \Omega i/\omega) = 0, \tag{26}$$

and we get as a simplified formula

$$R = \omega \sum_{i=0}^{\omega-1} (P(i) - \Omega i/\omega)^2. \tag{27}$$

$(P(i) - \Omega i/\omega)$ has the period ω, since increasing i by ω increases both terms of the expression by Ω. The same thing is true of its sum; that is,

$$\sum_{i=0}^{i-1} (P(i) - \Omega i/\omega)$$

has the period ω. For, if we increase i by ω, we add

$$\sum_{i=i}^{i+\omega-1} (P(i) - \Omega i/\omega) = \sum_{i=0}^{\omega-1} (P(i) - \Omega i/\omega) = 0,$$

by (26).

Now let $P(i) - \Omega i/\omega = \Omega \Delta\theta(i)$, where $\Delta\theta(i)$ denotes the first difference of a function θ. We have just shown that θ has the period ω. From (23) and (27),

$$A_2 = \Omega^2 \left\{ \frac{\omega^2 - 1}{24} - \tfrac{1}{2}\omega \sum_{i=0}^{\omega-1} (\Delta\theta(i))^2 \right\}. \tag{28}$$

For brevity let $i/\omega + \Delta\theta(i) = Q(i)$; then (16) gives

$$A_n = a_n \Omega^n,$$

where

$$a_n = \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} \cdots \sum_{i_n=0}^{i_{n-1}-1} (1 - Q(i_1) + Q(i_n))(Q(i_1) - Q(i_2)) \cdots (Q(i_{n-1}) - Q(i_n)).$$

Hence,

$$a_3 = \tfrac{1}{2} \sum_{i_1=0}^{\omega-1} \sum_{i_2=0}^{i_1-1} \sum_{i_3=0}^{i_2-1} (1 - Q(i_1) + Q(i_3))(Q(i_1) - Q(i_2))(Q(i_2) - Q(i_3)). \tag{29}$$

It is possible to greatly simplify this expression. The summand reduces to

$$Q(i_1)\,Q(i_2)+Q(i_2)\,Q(i_3)-Q(i_1)\,Q(i_3)-(Q(i_2))^2+(Q(i_3)-Q(i_2))\,(Q(i_1))^2$$
$$+(Q(i_1)-Q(i_3))\,(Q(i_2))^2+(Q(i_2)-Q(i_1))\,(Q(i_3))^2.$$

Distribute the sign of summation and apply to each term summation by parts or perform obvious summation. We obtain the following results:

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,Q(i_2)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_2\,Q(i_1)\,Q(i_2),$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_2)\,Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(\omega-i_1-1)\,Q(i_1)\,Q(i_2),$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_1-i_2-1)\,Q(i_1)\,Q(i_2),$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}(Q(i_2))^2=\sum_{i=0}^{\omega-1}(\omega-i-1)\,i\,(Q(i))^2,$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}(Q(i_1))^2Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_1-i_2-1)\,(Q(i_1))^2Q(i_2),$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}(Q(i_1))^2Q(i_2)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_2\,(Q(i_1))^2Q(i_2),$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,(Q(i_2))^2=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,(Q(i_2))^2 i_2,$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1}(Q(i_2))^2Q(i_3)=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(\omega-i_1-1)\,(Q(i_1))^2Q(i_2),$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_2)\,Q(i_3)^2=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(\omega-i_1-1)\,Q(i_1)\,(Q(i_2))^2,$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}\sum_{i_3=0}^{i_2-1} Q(i_1)\,(Q(i_3))^2=\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_1-i_2-1)\,Q(i_1)\,(Q(i_2))^2.$$

Collecting,

$$a_3=\tfrac{1}{2}\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}[Q(i_1)\,Q(i_2)\,(\omega+2\,i_2-2\,i_1)+(Q(i_1))^2Q(i_2)\,(2\,i_1-2\,i_2-\omega)$$
$$+Q(i_1)\,(Q(i_2))^2(\omega+2\,i_2-2\,i_1)]-\tfrac{1}{2}\sum_{i=0}^{\omega-1}(\omega-i-1)\,i\,(Q(i))^2. \qquad (30)$$

By means of the formula for summation by parts, one proves easily:

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)Q(i_2)=\tfrac{1}{2}\Big(\sum_{i=0}^{\omega-1}Q(i)\Big)^2-\tfrac{1}{2}\sum_{i=0}^{\omega-1}(Q(i))^2,$$

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(Q(i_1))^2\,Q(i_2)\,i_2=\Big(\sum_{i=0}^{\omega-1}i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big)-\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1} i_1\,Q(i_1)\,(Q(i_2))^2$$
$$=\Big(\sum_{i=0}^{\omega-1}i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big)$$
$$-\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_1\,Q(i_1)\,(Q(i_2))^2-\sum_{i=0}^{\omega-1}i\,(Q(i))^3.$$

Similarly,

$$\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)(Q(i_2))^2 i_2 = \Big(\sum_{i=0}^{\omega-1} i\,(Q(i))^2\Big)\Big(\sum_{i=0}^{\omega-1} Q(i)\Big)$$
$$-\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} i_1\,(Q(i_1))^2 Q(i_2) - \sum_{i=0}^{\omega-1} i\,(Q(i))^3.$$

Substituting these values in (30),

$$a_3 = \frac{\omega}{4}\Big(\sum_{i=0}^{\omega-1} Q(i)\Big)^2 + \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_2-i_1)\,Q(i_1)\,Q(i_2)$$
$$+\Big(\sum_{i=0}^{\omega-1} i\,(Q(i))^2\Big)\Big(\sum_{i=0}^{\omega-1} Q(i)\Big) - \Big(\sum_{i=0}^{\omega-1} i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big)$$
$$+\frac{\omega}{2}\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,Q(i_2)\,(Q(i_2)-Q(i_1)) - \frac{\omega-1}{2}\sum_{i=0}^{\omega-1} i\,(Q(i))^2$$
$$-\frac{\omega}{4}\sum_{i=0}^{\omega-1}(Q(i))^2 + \tfrac{1}{2}\sum_{i=0}^{\omega-1} i^2\,(Q(i))^2. \tag{31}$$

Moreover, $Q(i) = \dfrac{i}{\omega} + \Delta\theta(i) = \dfrac{i}{\omega} + \Big(P(i) - \Omega\dfrac{i}{\omega}\Big)$. Hence, by (26),

$$\sum_{i=0}^{\omega-1} Q(i) = \sum_{i=0}^{\omega-1}\frac{i}{\omega} = \frac{\omega-1}{2}.$$

Substituting in (31),

$$a_3 = \frac{(\omega-1)^2\omega}{16} + \sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1}(i_2-i_1)\,Q(i_1)\,Q(i_2)$$
$$+\tfrac{1}{2}\sum_{i=0}^{\omega-1} i^2\,(Q(i))^2 - \Big(\sum_{i=0}^{\omega-1} i\,Q(i)\Big)\Big(\sum_{i=0}^{\omega-1}(Q(i))^2\Big)$$
$$+\frac{\omega}{2}\sum_{i_1=0}^{\omega-1}\sum_{i_2=0}^{i_1-1} Q(i_1)\,Q(i_2)\,(Q(i_2)-Q(i_1)) - \frac{\omega}{4}\sum_{i=0}^{\omega-1}(Q(i))^2. \tag{32}$$

We have only defined the function θ by its first difference. We can particularize by letting $\theta(0) = \theta(\omega) = 0$. Under this assumption, the above expression for a_3 permits of great simplification. We treat the different sums occurring separately, applying summation by parts, collecting and simplifying, arriving at the following result:

$$a_3 = \frac{1}{\omega}\Big[\frac{\omega^4}{720} - \frac{\omega^2}{144} + \frac{1}{180}\Big] + 2\sum_{i=0}^{\omega-1}(\theta(i))^2 - \frac{\omega^2+2}{12}\sum_{i=0}^{\omega-1}(\Delta\theta(i))^2$$
$$-\frac{\omega}{2}\sum_{i=0}^{\omega-1}\theta(i)\,(\Delta\theta(i))^2 - \frac{\omega}{2}\sum_{i=0}^{\omega-1}\theta(i+1)\cdot(\Delta\theta(i))^2. \tag{33}$$

We know that

$$A_3 = \Omega^3\,a_3. \tag{34}$$

To obtain the actual formulas desired, we proceed thus:

$$\Omega\,\Delta\theta(i) = P(i) - \Omega\frac{i}{\omega} = \sum_{i=0}^{i-1} p(i) - \Omega\frac{i}{\omega} + C,$$

where C is determined so that $\sum_{i=0}^{\omega-1} P(i) = \Omega \frac{\omega-1}{2}$.

$$\Omega \Delta^2 \theta(i) = p(i+1) - \frac{\Omega}{\omega}.$$

Let $\Omega \Delta^2 \theta(i) = \Delta^2 \phi(i+1)$, where $\Delta^2 \phi(i)$ denotes the second difference of a function $\phi(i)$. Let $\frac{\Omega}{\omega} = c$. Then $\Delta^2 \phi(i+1) = c\omega \Delta^2 \theta(i)$. Determine the arbitrary constants of summation so that $\Delta\phi(i+1) = c\omega \Delta\theta(i)$ and $\phi(i+1) = c\omega\theta(i)$. Like $\Delta\theta(i)$ and $\Delta^2\theta(i)$, $\Delta\phi(i)$ and $\Delta^2\phi(i)$ have the period ω. Bear in mind that, if $f(i)$ has the period ω, $\sum_{i=k}^{i=k+\omega-1} f(i)$ is independent of k, and substitute in (16) for A_1 and in (28) and (34) for A_2 and A_3 respectively, using the value of a_3 given by (33):

$$A_1 = \frac{c\omega^2}{2}, \tag{35}$$

$$A_2 = c^2\omega^2 \left[\frac{\omega^2-1}{24}\right] - \tfrac{1}{2}\omega \sum_{i=0}^{\omega-1} (\Delta\phi(i))^2, \tag{36}$$

$$A_3 = c^3\omega^2 \left[\frac{\omega^4}{720} - \frac{\omega^2}{144} + \frac{1}{180}\right] + 2c\omega \sum_{i=0}^{\omega-1} (\phi(i))^2 - c\frac{\omega^3+2\omega}{12} \sum_{i=0}^{\omega-1} (\Delta\phi(i))^2$$
$$- \frac{\omega}{2} \sum_{i=0}^{\omega-1} \phi(i)(\Delta\phi(i))^2 - \frac{\omega}{2} \sum_{i=0}^{\omega-1} (\Delta\phi(i))^2 \phi(i+1). \tag{37}$$

We proceed by determining $c = \frac{\Omega}{\omega}$, then $\Delta^2\phi(i) = p(i) - ci$, then $\Delta\phi(i) = \Sigma\Delta^2\phi(i)$, where that particular sum is chosen which will cause $\phi(i)$ to be periodic.

$\phi(i) = \Sigma\Delta\phi(i)$, determined so that $\phi(1) = 0$.

Formulas (35), (36) and (37) are easily applicable when p is expressed as a trigonometric sum, which development is always, theoretically at least, possible.

Finite Groups of Plane Birational Transformations with Eight Fundamental Points.

By F. R. Sharpe.

The enumeration of the finite groups of plane birational transformations is due to S. Kantor and A. Wiman. The former, in his Naples Prize Memoir,* showed that all periodic plane birational transformations could be reduced by combinations of quadratic transformations into one of a certain number of types having at most 8 fundamental points. The groups of periodic transformations having 3, 4, 5 or 6 fundamental points have been completely determined.† In the cases of 6, 7 or 8 fundamental points the Grassmann depiction of the plane on a cubic surface may be used to advantage. The groups of periodic transformations with 6 fundamental points are therefore isomorphic with the groups of linear transformations of the 27 lines of a cubic surface. In the cases of 7 or 8 fundamental points the cubic surface may be depicted upon a double plane by means of a certain (1, 2) correspondence. In the case of 8 fundamental points the curve of branch points on the double plane is a sextic with 2 coincident triple points.‡ A. Wiman § found the equation of this curve and determined its groups of transformations. The purpose of this paper (suggested by Prof. Virgil Snyder) is to determine the transformations in the simple plane in which the 8 fundamental points lie that correspond to a given group of transformations in the double plane. There are 120 conics that are tangent to the sextic at the triple points and also at 3 other points.|| It is shown that each conic leads to the partial determination of the cubic surface. The complete determination of the cubic surface then reduces to finding the bitangents to a certain quartic curve. To the triple infinity of conics that are tangent to the sextic at the triple point, correspond a triple infinity of quadrics that are tangent to the cubic surface at two points O, O'. This system of quadrics meets the cubic surface in space-sextic curves. To the 120 special conics in the double plane cor-

* "Prémiers fondaments pour un théorie des transformations périodiques univoques," 1891.
† A. Wiman, *Math. Annalen*, Band 48.
‡ S. Kantor, "Theorie der endlichen Gruppen," 1895.
§ *Loc. cit.*
|| Schottky, *Crelle*, Band 103.

respond 120 composite space sextics. Of these, 27 are the 27 lines on the cubic surface and 27 residual quintics; 2 are the points O, O' and 2 sextics; 54 are the residual conics found by passing planes through the 27 lines and O or O' and 54 residual quartics. In depicting the cubic surface on the simple plane, any line may be depicted as a point a_1. Any of the 15 lines skew to a_1 may be depicted as a conic b_1, not passing through a_1. Five other lines skew to a_1, but meeting b_1, are depicted as points a_2, a_3, a_4, a_5, a_6 on the conic b_1. O and O' are depicted as points, $a_1, a_2, a_3, a_4, a_5, a_6, O, O'$ being the 8 fundamental points. Given the transformation in the double plane, the transformation on the cubic surface can be found. There are two cases according as we do or do not interchange the two sheets of the double plane. To this interchange corresponds the interchange of the 2 parts of each composite sextic of the cubic surface and the Bertini transformation of order 17 in the simple plane. We can now determine the images of 6 a lines and of O and O' on the cubic surface, and hence of the 8 fundamental points in the simple plane. For a special case, the determination of the 120 conics, the cubic surface and the transformation in the simple plane is actually carried out.

By using homogeneous coordinates (x, y, z, u), letting O be $(0, 0, 1, 0)$ and O' $(0, 0, 0, 1)$, the equation of the cubic surface may be reduced to the form

$$z(zu + f_2) + u(zu + \phi_2) + zuf_1 + f_3 = 0, \tag{1}$$

where f_2, ϕ_2, f_1, f_3 are homogeneous functions of x and y of degrees $2, 2, 1, 3$ respectively. The quadric surfaces tangent to (1) at O, O' are

$$zu = Ax^2 + 2Hxy + By^2 = F_2. \tag{2}$$

We may write (1) in the form

$$\begin{aligned}\{z(zu+f_2)-u(zu+\phi_2)\}^2 &= (zuf_1+f_3)^2-4zu(zu+f_2)(zu+\phi_2)\\ &=4\left(-zu-\frac{4f_2+4\phi_2-f_1^2}{12}\right)^3-\left(-zu-\frac{4f_2+4\phi_2-f_1^2}{12}\right)\left\{2f_1f_3-4f_2\phi_2+\frac{(4f_2+4\phi_2-f_1^2)^2}{12}\right\}\\ &\quad-\left\{-f_3^2+\frac{(2f_1f_3-4f_2\phi_2)(4f_2+4\phi_2-f_1^2)}{12}+\frac{(4f_2+4\phi_2-f_1^2)^3}{216}\right\}.\end{aligned} \tag{3}$$

For a given value of $\frac{y}{x}$, (1) and (2) give 2 sets of values for $\frac{z}{x}, \frac{u}{x}$. If we set up a correspondence between a point (x, y, z, u) on the cubic surface and a point (x', y', z') on a plane, where

$$x' = x, \quad y' = y, \quad z'y' = -zu - \frac{4f_2 + 4\phi_2 - f_1^2}{12}, \tag{4}$$

any plane through O, O' will meet the curve of intersection of (1) and (2) in 2 points that will be represented on the plane by the same point. The plane is therefore a double plane.

The 2 points coincide if

$$z(zu+f_2)-u(zu+\phi_2)=0. \tag{5}$$

The curve of degree 9 in which (5) meets (1) is represented on the double plane by the sextic curve of branch points

$$4z'^3y'^3-z'y'f_4-f_6=0, \tag{6}$$

where, from (3),

$$\begin{aligned} f_4&=2f_1f_3-4f_2\phi_2+\frac{(4f_2+4\phi_2-f_1^2)^2}{12},\\ f_6&=-f_3^2+\frac{(2f_1f_3-4f_2\phi_2)(4f_2+4\phi_2-f_1^2)}{12}+\frac{(4f_2+4\phi_2-f_1^2)^3}{216}. \end{aligned} \tag{7}$$

From (2) and (4) the sextic curves of intersection of (1) and (2) are represented on the double plane by conics

$$z'y'=F_2'=px'^2+qx'y'+ry'^2, \tag{8}$$

that are tangent to (6) at $(0,0,1)$.

There are 120 sets of values of p, q, r such that, when (8) is substituted in (6), the left-hand member of (6) is of the form

$$(\alpha x'^3+\beta x'^2y'+\gamma x'y'^2+\delta y'^3)^2.$$

The conic (8) is then tangent to (6) at 3 points as well as at $(0,0,1)$.

Substituting any 1 of these sets of values of p, q, r in (8), let

$$\frac{4f_2+4\phi_2-f_1^2}{12}=-F_2'. \tag{9}$$

From (7) it follows that

$$\begin{aligned} f_3^2&=-f_6-(f_4-12F_2'^2)F_2'-8F_2'^3\\ &=4F_2'^3-f_4F_2'-f_6\\ &=(\alpha x'^3+\beta x'^2y'+\gamma x'y'^2+\delta y'^3)^2. \end{aligned}$$

Hence,

$$f_3=\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3. \tag{10}$$

From (7) and (9) we find

$$(f_2-\phi_2)^2=\left(\frac{f_1^2}{4}-3F_2'\right)^2-2f_1f_3-12F_2'^2+f_4. \tag{11}$$

If $z'=\frac{f_1}{2}$ is a bitangent to the quartic

$$(z'^2-3F_2')^2-4z'f_3-12F_2'^2+f_4=0, \tag{12}$$

then

$$(f_2-\phi_2)^2=(sx'^2+tx'y'+vy^2)^2, \tag{13}$$

and f_2, ϕ_2 can be found from (9) and (13) in the form

$$\frac{f_1^2}{8}-\frac{3}{2}F_2'\pm\frac{1}{2}(sx'^2+tx'y'+vy'^2). \tag{14}$$

It is to be remarked that f_4 and $-f_6$ are the invariants I and J of (12), regarded as a quartic in z', and that (6), regarded as a cubic in $z'y'$, is the reducing cubic of (12).

The determination of the 120 tritangent conics to (6) appears to be a matter of considerable difficulty. In the particular case $f_4 \equiv 0$, $f_6 \equiv -x'^6 - y'^6$ we have to determine p, q, r so that the substitution in (6) of

$$\sqrt[3]{4}\, z' y' = p x'^2 + q x' y' + r y'^2$$

gives (dropping the primes)

$$(p x^2 + q x y + r y^2)^3 + x^6 + y^6 = (\alpha x^3 + \beta x^2 y + \gamma x y^2 + r y^3)^2. \qquad (15)$$

The solutions are of 5 types:

$$p x^2 + q x y + r y^2 = -x^2 \text{ or } -y^2, \qquad \text{(I)}$$

and 4 other solutions found by substituting εx for x, εy for y, where $\varepsilon^6 = 1$.

$$p x^2 + q x y + r y^2 = \sqrt[3]{2}\, x y, \qquad \text{(II)}$$

and 5 other solutions found by the same substitution.

$$p x^2 + q x y + r y^2 = -x^2 - y^2 \sqrt[3]{4} \text{ or } -y^2 - x^2 \sqrt[3]{4}, \qquad \text{(III)}$$

and 16 others found as in the previous types.

$$p x^2 + q x y + r y^2 = \sqrt[3]{2}\, \{x y + k (x + y)^2\}, \qquad \text{(IV)}$$

where $k = -2 - \sqrt[3]{2} - \sqrt[3]{4}$, $-2 - \varepsilon^2 \sqrt[3]{2} - \varepsilon^4 \sqrt[3]{4}$, $-2 - \varepsilon^4 \sqrt[3]{2} - \varepsilon^2 \sqrt[3]{4}$, and 53 others of the same type. This result is obtainable by putting $\alpha = \delta$, $\beta = \gamma$ so that $(x + y)^2$ is a factor of the right-hand member of (15). We may now take

$$p x^2 + q x y + r y^2 = \sqrt[3]{2}\, \{x y + k (x + y)^2\},$$

since $p = r$ and $(x + y)^2$ is a factor, where k is chosen so that

$$\{(x + y)^2 - 3 x y\}^2 + 2 \{k^3 (x + y)^4 + 3 k^2 (x + y)^2 x y + 3 k x^2 y^2\}$$

is a perfect square. Hence,

$$9 (k^2 - 1)^2 = (1 + 2 k^3)(9 + 6 k);$$

that is,

$$k^3 + 6 k^2 + 6 k + 2 = 0,$$

which gives the stated values for k.

The remaining 36 solutions are of a more general type. From (15) we have

$$\alpha^2 = p^3 + 1, \quad 2 \alpha \beta = 3 p^2 q, \quad 2 \alpha \gamma + \beta^2 = 3 p^2 r + 3 p q^2,$$
$$\beta^2 = r^3 + 1, \quad 2 \gamma \delta = 3 r^2 q, \quad 2 \beta \delta + \gamma^2 = 3 r^2 p + 3 r q^2,$$
$$2 \alpha \delta + 2 \beta \gamma = q^3 + 6 p q r.$$

Eliminating α and δ,

$$\left.\begin{array}{c} 4\,(p^3+1)\,\beta^2 = 9\,p^4 q^2, \quad 4\,(p^3+1)\,\gamma^2 = (3\,p^2 r + 3\,p\,q^2 - \beta^2)^2, \\ 4\,(r^3+1)\,\gamma^2 = 9\,r^4 q^2, \quad 4\,(r^3+1)\,\beta^2 = (3\,p^2 r + 3\,r\,q^2 - \gamma^2)^2, \\ 4\,(p^3+1)\,(r^3+1) = (q^3 + 6\,p\,q\,r - 2\,\beta\,\gamma)^2. \end{array}\right\} \quad (16)$$

Eliminating β and γ,

$$\left.\begin{array}{c} 16\,(p^3+1)^3 r^4 q^2 = (r^3+1)\,\{4\,p\,(p\,r+q^2)\,(p^3+1) - 3\,p^4 q^2\}, \\ 16\,(r^3+1)^3 p^4 q^2 = (p^3+1)\,\{4\,r\,(p\,r+q^2)\,(r^3+1) - 3\,r^4 q^2\}. \end{array}\right\} \quad (17)$$

Divide the first of these by the second, extract the square root, and divide by $p^3\,(r^3+1) - r^3\,(p^3+1)$. This gives

$$4\,(p^3+1)\,(r^3+1)\,(p\,r+q^2) = 3\,q^2\,\{p^3\,(r^3+1) + r^3\,(p^3+1)\}.$$

Substituting back in (16) and (17), we find

$$16\,(p^3+1)\,(r^3+1) = 9\,p^2 q^2 r^2 = \beta^2\gamma^2 = 4\,q^6.$$

Hence,

$$\frac{p^3}{p^3+1} + \frac{r^3}{r^3+1} = \frac{20}{9}, \qquad \frac{p^3 r^3}{(p^3+1)\,(r^3+1)} = \frac{32}{27};$$

and therefore

$$\frac{p^3}{p^3+1} = \frac{4}{3}, \qquad \frac{r^3}{r^3+1} = \frac{8}{9},$$

so that

$$p = -\sqrt[3]{4}, \quad r = 2;$$

and therefore

$$q = \sqrt{-3}\,\sqrt[3]{2},$$

$$\alpha = \sqrt{-3}, \quad \beta = 3\,\sqrt[3]{4}, \quad \gamma = -2\,\sqrt{-3}\,\sqrt[3]{2}, \quad \delta = -3.$$

Hence,

$$p\,x^2 + q\,x\,y + r\,y^2 = -\sqrt[3]{4}\,x^2 + \sqrt{-3}\,\sqrt[3]{2}\,x\,y + 2\,y^2, \quad \text{(V)}$$

and 35 other solutions of the same type.

If we take the simplest solution

$$p\,x^2 + q\,x\,y + r\,y^2 \equiv -x^2,$$

then

$$F_2 \equiv \frac{-x^2}{\sqrt[3]{4}} \quad \text{and} \quad f_3 \equiv y^3.$$

Hence, from (12), $z = \frac{1}{2} f_1$ is bitangent to

$$\left(z^2 + \frac{3\,x^2}{\sqrt[3]{4}}\right)^2 - 4\,z\,y^3 - \frac{12\,x^4}{2\,\sqrt[3]{2}} = 0. \quad (18)$$

Writing x for $\dfrac{x}{\sqrt[3]{2}}$, this equation becomes

$$4z(z^3-y^3)-3(z^2-x^2)^2=0. \qquad (19)$$

Hence, $z=0$, $z=y$, $\varepsilon^2 y$, $\varepsilon^4 y$ are bitangents.

To find the remaining 24 bitangents, consider the cubic surface

$$zu^2+\sqrt{3}\,(z^2-x^2)\,u+z^3-y^3=0. \qquad (20)$$

The tangent cone vertex $V\,(0,0,0,1)$ meets $u=0$ in the quartic (19). The plane through V and any bitangent to (19) meets (20) in one of the 27 lines and a conic. Four planes of the pencil

$$u=k\,(z-y), \quad (k\neq 0), \qquad (21)$$

through the line $u=0$, $z=y$ meet (20) in a triangle. Since

$$k^2(z-y)\,z+\sqrt{3}\,k\,(z^2-x^2)+z^2+yz+z^2=0 \qquad (22)$$

passes through the intersection of (20) and (21), we can determine k by making

$$k^2(z-y)\,z+\sqrt{3}\,kz^2+z^2+yz+z^2=0$$

a perfect square. Hence,

$$(k^2-1)^2=4\,(k^2+\sqrt{3}\,k+1)\,;$$

that is,

$$(k+\sqrt{3}\,)\,(k^3-\sqrt{3}\,k^2-3k-\sqrt{3}\,)=0.$$

Hence,

$$k=-\sqrt{3}\,,\quad \frac{(1+\sqrt[3]{2}\,)^2}{\sqrt{3}},\quad \frac{(1+\varepsilon^2\sqrt[3]{2}\,)^2}{\sqrt{3}},\quad \frac{(1+\varepsilon^4\sqrt[3]{2}\,)^2}{\sqrt{3}}.$$

Substituting in (22), we find

$$y+\frac{1-k^2}{2}z=\sqrt{\sqrt{3}\,k}\cdot x.$$

Hence, the 28 bitangents of (18) are

$$z=0,\quad z=y,\quad z=y\pm\frac{\sqrt{-3}\,x}{\sqrt[3]{2}},$$

$$y-(1+\sqrt[3]{2}+\sqrt[3]{4}\,)\,z=\pm\frac{(1+\sqrt[3]{2}\,)\,x}{\sqrt[3]{2}},$$

$$y-(1+\varepsilon^2\sqrt[3]{2}+\varepsilon^4\sqrt[3]{4}\,)\,z=\pm\frac{(1+\varepsilon^2\sqrt[3]{2}\,)\,x}{\sqrt[3]{2}},$$

$$y-(1+\varepsilon^4\sqrt[3]{2}+\varepsilon^2\sqrt[3]{4}\,)\,z=\pm\frac{(1+\varepsilon^4\sqrt[3]{2}\,)\,x}{\sqrt[3]{2}},$$

and 18 others found by substituting $\varepsilon^2 y$ or $\varepsilon^4 y$ for y. Selecting $z=0$ as the simplest case, we have $f_1\equiv 0$. Hence, from (9) and (11),

$$f_2 + \phi_2 = \frac{3\,x^2}{\sqrt[3]{4}}, \qquad f_2 - \phi_2 = \frac{\sqrt{-3}\,x^2}{\sqrt[3]{4}},$$

$$f_2 = \frac{3 + \sqrt{-3}}{2\sqrt[3]{4}}\,x^2 = -\alpha\,\epsilon^2 x^2,$$

$$\phi_2 = \frac{3 - \sqrt{-3}}{2\sqrt[3]{4}}\,x^2 = \alpha\,\epsilon^4 x^2,$$

where $\alpha = \dfrac{\sqrt{-3}}{\sqrt[3]{4}}$. The cubic surface is therefore

$$z\,(z\,u - \alpha\,\epsilon^2 x^2) + u\,(z\,u + \alpha\,\epsilon^4 x^2) + y^3 = 0. \tag{23}$$

Theoretically, in order to determine the 27 lines on (23), we could find the intersection of (23) with the 120 quadrics derivable from the 120 conics previously found. Practically, it is easier to write (23) in the form

$$\alpha\,\epsilon^2 x^2\,(\epsilon^2 u - z) + y^3 + z^2 u + z\,u^2 = 0. \tag{24}$$

The plane $z = \epsilon^2 u$ meets (24) in the 3 lines

$$z = \epsilon^2 u, \qquad y = u, \qquad \epsilon^2 u \text{ or } \epsilon^4 u.$$

Proceeding as in the determination of the bitangents to the quartic, we find that through the intersection of (24) and the plane

$$\epsilon^2 u - z = k\,(y - u) \tag{25}$$

passes the cone

$$\alpha\,\epsilon^2 k\,x^2 + y^2 + u\,y\,(1 + k^2) + (1 - k - 2\,k\,\epsilon^2 - k^2)\,u^2 = 0. \tag{26}$$

This cone breaks up into 2 planes if

$$(1 + k^2)^2 = 4\,(1 - 2\,k\,\epsilon^2 - k^2).$$

Hence,

$$k = -\sqrt{-3}, \quad \frac{(1 + \sqrt[3]{2}\,)^2}{-\sqrt{-3}}, \quad \frac{(1 + \epsilon^2\sqrt[3]{2}\,)^2}{-\sqrt{-3}}, \quad \frac{(1 + \epsilon^4\sqrt[3]{2}\,)^2}{-\sqrt{-3}}.$$

Substituting the values of k in (25) and (26), we find the 4 lines

$$\begin{cases} y - u = \pm\dfrac{\sqrt{-3}}{\sqrt[3]{2}}\,\epsilon\,x, \\[2ex] y - \epsilon^4 z = \pm\dfrac{\sqrt{-3}}{\sqrt[3]{2}}\,\epsilon^3 x, \end{cases}$$

$$\begin{cases} y - (1 + \sqrt[3]{2} + \sqrt[3]{4}\,)\,u = \pm\dfrac{(1 + \sqrt[3]{2}\,)\,\epsilon\,x}{\sqrt[3]{2}}, \\[2ex] y - \epsilon^4\,(1 + \epsilon^2\sqrt[3]{2} + \epsilon^4\sqrt[3]{4}\,)\,z = \pm\dfrac{(1 + \epsilon^2\sqrt[3]{2}\,)\,\epsilon^3 x}{\sqrt[3]{2}}, \end{cases}$$

and 4 similar lines found by changing $\sqrt[3]{2}$ into $\varepsilon^2\sqrt[3]{2}$ or $\varepsilon^4\sqrt[3]{2}$ except in the denominator of the right-hand members. By changing y into $\varepsilon^2 y$ or $\varepsilon^4 y$ we find the remaining 16 lines. For brevity, denote by $\sqrt{-3}$, $\sqrt[3]{2}$, $\varepsilon^2\sqrt[3]{2}$, $\varepsilon^4\sqrt[3]{2}$ the value of k used, by y, $\varepsilon^2 y$, $\varepsilon^4 y$ the value of y, and by $+x$, $-x$ the value of x. The 27 lines can now be chosen as follows:

a_1, y, $\sqrt{-3}$, x; a_2, y, $\sqrt[3]{2}$, x; a_3, $\varepsilon^2 y$, $\varepsilon^4\sqrt[3]{2}$, $-x$;
a_4, $\varepsilon^2 y$, $\sqrt[3]{2}$, $-x$; a_5, $\varepsilon^4 y$, $\sqrt{-3}$, $-x$; a_6, $\varepsilon^4 y$, $\varepsilon^4\sqrt[3]{2}$, x;
b_1, y, $\sqrt[3]{2}$, $-x$; b_2, y, $\sqrt{-3}$, $-x$; b_3, $\varepsilon^2 y$, $\sqrt[3]{2}$, x;
b_4, $\varepsilon^2 y$, $\varepsilon^4\sqrt[3]{2}$, x; b_5, $\varepsilon^4 y$, $\varepsilon^4\sqrt[3]{2}$, $-x$; b_6, $\varepsilon^4 y$, $\sqrt{-3}$, x;
c_{13}, $\varepsilon^4 y$, $\varepsilon^2\sqrt[3]{2}$, $-x$; c_{14}, $\varepsilon^4 y$, $\sqrt[3]{2}$, $-x$; c_{15}, $\varepsilon^2 y$, $\varepsilon^2\sqrt[3]{2}$, x;
c_{24}, $\varepsilon^4 y$, $\varepsilon^2\sqrt[3]{2}$, x; c_{23}, $\varepsilon^4 y$, $\sqrt[3]{2}$, x; c_{26}, $\varepsilon^2 y$, $\varepsilon^2\sqrt[3]{2}$, $-x$;
c_{16}, $\varepsilon^2 y$, $\sqrt{-3}$, x; c_{25}, $\varepsilon^2 y$, $\sqrt{-3}$, $-x$; c_{35}, y, $\varepsilon^4\sqrt[3]{2}$, $-x$;
c_{36}, y, $\varepsilon^2\sqrt[3]{2}$, x; c_{45}, y, $\varepsilon^2\sqrt[3]{2}$, $-x$; c_{46}, y, $\varepsilon^4\sqrt[3]{2}$, x;
$c_{12}\begin{cases}\varepsilon^2 z - u = 0\\ y = u\end{cases}$; $c_{34}\begin{cases}\varepsilon^2 u - z = 0\\ y = \varepsilon^2 u\end{cases}$; $c_{56}\begin{cases}\varepsilon^2 u - z = 0\\ y = \varepsilon^4 u\end{cases}$.

This choice was determined by first choosing c_{12}, c_{34}, c_{56}. Then a_1 was chosen, which fixes b_2; and a_2, which fixes b_1. The others were then found, first the a's, then the b's and c's. The transformation $y=\varepsilon y'$, $z=\varepsilon^5 z'$ in the double plane gives, from (4), $zu=z'u'$ on the cubic surface. Also, from (3),

$$z\,(zu+f_2)-u\,(zu+\phi_2)=\mp\,\{z'\,(z'u'+f'_2)-u'\,(z'u'+\phi'_2)\},$$

according as we do or do not interchange the sheets in the double plane.

From (1), since $\varepsilon^3=-1$, we find

$$z\,(zu+f_2)+u\,(zu+\phi_2)=-\,\{z'\,(z'u'+f'_2)+u'\,(z'u'+\phi'_2)\}.$$

Hence, if we interchange the sheets in the double plane, the transformation is linear on the surface, and is

$$z=-z',\quad u=-u',\quad x=x',\quad y=\varepsilon y';$$

that is,

$$z=z',\quad u=u',\quad x=-x',\quad y=\varepsilon^4 y'.$$

This transformation sends a_1 into a_5, a_2 into c_{14}, a_3 into c_{46}, a_4 into a_2, a_5 into c_{16}, a_6 into a_3. It leaves O and O' invariant. It is therefore a quadratic transformation of Kantor's B_6 type "b' en c, c' en a, a' en b," a, b, c, a', b', c' being a_2, a_3, a_5, a_6, a_1, a_4, respectively. If we do not interchange the sheets,

the transformation is of degree 16 having a_1, a_4, a_6, O, O' for 6-fold points, and a_2, a_3, a_5 for 5-fold points.

The transformation $x = \epsilon x'$ in the double plane gives

$$zu + \frac{x^2}{\sqrt[3]{4}} = z'u' + \frac{x'^2}{\sqrt[3]{4}}$$

on the cubic surface. Hence,

$$zu = z'u' + \phi_2'.$$

Also

$$z(zu + f_2) + u(zu + \phi_2) = -f_3 = z'(z'u' + f_2') + u'(z'u' + \phi_2'),$$

and

$$z(zu + f_2) - u(zu + \phi_2) = \mp\{z'(z'u' + f_2') - u'(z'u' + \phi_2')\},$$

according as we do or do not interchange the sheets on the double plane. If we do interchange,

$$z = \frac{u'(z'u' + \phi_2')}{zu + f_2} = \frac{u'(z'u' + \phi_2')}{z'u'},$$

$$u = \frac{z'(z'u' + f_2')}{zu + \phi_2} = \frac{z'(z'u' + f_2')}{z'u' + f_2'}.$$

Hence,

$$x = \epsilon x' \cdot z', \quad y = y' \cdot z', \quad z \mp u'(z'u' + \phi_2'), \quad u = z'^2.$$

This is a quadric transformation, the image of $O\,(0, 0, 1, 0)$ being the plane $z' = 0$. Any plane through O is transformed into a plane through O'. Hence, the image of any line on the cubic surface is found by first transforming each line by changing the first plane on which it lies by $u = z'$, $x = \epsilon x'$ into the second plane on which another line lies. This is equivalent to changing y into $\epsilon^4 y'$, x into $-x'$. The image of the first line is then the residual conic found by passing a plane through the second line and O'. Hence,

a_1 is transformed into a conic residual to a_5,
a_2 " " " " " " " c_{14},
a_3 " " " " " " " c_{46},
a_4 " " " " " " " a_2,
a_5 " " " " " " " c_{16},
a_6 " " " " " " " a_3.

O is transformed into $z' = 0$, the section of the cubic surface by the tangent plane at O'. O' is transformed into O. Hence, in the plane,

a_1	is	transformed	into	the	cubic	$\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & O, & O' \\ 1, & 1, & 1, & 1, & 2, & 1, & 0, & 1 \end{pmatrix}$,
a_2	"	"	"	"	conic	(0, 1, 1, 0, 1, 1, 0, 1),
a_3	"	"	"	"	conic	(1, 1, 1, 0, 1, 0, 0, 1),
a_4	"	"	"	"	cubic	(1, 2, 1, 1, 1, 1, 0, 1),
a_5	"	"	"	"	conic	(0, 1, 1, 1, 1, 0, 0, 1),
a_6	"	"	"	"	cubic	(1, 1, 2, 1, 1, 1, 0, 1),
O	"	"	"	"	cubic	(1, 1, 1, 1, 1, 1, 0, 2),
O'	"	"	"	"	the point O.	

This is a transformation of degree 7. If we do not interchange the sheets, we find a transformation of degree 11 having 4 3-fold, 3 4-fold and 1 6-fold point.

CORNELL UNIVERSITY, *November*, 1913.

Conics through Inflections of Self-projective Quartics.

By F. R. Sharpe.

The line through any 2 of the 9 inflections of a plane cubic curve meets the curve again in another inflection. In 1875 J. Grassmann * stated the analogous theorem for quartic curves: "The conic through any 5 of the 24 inflections of a quartic curve meets the curve again in 3 other inflections. There are therefore 759 such conics." This theorem is now known to be untrue. In 1899 Ciani,† in a paper on quartics invariant under homologies, found quartics having as many as 51 conics each passing through 8 inflections. In 1901 the same author ‡ showed that the Klein quartic had 147 such conics as well as 112 conics through 6 inflections. His method depended chiefly on the use of groups of collineations under which the quartics and conics were invariant. In this paper more use is made of the points of inflection themselves. The results confirm those of Ciani as regards the numbers of conics through 8 inflections. They also show the existence of additional conics through 6 inflections, 220 in the case of a quartic invariant under 1 homology and 2100 for the Klein quartic.

The different cases were classified by Ciani according to the number of homologies under which the quartic is invariant. For the sake of clearness let us begin with the quartic

$$x^4 + y^4 + z^4 - 3\mu(y^2z^2 + z^2x^2 + x^2y^2) = 0, \tag{10}$$

which is invariant under a G_{24} of collineations 9 of which are homologies. If (f, g, h) are the coordinates of one point of inflection, the 24 inflections are §

$$\begin{array}{lll}
A \equiv (f, g, h), & B \equiv (g, h, f), & C \equiv (h, f, g), \\
D \equiv (-f, g, h), & K \equiv (-g, h, f), & R \equiv (-h, f, g), \\
S \equiv (f, -g, h), & E \equiv (g, -h, f), & L \equiv (h, -f, g), \\
N \equiv (f, g, -h), & U \equiv (g, h, -f), & G \equiv (h, f, -g), \\
X \equiv (f, h, g), & J \equiv (g, f, h), & Q \equiv (h, g, f), \\
O \equiv (-f, h, g), & V \equiv (-g, f, h), & H \equiv (-h, g, f), \\
M \equiv (f, -h, g), & T \equiv (g, -f, h), & F \equiv (h, -g, f), \\
P \equiv (f, h, -g), & W \equiv (g, f, -h), & I \equiv (h, g, -f).
\end{array}$$

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* Dissertation, Berlin. † *Rend. Circ. Mat. di Palermo.* ‡ *Annali di Mat.*

§ Ciani, *Annali di Mat.*, 1901, p. 53.

The reason for the particular choice of letters will appear later on in the work.

Corresponding to the 9 homologies there are 9 systems of conics:

$$p x^2 + q y^2 + r z^2 + s y z = 0, \tag{1}$$
$$p y^2 + q z^2 + r x^2 + s z x = 0, \tag{2}$$
$$p z^2 + q x^2 + r y^2 + s x y = 0, \tag{3}$$
$$p\,(y^2 + z^2) + q x^2 + r y z + s x\,(y + z) = 0, \tag{4}$$
$$p\,(z^2 + x^2) + q y^2 + r z x + s y\,(z + x) = 0, \tag{5}$$
$$p\,(x^2 + y^2) + q z^2 + r x y + s z\,(x + y) = 0, \tag{6}$$
$$p\,(y^2 + z^2) + q x^2 + r y z + s x\,(y - z) = 0, \tag{7}$$
$$p\,(z^2 + x^2) + q y^2 + r z x + s y\,(z - x) = 0, \tag{8}$$
$$p\,(x^2 + y^2) + q z^2 + r x y + s z\,(x - y) = 0, \tag{9}$$

each being invariant under 1 homology. The inflections may be arranged in the corresponding invariant pairs of points:

$$(1')\ \begin{array}{c|c|c|c} AD & SN & XO & MP \\ BK & EU & JV & TW \\ CR & LG & QH & FI \end{array} \qquad (2')\ \begin{array}{c|c|c|c} AS & DN & XM & OP \\ BE & KU & JT & VW \\ CL & RG & QF & HI \end{array} \qquad (3')\ \begin{array}{c|c|c|c} AN & SD & XP & OM \\ BU & EK & JW & VT \\ CG & LR & QI & HF \end{array}$$

$$(4')\ \begin{array}{c|c|c|c} AX & DO & SP & NM \\ BJ & KV & EW & UT \\ CQ & RH & LI & GF \end{array} \qquad (5')\ \begin{array}{c|c|c|c} AQ & DI & SF & NH \\ BX & KP & EM & UO \\ CJ & RW & LT & GV \end{array} \qquad (6')\ \begin{array}{c|c|c|c} AJ & DT & SV & NW \\ BQ & KF & EH & UI \\ CX & RM & LO & GP \end{array}$$

$$(7')\ \begin{array}{c|c|c|c} AO & DX & SM & NP \\ BV & KJ & ET & UW \\ CH & RQ & LF & GI \end{array} \qquad (8')\ \begin{array}{c|c|c|c} AF & DH & SQ & NI \\ BM & KO & EX & UP \\ CT & RV & LJ & GW \end{array} \qquad (9')\ \begin{array}{c|c|c|c} AW & DV & ST & NJ \\ BI & KH & EF & UQ \\ CP & RO & LM & GX \end{array}$$

If from any 1 of these 9 sets of pairs of points we select any 3 pairs, the 6 points lie on a conic of the corresponding type. We can now take up with advantage the various types of quartics, beginning with

CASE I. One homology, $\begin{pmatrix} x\ y\ z \\ -x\ y\ z \end{pmatrix}$, the quartic being

$$x^4 + x^2 f_2\,(y, z) + f_4\,(y, z) = 0.$$

Using the arrangement (1′) there are 220 conics of type (1) that pass through 6 inflections.

CASE II. Three homologies, $\begin{pmatrix} x\ y\ z \\ x\ z\ y \end{pmatrix}, \begin{pmatrix} x\ y\ z \\ z\ y\ x \end{pmatrix}, \begin{pmatrix} x\ y\ z \\ y\ x\ z \end{pmatrix}$, with their centers on $x + y + z = 0$ and their axes concurrent at $(1, 1, 1)$. The quartic is of the symmetric form

$$a \Sigma x^4 + b \Sigma y^2 z^2 + c x y z \Sigma x + d \Sigma x y \cdot \Sigma x^2 = 0.$$

Using the arrangements (4′), (5′), (6′) and the corresponding types of conics,

the 4 sets of 6 points $ABCJQX$, $DOLITU$, $FGPKVS$, $EHMNRW$ are common to the 3 arrangements and lie on 4 conics of the symmetric type $a\Sigma x^2 + b\Sigma xy = 0$. The line $x + y + z = 0$ is a bitangent to the quartic, the points of contact being $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, where $\omega^3 = 1$, which also lie on the 4 conics. Hence, the inflections lie on 3 sets of 220 conics, 4 of which are common to the 3 sets and pass through the points of contact of a bitangent.

CASE III. Three homologies, $\begin{pmatrix} x & y & z \\ -x & y & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ x & -y & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ x & y & -z \end{pmatrix}$, the quartic being

$$a x^4 + b y^4 + c z^4 + d y^2 z^2 + e z^2 x^2 + f x^2 y^2 = 0.$$

The inflections may be arranged in 6 sets of 4,

$$ADNS,\ BKEU,\ CRLG,\ XOMP,\ JVTW,\ QHFI,$$

such that a conic of the form

$$p x^2 + q y^2 + r z^2 = 0 \tag{10}$$

passes through each pair of sets. Hence, the inflections lie in eights on 15 conics. Among the conics of types (1), (2), (3) each of these conics occurs 4 times. Hence, the inflections lie in sixes on 3 sets of 160 conics.

CASE IV. Five homologies, the quartic being

$$a(x^4 + y^4) + b z^4 + c z^2 (x^2 + y^2) + d x^2 y^2 = 0.$$

In addition to the 3 homologies of Case III, we have also $\begin{pmatrix} x & y & z \\ y & x & z \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ y & x & -z \end{pmatrix}$. The inflections may be arranged in 6 new sets of 4,

$$AJNW,\ BIQU,\ CGPX,\ DSTV,\ EFHK,\ LMOR,$$

such that a conic of the form

$$p(x^2 + y^2) + q z^2 + r x y = 0 \tag{11}$$

passes through each pair of sets. Three of these pairs coincide with 3 of the pairs of Case III. Hence, the inflections lie in eights on 27 conics. In counting the conics through 6 inflections of type (1) or (2), each of the 15 conics of type (10) through 8 inflections is counted 4 times, and similarly for (6) or (9) or (11). In the case of type (3) each of the 27 conics of type (10) or (11) is counted 4 times. Hence, the inflections lie in sixes on 4 sets of 160 conics and 1 set of 112 conics.

CASE V. Seven homologies, the quartic being

$$a(x^4 + y^4) + b z^4 + c x^2 y^2 = 0.$$

The 2 additional homologies are

$$\begin{pmatrix} x & y & z \\ y & -x & iz \end{pmatrix}, \begin{pmatrix} x & y & z \\ -y & x & iz \end{pmatrix}.$$

The 24 inflections consist of 4 sets of 4,

$$ANBU,\ DSKE,\ JWQI,\ VTHF,$$

lying on 4 lines through $(0, 0, 1)$, and 4 undulation points on $z = 0$,

$$\begin{cases} C \\ G \end{cases} \begin{cases} L \\ R \end{cases} \begin{cases} O \\ M \end{cases} \begin{cases} X \\ P \end{cases}$$

each of which counts for 2 inflections. The first and fourth lines, the second and third, and each of the 4 taken with $z=0$ twice, form 10 new degenerate conics through 8 inflections. Hence, the inflections lie in eights on 37 conics. In counting the conics through 6 inflections we find, of type (1), (2), (6) or (9), 32 not passing through undulations, 48 through 2 undulations, 16 through 4 undulations, the 2 latter types being counted twice. Among the conics of type (3) are 8 not through undulations, 80 tangent to the quartic at 1 undulation and 16 tangent at 2 undulations. There are also, corresponding to the 2 new homologies, the 2 new types of conics

$$p\,z^2 + q\,(x^2 - y^2) + r\,x\,y + s\,z\,(x - i\,y) = 0,$$
$$p\,z^2 + q\,(x^2 - y^2) + r\,x\,y + s\,z\,(y - i\,x) = 0,$$

and the corresponding pairs of invariant points

$$\begin{array}{llll} AF & HN & \begin{cases} C \\ G \end{cases} & \begin{cases} M \\ O \end{cases} \\ BV & IS & & \\ DQ & KW & \begin{cases} X \\ P \end{cases} & \begin{cases} R \\ L \end{cases} \\ EJ & TU & & \end{array} \qquad \begin{array}{llll} AH & EW & \begin{cases} C \\ G \end{cases} & \begin{cases} M \\ O \end{cases} \\ BT & JK & & \\ DI & QS & \begin{cases} X \\ P \end{cases} & \begin{cases} R \\ L \end{cases} \\ FN & UV & & \end{array}$$

In either case, if we select 3 out of the first 8 pairs and reject the 2 conics through 8 points *ANBUVTHF*, *DSKEJWQI*, we find 48 conics. If we select 2 of the first 8 pairs and 2 of the undulations *CM* or *XR*, each counted twice, we find 56 repeated conics. If we select 1 of the first 8 pairs and 4 undulations, we find 8 conics repeated 4 times or 16 conics tangent at 2 undulations. Hence, the inflections lie in sixes on 4 sets of 160 conics, 1 set of 104 conics and 2 sets of 152 conics.

CASE VI. Nine homologies, the quartic being

$$\Sigma\,x^4 - 3\,\mu\,\Sigma\,y^2 z^2 = 0.$$

This is a combination of II and III, and differs from IV in having 2 new sets of conics through 8 inflections:

$$p\,(y^2 + z^2) + q\,x^2 + r\,y\,z = 0, \tag{12}$$

with the 6 sets of 4 points

$$ADXO,\ BKJV,\ CRQH,\ SNMP,\ EUTW,\ LGFI,$$

and

$$p\,(z^2 + x^2) + q\,y^2 + r\,x\,z = 0, \tag{13}$$

with the corresponding sets of points

$$ASQF,\ BEXM,\ CLJI,\ POKU,\ VWGR,\ DNIH,$$

each giving 12 new and 3 old conics through 8 inflections. Hence, the inflections lie in eights on 51 conics. The sets of conics through 6 inflections of types (4), (8), (9) have 4 conics in common, of the form

$$p\,(x^2+y^2+z^2)+q\,(-yz+zx+xy)=0,$$

that pass through the points of contact of the bitangent $y+z-x=0$; namely,

$$AXFEWG,\ BJMLNI,\ CQTSPU,\ DOHKVR.$$

Similarly, for (5), (7), (9) and the bitangent $x+z-y=0$ we find

$$AQORUW,\ BXVDGI,\ CJHKPN,\ SFMLTE;$$

and for (6), (7), (8) and the bitangent $x+y-z=0$,

$$AJOKLF,\ BQVRMS,\ CXHDTE,\ NWUIGP.$$

Hence, the inflections lie in sixes on 3 sets of 112 conics, on 6 sets of 152 conics and on 4 sets of 4 conics that pass through the points of contact of a bitangent.

CASE VII. Twenty-one homologies, the quartic being

$$\Sigma x^4-3\mu\Sigma y^2z^2=0, \tag{13}$$

where $\mu^2-\mu+2=0$.

The quartic of VI is invariant under a G_{24} of collineations, while (13), for the special values of μ, is invariant under a G_{168} of collineations. To determine the systems of conics, we must know how one of the 48 collineations of period 7 interchanges the 24 inflections. The Klein quartic

$$x^3z+y^3x+z^3y=0 \tag{14}$$

has $A\,(1,0,0)$, $B\,(0,1,0)$, $C\,(0,0,1)$ for 3 inflections. It is invariant under 21 homologies, 1 of which is, in the notation of Weber ("Algebra," Band II),

$$\left.\begin{aligned} x' &= \alpha x+\beta y+\gamma z,\\ y' &= \beta x+\gamma y+\alpha z,\\ z' &= \gamma x+\alpha y+\beta z, \end{aligned}\right\} \tag{15}$$

where

$$\alpha=h\,(\varepsilon^4-\varepsilon^3),\quad \beta=h\,(\varepsilon^2-\varepsilon^5),\quad \gamma=h\,(\varepsilon-\varepsilon^6),$$

$$h=-\frac{1}{\sqrt{-7}},\quad \varepsilon^7=1,\quad \alpha\varepsilon^2+\beta\varepsilon+\gamma\varepsilon^4=0,\quad \alpha\varepsilon^5+\beta\varepsilon^6+\gamma\varepsilon^3=0.$$

The center of this homology is $a\,(\beta\gamma,\ \alpha\beta,\ \alpha\gamma)$, and the axis bc is

$$\beta\gamma x+\alpha\beta y+\alpha\gamma z=0. \tag{16}$$

Transforming by the collineations

$$\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} x & y & z \\ \varepsilon x & \varepsilon^2 y & \varepsilon^4 z \end{pmatrix}, \tag{18}$$

we can find 20 other axes and centers of homology. Four centers lie on each axis, 4 axes pass through each center. Transforming A, B, C by means of (15) and (18), we can find the remaining 21 inflections. The quartic (14), when referred to $a\,b\,c$ as triangle of reference (where b, c are 2 of the 4 centers on (16) such that each lies on the axis corresponding to the other), will have (13) for its equation, provided that the other 2 centers, d, d' on $b\,c$ and e, e' on $a\,c$, are transformed into $(0, 1, \pm 1)$, $(1, 0, \pm 1)$. We can then determine the effect of (18) on the inflections when referred to the triangle of reference $a\,b\,c$.

If

$$b \text{ is } (\alpha\beta\varepsilon,\ \alpha\gamma\varepsilon^2,\ \beta\gamma\varepsilon^4),$$
$$a\,c \text{ is } \alpha\beta\varepsilon^6 x + \alpha\gamma\varepsilon^5 y + \beta\gamma\varepsilon^3 z = 0,$$

then

$$c \text{ is } (\alpha\gamma\varepsilon^3,\ \beta\gamma\varepsilon^6,\ \alpha\beta\varepsilon^5),$$

and

$$a\,b \text{ is } \alpha\gamma\varepsilon^4 x + \beta\gamma\varepsilon y + \alpha\beta\varepsilon^2 z = 0.$$

These results follow from the identities (15). Hence, the transformation which sends (14) into (13) is

$$\begin{aligned} x' &= p\,(\beta\gamma x + \alpha\beta y + \alpha\gamma z), \\ y' &= q\,(\alpha\beta\varepsilon^6 x + \alpha\gamma\varepsilon^5 y + \beta\gamma\varepsilon^3 z), \\ z' &= r\,(\alpha\gamma\varepsilon^4 x + \beta\gamma\varepsilon y + \alpha\beta\varepsilon^2 z). \end{aligned}$$

The 2 remaining centers on $b\,c$ are

$$d\,(\alpha\gamma\varepsilon^4,\ \beta\gamma\varepsilon,\ \alpha\beta\varepsilon^2) \quad \text{and} \quad d'\,(\alpha\beta\varepsilon^6,\ \alpha\gamma\varepsilon^5,\ \beta\gamma\varepsilon^3),$$

and on $a\,c$

$$e\,(\beta\gamma\varepsilon^2,\ \alpha\beta\varepsilon^4,\ \alpha\gamma\varepsilon) \quad \text{and} \quad e'\,(\alpha\gamma\varepsilon^6,\ \beta\gamma\varepsilon^5,\ \alpha\beta\varepsilon^3).$$

Using the additional relations

$$\left.\begin{array}{c} -\mu = \varepsilon^6 + \varepsilon^5 + \varepsilon^3, \quad -\mu' = \varepsilon + \varepsilon^2 + \varepsilon^4, \\ \mu + \mu' = 1, \quad \mu\mu' = 2, \quad h\,(\mu - \mu') = 1, \quad \gamma = \alpha\beta - \gamma^2, \\ \alpha\varepsilon + \beta\varepsilon^4 + \gamma\varepsilon^2 = 1, \quad \alpha = \beta\gamma - \alpha^2, \\ \alpha\varepsilon^6 + \beta\varepsilon^3 + \gamma\varepsilon^5 = 1, \quad \beta = \alpha\gamma - \beta^2, \end{array}\right\} \tag{19}$$

it can be shown that d is transformed into

$$(0,\quad q\,h\,(3 + \mu),\quad r\,h\,(3 + \mu)).$$

Hence $q = r$, and similarly, by considering e, we find that $p = r$. The required transformation is therefore

$$\left.\begin{aligned} x' &= \beta\gamma x + \alpha\beta y + \alpha\gamma z, \\ y' &= \alpha\beta\varepsilon^6 x + \alpha\gamma\varepsilon^5 y + \beta\gamma\varepsilon^3 z, \\ z' &= \alpha\gamma\varepsilon^4 x + \beta\gamma\varepsilon y + \alpha\beta\varepsilon^2 z. \end{aligned}\right\} \tag{20}$$

Denote $\beta\gamma$, $\alpha\beta\varepsilon^6$, $\alpha\gamma\varepsilon^4$ by f, g, h. The collineation (15) sends

$$A\ (1, 0, 0), \quad B\ (0, 1, 0), \quad C\ (0, 0, 1)$$

into

$$D\ (\alpha, \beta, \gamma), \quad E\ (\beta, \gamma, \alpha), \quad C\ (\gamma, \alpha, \beta).$$

Repeated applications (18) send

D into $EFGHIJD$,
K into $LMNOPQK$,
R into $STUVWXR$.

Referred to the new coordinate system, we find the coordinates to be as in Case VI. The reason for the choice of letters is now apparent.

The transformation (18) sends $a(\beta\gamma, \alpha\beta, \alpha\gamma)$ into $(\beta\gamma\varepsilon, \alpha\beta\varepsilon^2, \alpha\gamma\varepsilon^4)$. Referred to the new coordinate system, it sends $a\,(1, 0, 0)$ into $(\mu, -1, 1)$, and similarly,

$b\ (0, 1, 0)$ into $(\mu, 1, -1)$, $\quad c\ (0, 0, 1)$ into $(0, 1, 1)$,
$d\ (0, 1, 1)$ into $(-1, \mu, 1)$, $\quad e\ (1, 0, 1)$ into $(1, \mu, 1)$.

These results follow from (20) by using the identities (15) and (19). Hence, proceeding as in the determination of (20), we find the transformation of period 7 to be

$$\left.\begin{aligned} x &= -2x' + (y' - z')\,\mu, \\ y &= 2x' + (y' - z')\,\mu, \\ z &= \mu^2 (y' + z'). \end{aligned}\right\} \tag{21}$$

Instead of transforming the conics directly by (21), it is more convenient to use the points of inflection themselves. Through A pass 3 conics

$$ADNSXOMP, \quad ADNSJTVW, \quad ADNSQHFI$$

of type (10). If we transform these by (21) and $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$, we find 63 conics that are repeated in the types (11), (12), (13) and their transforms. The 24 inflections may be arranged as the vertices of 8 inflectional triangles

$$ABC, \ DKR, \ SEL, \ TFM, \ UGN, \ VHO, \ WIP, \ XJQ.$$

Hence, the inflections lie in eights on 63 conics that pass through one vertex of each triangle.

Through A and B pass 2 conics $ADNSBEKU$, $AJNWBIQU$ of types (10) and (11). If we transform these by (21) and $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$, we have 42 conics.

If we transform these by $\binom{x\,y\,z}{x\,z\,y}$, we have in all 84 conics that are repeated in (10), (11), (12), (13). Hence, the inflections lie in eights on 84 conics that pass through 2 vertices of 4 triangles.

If we transform (4′), (5′), (6′), (7′), (8′) and (9′) by repeated application of (21), (6′) and (8′) are transformed into (1′), (4′) and (9′) into (2′), (5′) and (7′) into (3′).

The 4 conics common to (4′), (5′), (6′) are transformed into 28 conics common to (1′), (2′), (3′) that pass through the 6 vertices of 2 triangles and the points of contact of a bitangent. The 4 conics common to (4′), (8′), (9′) are similarly transformed into 28 conics common to (1′), (2′). Similarly for (5′), (7′), (9′) and (2′), (3′), also (6′), (7′), (8′) and (1′), (3′). In each case the conics pass through 1 vertex of 6 triangles and the points of contact of a bitangent.

This accounts for the transformations of 12 of the 112 conics through 6 inflections of types (1′), (2′), (3′). The remaining conics can be divided into 2 types. The first type pass through 2 vertices of 2 triangles and 1 vertex of 2 other triangles. Type (1′) has 60 of these conics. If we transform these by (21) and $\binom{x\,y\,z}{y\,z\,x}$, we find in all 1260 of these conics. The second type pass through 1 vertex of 6 triangles. Type (1′) has 40 of these conics, excluding the 8 already considered. Transforming these by (21) and $\binom{x\,y\,z}{y\,z\,x}$, we find 840 conics. Hence, the inflections lie in sixes on 4 sets of 28 conics, on 1 set of 1260 conics and on 1 set of 840 conics.

CORNELL UNIVERSITY, *November*, 1913.

Concerning an Analogy Between Formal Modular Invariants and the Class of Algebraical Invariants Called Booleans.

BY O. E. GLENN.

In the general theory of algebraic invariants and covariants the coefficients of the forms involved are considered to be arbitrary variables. Likewise the coefficients in the set of linear transformations to which the forms are subjected are arbitrary variable parameters.

Hurwitz published, in 1903, a paper* in which he defined a type of invariant essentially different from the ordinary algebraic type. The distinction consists in this: Whereas the coefficients of the forms are still arbitrary variables, the coefficients of the linear transformations involved are parameters representing integers belonging to the residue system modulo p, p being a prime number.

Professor L. E. Dickson has called my attention to the present state of this theory. No proof of the finiteness of the formal modular concomitants described above has been published. Hurwitz has emphasized the difficulty of this problem in the case of the invariants.

In a recent paper Miss Sanderson proves a fundamental relation between the formal modular type and *modular*† invariants and covariants‡ as defined and extensively investigated by Dickson. The latter author has recently contributed a general invariant theory, the methods of which apply both to the modular and to the algebraic types of invariant. He has made applications of this method to the formal modular case.

§ 1. *Extension of a Principle Due to Boole.*

It is the purpose of this paper to show how a principle due to Boole,§ applied by him to the problem of finding a fundamental system of invariants and covariants of a binary form under the restricted substitutions

* Hurwitz, "Ueber höhere Kongruenzen," *Archiv der Math. und Phys.*, Ser. 3, Vol. V (1903).

† Dickson, *Transactions Amer. Math. Society*, Vol. X (1909) and Vol. XIV (1913); AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI (1909); *Proceedings London Math. Society*, Ser. 2, Vol. VII (1909); etc. An elementary account of Dickson's modular invariant and covariant theory and of his general invariant theory will be found in his "Colloquium Lectures," delivered at the Madison Colloquium of the American Mathematical Society in September, 1913.

‡ Sanderson, *Transactions Amer. Math. Society*, Vol. XIV (1913).

§ Boole, *Cambridge Mathematical Journal*, Vol. III.

$$x_1 = x_1' \frac{\sin(\omega-\alpha)}{\sin\omega} + x_2' \frac{\sin(\omega-\beta)}{\sin\omega}, \quad x_2 = x_1' \frac{\sin\alpha}{\sin\omega} + x_2' \frac{\sin\beta}{\sin\omega}, \tag{1}$$

may be applied in the case of formal modular concomitants.

The essential features of the method of Boole result from the fact that the form $f = x_1^2 + 2x_1x_2 \cos\omega + x_2^2$ is a universal covariant* under the transformation (1). That is,

$$x_1'^2 + 2x_1'x_2' \cos\omega' + x_2'^2 = x_1^2 + 2x_1x_2 \cos\omega + x_2^2,$$

where ω is the inclination of the old axes, and $\omega' = \beta - \alpha$. Then the Boolean concomitants of a binary n-ic F_n are precisely the simultaneous concomitants of f and F_n.

Likewise, if we possess for the restricted substitutions consisting of the general binary linear transformations with coefficients reduced modulo p, a universal covariant C, then the members of the simultaneous system of C and F_n will be, as a rule,† formal modular concomitants of F_n.

A fundamental system of universal covariants of the general linear transformations in m variables, with coefficients belonging to a general finite field, has been discovered by Dickson and published in his memoir, "A Fundamental System of Invariants of the General Modular Linear Group, with a Solution of the Form Problem," *Transactions of the American Mathematical Society*, Vol. XII (1911). In view of the principle explained above we may, in the present case, employ these universal covariants in place of C, restricting the finite field to the field of integers modulo p; $m=2$. Results of this combination are given in §§ 3, 4, 5.

§ 2. *Defining Properties of Formal Concomitants; (a), (b), (c).*

The definitions formulated by Hurwitz relate to absolute formal modular invariants only. We define the covariants of a form as follows: Let F_n be a binary form of order n with arbitrary coefficients;

$$F_n = a_0 x_1^n + a_1 x_1^{n-1} x_2 + \dots + a_n x_2^n.$$

Let F_n, transformed by the general linear binary transformation with coefficients belonging to the field of integers modulo p, go over into

$$F_n' = a_0' x_1'^n + a_1' x_1'^{n-1} x_2' + \dots + a_n' x_2'^n.$$

Let $\rho \not\equiv 0 \pmod p$ be the modulus of the transformation. Then

(a) *Any function $F(a_0, a_1, \dots, a_n, x_1, x_2)$ of the coefficients and variables of F_n which identically satisfies the congruence*

$$F(a_0', a_1', \dots, a_n', x_1', x_2') \equiv \rho^k F(a_0, a_1, \dots, a_n, x_1, x_2) \pmod p,$$

* Study, *Leipz. Ber.*, XL (1897).

† Compare § 5.

will be called a formal modular covariant of F_n. It will be a relative covariant except when $\rho^k \equiv 1 \pmod{p}$, when it will be an absolute covariant.

Without loss of generality we may assume that F is homogeneous both in the variables and in the coefficients. If its order is zero, it is an invariant.

(b) *The weight w of a term of F, and the index k, order ω, and the degree i of F satisfy the following congruences:*

$$\left.\begin{aligned} in-\omega &\equiv 2k \\ in+\omega &\equiv 2w \end{aligned}\right\} \pmod{p-1}. \tag{2}$$

For under the particular substitution $x_1 = \lambda x_1'$, $x_2 = \lambda x_2'$ ($\lambda =$ a primitive root) of determinant λ^2, we have $a_r' = \lambda^n a_r$, $x_i' = \lambda^{-1} x_i$, from which the first congruence readily follows. To prove the second, take $x_1 = x_1'$, $x_2 = \lambda x_2'$, whence $a_r' = \lambda^r a_r$, $x_2' = \lambda^{-1} x_2$. Let a typical term of F be

$$T = a_r^\rho\, a_s^\sigma\, a_t^\tau \ldots . x_1^\nu\, x_2^{\omega-\nu}.$$

Then,

$$T' = \lambda^{r\rho + s\sigma + t\tau + \ldots . + \nu - \omega}\, T \equiv \lambda^k\, T \pmod{p}.$$

Hence,

$$2(w-\omega) \equiv in - \omega \pmod{p-1},$$

which proves the second congruence.

(c) *The annihilators δ of a formal invariant $\phi(a)$ of degree i are partial differential operators of order i.* In fact, if F_n is transformed into F_n' by the particular substitution $S: x_1 = x_1' + t x_2'$, $x_2 = x_2'$ (t a residue mod p), then $\phi(a')$ may be expanded by Taylor's theorem;* and after powers of t are reduced by Fermat's theorem (mod p), the result takes the form†

$$\phi(a') - \phi(a) \equiv \delta'\phi(a)\, t + \frac{\delta'^2}{\underline{|2}}\phi(a)\, t^2 + \ldots . + \frac{\delta'^{p-1}\phi(a)}{\underline{|p-1}}\, t^{p-1} \pmod{p}.$$

Here δ' is an operator of order $> i$. Then a necessary and sufficient condition for the invariancy of ϕ under S is given by $\delta'\phi(a) = 0$. If in δ' we delete all partial derivatives of orders $> i$, we obtain an annihilator δ as described above. We give below the explicit form of δ for $n=2$, $i=4$, $p=3$.

$$\begin{aligned} \delta\phi(a) \equiv\; & 2a_0\,\phi_{a_1} + a_1\,\phi_{a_2} + 2a_0^2\,\phi_{a_1 a_2} + a_0 a_1\,\phi_{a_2^2} + \frac{4}{3}a_0^3\,\phi_{a_1^3} + 2a_0^2 a_1\,\phi_{a_1^2 a_2} \\ & + (a_0^3 + a_0 a_1^2)\,\phi_{a_1 a_2^2} + \left(\frac{1}{6}a_1^3 + \frac{1}{2}a_0^2 a_1\right)\phi_{a_2^3} + \frac{4}{3}a_0^4\,\phi_{a_1^3 a_2} + 2a_0^3 a_1\,\phi_{a_1^2 a_2^2} \\ & + a_0^2 a_1^2\,\phi_{a_1 a_2^3} + \frac{1}{3}a_0^4\,\phi_{a_1 a_2^3} + \frac{1}{6}(a_0 a_1^3 + a_0^3 a_1)\,\phi_{a_2^4}. \end{aligned} \tag{3}$$

In this operator $\phi_{a_1 a_2^2} = \dfrac{\partial^3\phi}{\partial a_1 \partial a_2^2}$, etc.

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*The method is due to Dickson. who developed it for modular invariants in *Transactions Amer. Math. Society*, Vol. VIII (1907), p. 209. The essential point of difference is that in the present case there is no greatest value for i, and δ is more complicated.

† Cf. *Transactions Amer. Math. Society*, Vol. XV, p. 72, lemma in § 1.

For illustration, we determine all invariants of F_2, of degree 4, modulo 3. These will be determined again by the method of § 1 (Cf. § 5, Table II). Let

$$\phi(a)=A^{(0)}_1 a_0^4+A^{(2)}_1 a_0^3 a_2+A^{(2)}_2 a_0^2 a_1^2+A^{(4)}_1 a_0^2 a_2^2+A^{(4)}_2 a_0 a_1^2 a_2 \\ +A^{(4)}_3 a_1^4+A^{(6)}_1 a_0 a_2^3+A^{(6)}_2 a_1^2 a_2^2+A^{(8)}_1 a_2^4.$$

By formula (2) this is the general form of $\phi(a)$. Operating by δ we obtain

$$\left.\begin{array}{c} A^{(4)}_2-A^{(4)}_3+A^{(6)}_1\equiv 0; \quad A^{(4)}_1\equiv A^{(4)}_2; \quad A^{(6)}_2\equiv A^{(8)}_1\equiv 0; \\ A^{(2)}_1+A^{(2)}_2+A^{(4)}_1-A^{(4)}_3\equiv 0 \end{array}\right\}(\text{mod } 3).$$

Also, since $\phi(a)$ must remain unaltered by the substitution $(a_0 a_2)(-a_1 a_1)$, we have

$$A^{(0)}_1\equiv A^{(2)}_2\equiv 0; \quad A^{(2)}_1\equiv A^{(6)}_1 \ (\text{mod } 3).$$

Hence,

$$\phi(a)\equiv A^{(4)}_1(a_0^2 a_2^2+a_0 a_1^2 a_2-a_0^3 a_2-a_0 a_2^3)+A^{(4)}_3(a_0^3 a_2+a_1^4+a_0 a_2^3) \\ \equiv A^{(4)}_1 J+A^{(4)}_3 I.$$

Two linearly independent formal modular invariants of degree 4 are thus I and J. One of these is reducible. In fact,

$$I+J\equiv D^2 (\text{mod } 3),$$

where D is the discriminant of F_2.

§ 3. *Universal Covariants in Two Variables.*

The universal covariants forming a fundamental system for the general binary linear group, modulo p, are two in number. They are (see § 1)

$$L=x_1^p x_2-x_1 x_2^p, \quad Q=(x_1^{p^2} x_2-x_1 x_2^{p^2})\div L.$$

We now prove a theorem concerning Q. Let $J_1(f\phi)$ be the functional determinant of f and ϕ. Let $J_2(f\phi)$ be the functional determinant of J_1 and ϕ, or the second iteration of the functional determinant of f and ϕ. In ordinary notation for transvectants,

$$J_2=((f\phi)\phi).$$

In general, let J_k be the k-th iteration; that is,

$$J_k=(\ldots\ldots(f\overbrace{\phi)\phi)\ldots\ldots\phi}^{k}).$$

Then we have

THEOREM: *The covariant Q is a modular covariant of L, and is precisely the $(p-2)$-th iteration of the functional determinant of the Hessian of L and L itself, modulo p.*

In proof, let $H=(LL)^2$. Then,

$$J_1(HL)\equiv -2x_1^{3(p-1)}+x_1^{2(p-1)}x_2^{(p-1)}+x_1^{p-1}x_2^{2(p-1)}-2x_2^{3(p-1)} (\text{mod } p).$$

Proceeding by induction, assume

$$J_{k-2}(HL)\equiv a^{(k)}_0 x_1^{k(p-1)}+a^{(k)}_1 x_1^{(k-1)(p-1)}x_2^{p-1}+\ldots\ldots+a^{(k)}_k x_2^{k(p-1)} (\text{mod } p).$$

Then,

$$+J_{k-1}(HL)\equiv\begin{vmatrix}\dfrac{\partial J_{k-2}}{\partial x_1} & x_2^p \\ \dfrac{\partial J_{k-2}}{\partial x_2} & -x_1^p\end{vmatrix}\equiv a_0^{(k+1)}x_1^{(k+1)(p-1)}+a_1^{(k+1)}x_1^{k(p-1)}x_2^{p-1}+\ldots\ldots+a_{k+1}^{(k+1)}x_2^{(k+1)(p-1)}\,(\text{mod } p). \quad (4)$$

From this congruence we obtain the recursion formula

$$a_h^{(k+1)}\equiv(k-h)a_h^{(k)}+(h-1)a_{h-1}^{(k)}\,(\text{mod } p), \quad (h=0, 1, \ldots, k+1). \quad (5)$$

Let us now define two new a numbers as follows:

$$a_{-1}^{(k)}\equiv 0; \quad a_{k+1}^{(k)}\equiv 0 \ (\text{mod } p).$$

Then we obtain readily from (5)

$$\left.\begin{array}{l} a_0^{(k+1)}\equiv -\underline{|k}, \quad a_{k+1}^{(k+1)}\equiv -\underline{|k} \\ a_h^{(k+1)}\equiv +\underline{|k-1} \quad (h=1, \ldots, k)\end{array}\right\}(\text{mod } p).$$

Assume $k=p-1$. Then by virtue of Wilson's theorem

$$J_{p-2}(HL)\equiv x_1^{p(p-1)}+x_1^{(p-1)(p-1)}x_2^{p-1}+\ldots\ldots+x_2^{p(p-1)}\equiv Q\,(\text{mod } p),$$

which was to be proved.

Since, then, Q is a covariant of L, it suffices to employ L for C (§ 1) in obtaining by the method of § 1 the concomitants of F_n modulo p.

§ 4. *Concomitants of F_3 Modulo* 2.

Let F_3 be the general binary cubic form

$$F_3=a_0x_1^3+3a_1x_1^2x_2+3a_2x_1x_2^2+a_3x_2^3,$$

and let $p=2$. Then the fundamental system* of F_3 and L, taken from the algebraical standpoint, is a closed set, and the system of irreducible concomitants obtained from this set by deleting the members which are reducible, modulo 2, constitutes a formal system for F_3 modulo 2. No proof that this is a fundamental system is given in this paper, however. These irreducible concomitants† are given in Table I. All transvectants in the simultaneous system of F_3 and L not given in the table prove to be reducible. For instance,

$$(HQ)\equiv H+R^{(2)}Q, \quad (Q(F_3L)^2)\equiv C_2^{(1)}+R^{(1)}Q, \qquad (\text{mod } 2).$$

§ 5. *Concomitants of F_2 Modulo* 3.

We give in Table II similar results for the case $p=3$, $n=2$. The mere process of transvection here sometimes fails to give a formal concomitant (mod 3), owing primarily to the fact that some of the numerical coefficients involved are divisible by the modulus. No proof that the system given by the method is or is not coextensive with the totality of formal concomitants of $f(=F_2)$ is here attempted. It is, however, worthy of note that we obtain by

* Faa di Bruno, Walter, "Theorie der binären Formen," p. 209.

† With each transvectant has been introduced the smallest numerical factor which has the effect of removing extraneous numerical factors when the concomitant is expressed in terms of the actual coefficients.

transvection between f and Q alone a complete set of concomitants, which, when reduced by Fermat's theorem, modulo 3, give the fundamental system of *modular* concomitants of f.* These are (see Table II) q, Δ, f, L, Q, C_1, C_2, f_4.

TABLE I.

Notation	Transvectant	Concomitant (mod 2)
$R^{(1)}$	$(F_3L)^3$	a_1+a_2
$R^{(2)}$	$(HQ)^2$	$a_0a_3+a_1a_2$
$R^{(3)}$		$a_0(a_0+a_1+a_2+a_3)a_3$
L		$x_1^2x_2+x_1x_2^2$
Q	$(LL)^2$	$x_1^2+x_1x_2+x_2^2$
F_3		$a_0x_1^3+a_1x_1^2x_2+a_2x_1x_2^2+a_3x_2^3$
H	$(F_3F_3)^2$	$(a_0a_2+a_1^2)x_1^2+(a_0a_3+a_1a_2)x_1x_2+(a_1a_3+a_2^2)x_2^2$
$C_1^{(1)}$	$(QF_3)^2$	$(a_0+a_1+a_2)x_1+(a_1+a_2+a_3)x_2$
$C_2^{(1)}$	$(F_3L)^2$	$a_0x_1^2+(a_1+a_2)x_1x_2+a_3x_2^2$
$C_4^{(1)}$	(F_3L)	$a_0x_1^4+(a_1+a_2)x_1^2x_2^2+a_3x_2^4$
$C_1^{(2)}$	$(HL)^2$	$(a_0a_3+a_1a_2+a_0a_2+a_1^2)x_1+(a_0a_3+a_1a_2+a_1a_3+a_2^2)x_2$
$C_2^{(3)}$	$(H(F_3L)^2)$	$(a_0^2a_3+a_0a_2^2+a_1^3+a_1^2a_2)x_1^2+(a_0a_3^2+a_1^2a_3+a_1a_2^2+a_2^3)x_2^2$

TABLE II.

Notation	Transvectant	Concomitant (mod 3)
Δ	$(ff)^2$	$a_1^2-a_0a_2$
q	$(f^3Q)^6$	$a_0^2a_2+a_0a_2^2+a_0a_1^2+a_1^2a_2-a_0^3-a_2^3$
I	$(f^4L^2)^8$	$a_0^3a_2+a_0a_2^3+a_1^4$
L		$x_1^3x_2-x_1x_2^3$
Q	$((LL)^2L)$	$x_1^6+x_1^4x_2^2+x_1^2x_2^4+x_2^6$
f		$a_0x_1^2+2a_1x_1x_2+a_2x_2^2$
f_4	$(fQ)^2$	$a_0x_1^4+a_1x_1^3x_2+a_1x_1x_2^3+a_2x_2^4$
C_1	$(f^3Q)^5$	$(a_0^2a_1-a_1^3)x_1^2+(a_0-a_2)(a_1^2+a_0a_2)x_1x_2+(a_1^3-a_1a_2^2)x_2^2$
C_2	$(f^2Q)^4$	$(a_0^2+a_1^2-a_0a_2)x_1^2+a_1(a_0+a_2)x_1x_2+(a_1^2+a_2^2-a_0a_2)x_2^2$
C_3	$(fL^2)^2$	$a_0x_1^6+2a_1x_1^3x_2^3+a_2x_2^6$
C_4	$(f^2L^2)^4$	$(a_0^2+a_1^2-a_0a_2)x_1^4+2a_1(a_0+a_2)(x_1^3x_2+x_1x_2^3)+(a_1^2+a_2^2-a_0a_2)x_2^4$
C_5	$(f^3L^2)^6$	$a_0^3x_1^2+2a_1^3x_1x_2+a_2^3x_2^2$
C_6	$(f^4L^2)^7$	$(a_0a_1^3-a_0^3a_1)x_1^2+(a_0a_2^3-a_0^3a_2)x_1x_2+(a_1a_2^3-a_1^3a_2)x_2^2$

UNIVERSITY OF PENNSYLVANIA, *October*, 1913.

* Dickson, *Transactions Amer. Math. Society*, Vol. XIV (1913), p. 310.

Periodic Orbits on a Smooth Surface.

By Daniel Buchanan.

§ 1. *Introduction.*

In the third chapter of Moulton's "Periodic Orbits," which will appear shortly, the general solutions of the problem of the spherical pendulum are determined as power series in a parameter in which the coefficients are transcendental functions of the time. The solution obtained for the vertical motion is periodic, but the solutions for the horizontal are not periodic in general. The problem discussed in this paper is a generalization of the problem of the spherical pendulum. The path described by the bob of the pendulum may be considered as the orbit described by a particle which moves, subject to gravity, on the surface of a smooth sphere whose radius is equal to the length of the pendulum and whose center is at the point of suspension. The object of this paper is to show the existence and to give a method for the determination of periodic orbits described by a particle which moves under similar conditions on a smooth surface of more general character. The orbits determined have the same period as the solution for the vertical motion of the spherical pendulum. The equation of the surface is the same as that used by Poincaré in his memoir "Sur les Lignes Géodésiques des Surfaces Convexes," *Trans. Am. Math. Soc.*, Vol. VI (July, 1905). Periodic orbits play the same rôle in the present paper as closed geodesic lines in Poincaré's memoir.

§ 2. *The Differential Equations.*

Let us take a system of rectangular axes with the positive z-axis directed upwards. We shall denote the surface upon which the particle is constrained to move by the equation

$$F(x, y, z) \equiv x^2 + y^2 + z^2 - l^2 + 2\varepsilon f(x, y, z) = 0, \tag{1}$$

where ε is a parameter and $f(x, y, z)$ is a power series of the form

$$f(x, y, z) = \sum_{i,j,k=0}^{\infty} f_{ijk}\, x^i y^j z^k, \tag{2}$$

the f_{ijk} being constants, and it converges for $|x|$, $|y|$ and $|z|$ sufficiently small.

If (2) is a polynomial, then no restrictions need be placed on x, y and z, except that they shall be finite. For $\varepsilon = 0$ the equation (1) reduces to the equation of a sphere with center at the origin and radius l.

Let us choose the unit of mass so that the mass of the particle is unity. Then, denoting derivatives with respect to the time by accents, we obtain as the differential equations of motion for the particle,

$$x'' = X, \quad y'' = Y, \quad z'' = Z - g, \tag{3}$$

where X, Y, Z are the normal reactions due to the surface. Since the surface is assumed to be smooth, the normal reactions at any point are proportional to the direction cosines of the normal at that point. Hence,

$$\left.\begin{aligned} X &= \lambda F_x = 2\lambda\,[x + \varepsilon f_x], \\ Y &= \lambda F_y = 2\lambda\,[y + \varepsilon f_y], \\ Z &= \lambda F_z = 2\lambda\,[z + \varepsilon f_z], \end{aligned}\right\} \tag{4}$$

where λ is a factor of proportionality. When equations (4) are substituted in (3), the differential equations become

$$x'' = 2\lambda\,[x + \varepsilon f_x], \quad y'' = 2\lambda\,[y + \varepsilon f_y], \quad z'' = 2\lambda\,[z + \varepsilon f_z] - g. \tag{5}$$

These equations admit the integral

$$x'^2 + y'^2 + z'^2 = g\,(-2z + c_1), \tag{6}$$

where c_1 is the constant of integration.

In order to determine λ, we find the second derivative of the equation of constraint with respect to t and eliminate x'', y'', z'', and $x'^2 + y'^2 + z'^2$ by (5) and (6). Since $F(x, y, z)$ is independent of t, we have $F'' = 0$, and therefore, when the differentiations and eliminations are made, we obtain

$$2\lambda = \frac{g\,(3z - c_1) - \varepsilon\,[x'^2 f_{xx} + y'^2 f_{yy} + z'^2 f_{zz} + 2y'z' f_{yz} + 2x'z' f_{xz} + 2x'y' f_{xy} - g f_z]}{l^2 + 2\varepsilon\,[x f_x + y f_y + z f_z - f] + \varepsilon^2\,[f_x^2 + f_y^2 + f_z^2]}. \tag{7}$$

After the values of the partial derivatives are obtained from (2) and substituted in (7), the expression for 2λ can be arranged as a power series of the form

$$2\lambda = \frac{g}{l^2}\,(3z - c_1) + \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \ldots, \tag{8}$$

in which the coefficients λ_j are power series in x, y, z and contain x', y', z' to the second degree considered together. If $f(x, y, z)$ is a polynomial, then the convergence of (8) can be controlled by ε alone, provided x, y and z are finite. If $f(x, y, z)$ is a power series, then x, y and z must lie in the region for which (2) converges. Upon substituting (8) in (5), the differential equations take the form

$$\left.\begin{aligned} x'' &= \frac{g}{l^2}(3z - c_1)\,x + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots, \\ y'' &= \frac{g}{l^2}(3z - c_1)\,y + \varepsilon Y_1 + \varepsilon^2 Y_2 + \ldots, \\ z'' &= \frac{g}{l^2}(3z - c_1)\,z - g + \varepsilon Z_1 + \varepsilon^2 Z_2 + \ldots, \end{aligned}\right\} \tag{9}$$

where X_j, Y_j, Z_j are similar in form to the λ_j in (8).

§ 3. *The Spherical Pendulum.*

For $\varepsilon = 0$ the equations (9) reduce to the differential equations of the spherical pendulum, which are

$$x'' = \frac{g}{l^2}(3z - c_1)\,x, \quad y'' = \frac{g}{l^2}(3z - c_1)\,y, \quad z'' = \frac{g}{l^2}(3z - c_1)\,z - g. \tag{10}$$

The last equation of (10) is independent of the other two and is solved first. It admits the integral

$$\begin{aligned} z'^2 &= \frac{g}{l^2}(2z - c_1)\,z^2 - g\,(2z - c_2) \\ &= \frac{2g}{l^2}(z - \alpha_1)(z - \alpha_2)(z - \alpha_3), \qquad (\alpha_1 \geqq \alpha_2 \geqq \alpha_3), \end{aligned} \tag{11}$$

where c_2 is the constant of integration. The periodic solution of the last equation of (10) has been obtained in Professor Moulton's memoir, to which reference has already been made. The solution is

$$z = \psi(\tau) = \alpha_3 + (\alpha_1 - \alpha_3)\left[\frac{1}{2}(1 - \cos 2\tau)\,\mu + \frac{1}{32}(1 - \cos 4\tau)\,\mu^2 + \ldots\right], \tag{12}$$

where

$$\left.\begin{aligned} \tau &= \sqrt{\frac{g(\alpha_1 - \alpha_3)}{2l^2(1+\delta)}}\,(t - t_0), \\ \delta &= \frac{1}{2}\mu + \frac{11}{32}\mu^2 + \ldots, \\ \mu &= \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}, \qquad (0 \leqq \mu \leqq 1). \end{aligned}\right\} \tag{13}$$

In the physical problem of the spherical pendulum, excluding the case of revolution in the xy-plane with infinite speed and that of the simple pendulum, the three constants α_1, α_2, α_3 satisfy the inequalities

$$-l < \alpha_3 < 0, \quad -l < \alpha_2 < +l, \quad \alpha_1 > +l.$$

On comparing the equations in (11), it is seen that

$$2(\alpha_1 + \alpha_2 + \alpha_3) = c_1, \quad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = -l^2, \quad 2\alpha_1\alpha_2\alpha_3 = -c_2 l^2. \tag{14}$$

§ 4. *The Solutions of the Equations of Variation.*

Let
$$z = \psi + w, \tag{15}$$
where ψ is the periodic function defined in (12). The function w is undetermined except that it vanishes with ε. Now let (15) and (13) be substituted in (9). If derivatives with respect to τ are denoted by dots, then the differential equations (9) become

$$\left.\begin{aligned}
\ddot{x} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots\dots]\,x &= \frac{6\,(1+\delta)\,xw}{\alpha_1 - \alpha_3} + \varepsilon\,\bar{X}_1 + \varepsilon^2\,\bar{X}_2 + \dots\dots,\\
\ddot{y} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots\dots]\,y &= \frac{6\,(1+\delta)\,yw}{\alpha_1 - \alpha_3} + \varepsilon\,\bar{Y}_1 + \varepsilon^2\,\bar{Y}_2 + \dots\dots,\\
\ddot{w} + [b^2 + \theta_1^{(2)}\mu + \theta_2^{(2)}\mu^2 + \dots\dots]\,w &= \frac{6\,(1+\delta)\,w^2}{\alpha_1 - \alpha_3} + \varepsilon\,W_1 + \varepsilon^2\,W_2 + \dots\dots,
\end{aligned}\right\} \tag{16}$$

where a^2 and b^2 are constants independent of μ; and $\theta_j^{(i)}\,(i=1,2;\ j=1,\dots\dots,\infty)$ are sums of cosines of even multiples of τ and are therefore periodic with the period 2π. The $\bar{X}_j$, $\bar{Y}_j$, W_j appearing in the right members of (16) are power series in x, y, w in which the coefficients are similar in form to $\psi(\tau)$ except that they contain additional terms in $\dot{x}$, $\dot{y}$, $\dot{w}$, $\dot{\psi}$ to the second degree considered together.

By putting $\varepsilon = 0$ in (16), and taking only the linear terms in x, y, and w, we obtain the equations of variation, which are

$$\left.\begin{aligned}
\ddot{x} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots\dots]\,x &= 0,\\
\ddot{y} + [a^2 + \theta_1^{(1)}\mu + \theta_2^{(1)}\mu^2 + \dots\dots]\,y &= 0,\\
\ddot{w} + [b^2 + \theta_1^{(2)}\mu + \theta_2^{(2)}\mu^2 + \dots\dots]\,w &= 0.
\end{aligned}\right\} \tag{17}$$

Obviously the solutions of the first two equations of (17) can differ only in the arbitrary constants. The solutions of these two equations have been obtained in Professor Moulton's memoir, where it is shown that there are three forms of the solutions according to the values of a:

CASE I. $a \neq 0$ and $2a$ not an integer.
CASE II. $a \neq 0$ and $2a$ an integer.
CASE III. $a = 0$.

CASE I. This is considered as the general case. The solutions in this case have the form

$$\left.\begin{aligned}
x &= A_1 e^{\alpha\sqrt{-1}\,\tau}\xi_1 + A_2 e^{-\alpha\sqrt{-1}\,\tau}\xi_2,\\
y &= A_3 e^{\alpha\sqrt{-1}\,\tau}\xi_1 + A_4 e^{-\alpha\sqrt{-1}\,\tau}\xi_2,
\end{aligned}\right\} \tag{18}$$

where A_i $(i = 1, \dots\dots, 4)$ are the constants of integration; α is a power series

in μ with constant coefficients; and ξ_1, ξ_2 are power series in μ in which the coefficients have the form

$$\xi_1 = \sum_j [a_j \cos 2\, j\tau + \sqrt{-1}\, b_j \sin 2\, j\tau],$$
$$\xi_2 = \sum_j [a_j \cos 2\, j\tau - \sqrt{-1}\, b_j \sin 2\, j\tau],$$

a_j and b_j denoting real constants. These solutions have the additional property

$$\xi_1(0) = \xi_2(0) = 1.$$

CASE II. In this case the solutions are similar in form to (18), but they contain, in addition, terms in $\cos 2\, a\, j\tau$ and $\sqrt{-1} \sin 2\, a\, j\tau$. The constants of integration are determined so that

$$\dot{\xi}_1(0) = \dot{\xi}_2(0) = 1.$$

CASE III. For $a = 0$ the solutions have the same form as (18) except that α, ξ_1, ξ_2 are power series in $\sqrt{\mu}$ instead of μ.

Unless otherwise stated we shall suppose that we are dealing with Case I.

We shall now derive the solutions of the last equation of (17). Since the differential equations (9) do not contain t explicitly, it is known, *à priori,** that two of the characteristic exponents which belong to the solutions of equations (17) are zero. These two zero exponents belong to the solution of the last equation of (17). The generating solution of this equation is $\psi(\tau)$, defined in (12), and it contains two arbitrary constants, viz., the initial time t_0, and the scale factor l, the latter occurring only in the second degree. Hence, the two solutions of the last equation of (17) are†

$$w_1 = \frac{\partial}{\partial t_0}\psi\left\{\sqrt{\frac{g\,(\alpha_1 - \alpha_3)}{2\, l^2(1+\delta)}}\,(t - t_0)\right\} = -\frac{\partial}{\partial \tau}\psi(\tau)$$
$$= \mu\,(\alpha_3 - \alpha_1)\left[\sin 2\,\tau + \frac{1}{8}\mu \sin 4\,\tau + \ldots.\right],$$
$$w_2 = \frac{\partial \psi}{\partial l^2}.$$

Since each of these solutions is multiplied later by an arbitrary constant, the factor $\mu\,(\alpha_3 - \alpha_1)$ may be absorbed by the arbitrary constant, and the first solution becomes

$$\bar{w}_1 = \phi = \sin 2\tau + \frac{1}{8}\mu \sin 4\tau + \ldots.. \tag{19}$$

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* Moulton, "Periodic Orbits," § 33. Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol. I, Chap. 4.

† Moulton, *loc. cit.*, § 32. Poincaré, *loc. cit.*, Vol. I, Chap. 4.

It is observed from equations (12) and (13) that ψ is a function of α_1, α_3, μ and τ, while τ is a function of α_1, α_3, δ and l^2, and δ is a function of μ. Now when α_2 is eliminated from (14) by the substitution

$$\alpha_2 = \alpha_3 + \mu\,(\alpha_1 - \alpha_3),$$

the first equation of (14) expresses μ as a function of α_1 and α_3, and the other two equations express α_1 and α_3 as functions of l^2. The constants c_1 and c_2 enter into these relations, but as they are constants of integration, they are independent of the scale factor. Hence,

$$\begin{aligned}\frac{\partial\psi}{\partial l^2} = \left(\frac{\partial\psi}{\partial\alpha_1}\right)\frac{\partial\alpha_1}{\partial l^2} + \left(\frac{\partial\psi}{\partial\alpha_3}\right)\frac{\partial\alpha_3}{\partial l^2} + \left(\frac{\partial\psi}{\partial\mu}\right)\left[\frac{\partial\mu}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial l^2} + \frac{\partial\mu}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial l^2}\right]\\ + \frac{\partial\psi}{\partial\tau}\left[\left(\frac{\partial\tau}{\partial l^2}\right) + \frac{\partial\tau}{\partial\delta}\frac{\partial\delta}{\partial\mu}\left\{\frac{\partial\mu}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial l^2} + \frac{\partial\mu}{\partial\alpha_3}\frac{\partial\alpha_3}{\partial l^2}\right\} + \left(\frac{\partial\tau}{\partial\alpha_1}\right)\frac{\partial\alpha_1}{\partial l^2} + \left(\frac{\partial\tau}{\partial\alpha_3}\right)\frac{\partial\alpha_3}{\partial l^2}\right], \qquad (20)\end{aligned}$$

where the parentheses ( ) denote that the differentiation is performed only in so far as the variable enters the function explicitly. Upon performing the differentiations expressed in (20), we obtain

$$\begin{aligned}&\left(\frac{\partial\psi}{\partial\alpha_1}\right) = 1 - \left(\frac{\partial\psi}{\partial\alpha_3}\right) = \frac{1}{2}(1-\cos 2\tau)\,\mu + \frac{1}{32}(1-\cos 4\tau)\,\mu^2 + \ldots\ldots,\\ &\frac{\partial\alpha_1}{\partial l^2} = \frac{1}{(\alpha_1-\alpha_3)(1-\mu)}, \quad \frac{\partial\alpha_3}{\partial l^2} = \frac{1}{\mu\,(\alpha_3-\alpha_1)} = \frac{2}{2\,\alpha_1 + 4\,\alpha_3 - c_1},\\ &\frac{\partial\mu}{\partial\alpha_1} = \frac{1+\mu}{\alpha_3-\alpha_1}, \quad \frac{\partial\mu}{\partial\alpha_3} = \frac{2-\mu}{\alpha_3-\alpha_1}, \quad \frac{\partial\psi}{\partial\tau} = \mu\,(\alpha_1-\alpha_3)\,\phi,\\ &\left(\frac{\partial\psi}{\partial\mu}\right) = (\alpha_1-\alpha_3)\left[\frac{1}{2}(1-\cos 2\tau) + \frac{1}{16}(1-\cos 4\tau)\,\mu + \ldots\ldots\right],\\ &\frac{\partial\tau}{\partial\delta} = -\frac{\tau}{2\,(1+\delta)}, \quad \frac{\partial\delta}{\partial\mu} = \frac{1}{2} + \frac{11}{16}\mu + \ldots\ldots,\\ &\left(\frac{\partial\tau}{\partial l^2}\right) = -\frac{\tau}{2\,l^2}, \quad \left(\frac{\partial\tau}{\partial\alpha_1}\right) = -\left(\frac{\partial\tau}{\partial\alpha_3}\right) = \frac{\tau}{2\,(\alpha_1-\alpha_3)}.\end{aligned}$$

When these results are substituted in (20), we get for the second solution

$$w_2 = \chi + A\,\tau\,\phi, \qquad (21)$$

where χ and A are power series in μ of the form

$$\begin{aligned}\chi = \frac{1}{2(\alpha_3-\alpha_1)(2\alpha_1+4\alpha_3-c_1)}\,[2\,\alpha_1 + 4\,\alpha_3 - c_1 - (6\,\alpha_1 - c_1)\cos 2\tau\\ + \frac{\mu}{8}\{22\alpha_1 + 32\alpha_3 - 9\,c_1 - 8\,(2\,\alpha_1 + 4\,\alpha_3 - c_1)\cos 2\tau - (6\,\alpha_1 - c_1)\cos 4\tau\} + \ldots\ldots],\end{aligned}$$

$$A = \frac{8\,(\alpha_3-\alpha_1)^2 - 9\,l^2}{16\,l^2(\alpha_3-\alpha_1)}\,\mu + \ldots\ldots$$

The determinant of the two solutions (19) and (21) at $\tau = 0$ is

$$D = -\dot{\phi}(0)\,\chi(0) = -\frac{4}{2\,\alpha_1 + 4\,\alpha_3 - c_1} + \text{terms in } \mu, \tag{22}$$

which is not zero for μ sufficiently small unless α_1, α_3 or c_1 is infinite. Now α_3 is finite because of the inequality $-l < \alpha_3 < 0$. It is shown in Professor Moulton's memoir that if c_1 is infinite, the particle revolves in the xy-plane with infinite speed, and, of course, this case can not be realized physically. Then c_1 is finite, and since $-l < \alpha_3 < 0$, $-l < \alpha_2 < +l$, it follows from the first equation of (14) that α_1 also is finite. Hence, $D \neq 0$ for μ sufficiently small, and the two solutions (19) and (21) constitute a fundamental set. The general solution of the last equation of (17) is therefore

$$w = A_5\phi + A_6\,[\chi + A\,\tau\,\phi], \tag{23}$$

where A_5, A_6 are the arbitrary constants. The functions ϕ and χ are periodic in τ with the period 2π.

§ 5. *Existence of Periodic Solutions.*

Let us take the initial conditions at $\tau = 0$,

$$x = a_1, \quad \dot{x} = a_2, \quad y = a_3, \quad \dot{y} = a_4, \quad w = a_5, \quad \dot{w} = a_6. \tag{24}$$

If the surface of constraint is a closed surface, then the vertical velocity of the particle must vanish for some value of t and change sign. If the surface is not closed and if the motion of the particle is to be periodic, then x', y', z' must vanish and change signs for some values of t, otherwise the particle would recede to infinity. Then without loss of generality t_0 can be chosen as the time when the vertical velocity is zero. Hence, $\dot{z} = 0$ at $\tau = 0$; and since $\dot{\psi}(0) = 0$, it follows from (15) that $\dot{w}(0) = 0$. Therefore, a_6 in (24) may be put equal to zero.

In order to prove the existence of periodic solutions of equations (16), we integrate (16) as power series in $a_i\,(i = 1, \ldots, 5)$ and ε, and impose necessary conditions that x, y and w shall be periodic in τ with the period 2π. Only the linear terms in a_i are required in explicit form. The solutions of (16), subject to the initial conditions (24), are

$$\left.\begin{aligned} x &= \left(\frac{a_1}{2} - \frac{a_2}{\Delta}\right) e^{a\sqrt{-1}\,\tau}\,\xi_1 + \left(\frac{a_1}{2} + \frac{a_2}{\Delta}\right) e^{-a\sqrt{-1}\,\tau}\,\xi_2 + \varepsilon P_1(a_i, \varepsilon;\ \tau), \\ y &= \left(\frac{a_3}{2} - \frac{a_4}{\Delta}\right) e^{a\sqrt{-1}\,\tau}\,\xi_1 + \left(\frac{a_3}{2} + \frac{a_4}{\Delta}\right) e^{-a\sqrt{-1}\,\tau}\,\xi_2 + \varepsilon P_2(a_i, \varepsilon;\ \tau), \\ w &= \frac{a_5}{\chi(0)}\,[\chi + A\,\tau\,\phi] + \varepsilon P_3(a_i, \varepsilon;\ \tau), \end{aligned}\right\} \tag{25}$$

where Δ is the determinant of the fundamental set of solutions (18) and is therefore different from zero. The terms P_1, P_2 and P_3 are power series in a_i and ε, and carry ε as a factor since the right members of (16) vanish with ε.

Sufficient conditions that x, y, w in (25) shall be periodic in τ with the period 2π are

$$\left.\begin{array}{lll} x(2\pi)-x(0)=0, & y(2\pi)-y(0)=0, & w(2\pi)-w(0)=0, \\ \dot{x}(2\pi)-\dot{x}(0)=0, & \dot{y}(2\pi)-\dot{y}(0)=0, & \dot{w}(2\pi)-\dot{w}(0)=0. \end{array}\right\} \tag{26}$$

These six conditions are not independent, as we shall show, and the condition $w(2\pi)-w(0)=0$ is a consequence of the other five conditions. In order to show this we make use of the integral (6), which, on being transformed by the substitutions (13) and (15), takes the form

$$\dot{x}^2+\dot{y}^2+(\dot{\psi}+\dot{w})^2+\frac{2l^2(1+\delta)}{\alpha_1-\alpha_3}(2\psi+2w-c_1)=0. \tag{27}$$

Let us make in (27) the usual substitutions

$$\left.\begin{array}{lll} x=x(0)+\bar{x}, & y=y(0)+\bar{y}, & w=w(0)+\bar{w}, \\ \dot{x}=\dot{x}(0)+\dot{\bar{x}}, & \dot{y}=\dot{y}(0)+\dot{\bar{y}}, & \dot{w}=0+\dot{\bar{w}}, \end{array}\right\} \tag{28}$$

where $\bar{x}, \ldots, \dot{\bar{w}}$ vanish at $\tau=0$; and let the resulting equations be denoted by (27a). By putting $\tau=0$ in (27a) we obtain an equation (27b) connecting the constant terms of (27a) which are independent of $\bar{x}, \ldots, \dot{\bar{w}}$. When those constant terms are eliminated from (27a) by means of (27b), there results an equation of the form

$$G(\bar{x}, \bar{y}, \bar{w}, \dot{\bar{x}}, \dot{\bar{y}}, \dot{\bar{w}})=0, \tag{29}$$

in which, at $\tau=2\pi$, there are no terms independent of the arguments indicated. The coefficient of the linear term in $w(2\pi)$ is $4l^2(1+\delta)/(\alpha_1-\alpha_3)$, and it is different from zero. Hence, by the theory of implicit functions, equation (29) can be solved uniquely for $\bar{w}(2\pi)$ as a power series of the form

$$\bar{w}(2\pi)=Q\{\bar{x}(2\pi),\ \bar{y}(2\pi),\ \dot{\bar{x}}(2\pi),\ \dot{\bar{y}}(2\pi),\ \dot{\bar{w}}(2\pi)\}, \tag{30}$$

in which there is no constant term. Now if the conditions in (26), except the condition $w(2\pi)-w(0)=0$, are imposed, then

$$\bar{x}(2\pi)=\bar{y}(2\pi)=\dot{\bar{x}}(2\pi)=\dot{\bar{y}}(2\pi)=\dot{\bar{w}}(2\pi)=0, \tag{31}$$

and it follows from (30) and (31) that $\bar{w}(2\pi)=0$. Therefore, the condition $w(2\pi)-w(0)=0$ is a consequence of the other five conditions in (26) and may be suppressed.

When the necessary conditions of (26) are imposed upon the solutions (25), the equations which a_i $(i=1, \ldots, 5)$ must satisfy are found to be

$$\left.\begin{aligned}
0 &= \frac{a_1}{2}\left[e^{2a\sqrt{-1}\pi}+e^{-2a\sqrt{-1}\pi}-2\right]-\frac{a_2}{\Delta}\left[e^{2a\sqrt{-1}\pi}-e^{-2a\sqrt{-1}\pi}\right]+\text{terms in }\varepsilon,\\
0 &= -\frac{\Delta a_1}{4}\left[e^{2a\sqrt{-1}\pi}-e^{-2a\sqrt{-1}\pi}\right]+\frac{a_2}{2}\left[e^{2a\sqrt{-1}\pi}+e^{-2a\sqrt{-1}\pi}-2\right]+\text{terms in }\varepsilon,\\
0 &= \frac{a_3}{2}\left[e^{2a\sqrt{-1}\pi}+e^{-2a\sqrt{-1}\pi}-2\right]-\frac{a_4}{\Delta}\left[e^{2a\sqrt{-1}\pi}-e^{-2a\sqrt{-1}\pi}\right]+\text{terms in }\varepsilon,\\
0 &= -\frac{\Delta a_3}{4}\left[e^{2a\sqrt{-1}\pi}-e^{-2a\sqrt{-1}\pi}\right]+\frac{a_4}{2}\left[e^{2a\sqrt{-1}\pi}+e^{-2a\sqrt{-1}\pi}-2\right]+\text{terms in }\varepsilon,\\
0 &= \frac{2\pi a_5 A\dot{\phi}(2\pi)}{\chi(0)}+\text{terms in }\varepsilon.
\end{aligned}\right\}\quad(32)$$

The determinant of the linear terms in a_i $(i=1,\ldots,5)$ is

$$2\pi A\frac{\dot{\phi}(2\pi)}{\chi(0)}\left(e^{2a\sqrt{-1}\pi}-1\right)^2\left(e^{-2a\sqrt{-1}\pi}-1\right)^2, \tag{33}$$

and it is different from zero if a is not a real integer or zero. First, let us suppose that a is not an integer. Then the determinant (33) is not zero and the equations (32) can be solved uniquely for a_i as power series in ε. These series vanish with ε and converge for $|\varepsilon|$ sufficiently small. Hence, periodic solutions of (16) exist uniquely and have the form

$$x=\sum_{i=1}^{\infty}x_i\varepsilon^i,\quad y=\sum_{i=1}^{\infty}y_i\varepsilon^i,\quad w=\sum_{i=1}^{\infty}w_i\varepsilon^i, \tag{34}$$

where each x_i, y_i, w_i is separately periodic for $|\varepsilon|$ sufficiently small.

Now let us suppose that a is an integer or zero. Then the determinant (33) is zero, and in order to prove the existence of periodic solutions of (16) we require the explicit form of the quadratic terms of the first two equations of (25). Let us denote the linear and quadratic terms of these two equations by x_1, y_1 and x_2, y_2 respectively. Then the values of x_1 and y_1, subject to the initial conditions (24), are

$$\left.\begin{aligned}
x_1 &= \left(\frac{a_1}{2}-\frac{a_2}{\Delta}\right)e^{a\sqrt{-1}\tau}\xi_1+\left(\frac{a_1}{2}+\frac{a_2}{\Delta}\right)e^{-a\sqrt{-1}\tau}\xi_2,\\
y_1 &= \left(\frac{a_3}{2}-\frac{a_4}{\Delta}\right)e^{a\sqrt{-1}\tau}\xi_1+\left(\frac{a_3}{2}+\frac{a_4}{\Delta}\right)e^{-a\sqrt{-1}\tau}\xi_2.
\end{aligned}\right\}\quad(35)$$

The differential equations from which x_2 and y_2 are obtained, are the same as (17) except that the right members are not zero. If the right members are denoted by $X^{(2)}$ and $Y^{(2)}$ respectively, then

$$X^{(2)} = x_1 w_1 R_0 + \varepsilon x_1 R_1 + \varepsilon y_1 R_2 + \varepsilon w_1 R_3 + \varepsilon^2 R_4,$$
$$Y^{(2)} = y_1 w_1 R_0 + \varepsilon x_1 R_2 + \varepsilon y_1 R_5 + \varepsilon w_1 R_6 + \varepsilon^2 R_7,$$

where R_0 is a power series in μ with constant coefficients, and $R_i\,(i = 1, \ldots, 7)$ are power series in μ in which the coefficients are sums of cosines of even multiples of τ. The complementary functions of the differential equations defining x_2, y_2 are the same as (18); that is,

$$\left.\begin{aligned} x_2 &= a_1^{(2)} e^{\alpha\sqrt{-1}\,\tau}\xi_1 + a_2^{(2)} e^{-\alpha\sqrt{-1}\,\tau}\xi_2, \\ y_2 &= a_3^{(2)} e^{\alpha\sqrt{-1}\,\tau}\xi_1 + a_4^{(2)} e^{-\alpha\sqrt{-1}\,\tau}\xi_2, \end{aligned}\right\} \tag{36}$$

where $a_i^{(2)}\,(i = 1, \ldots, 4)$ are the arbitrary constants. Now regarding these constants as variables, according to the method of the variation of parameters, we have

$$\left.\begin{aligned} \dot{a}_1^{(2)} &= -\frac{1}{\Delta} e^{-\alpha\sqrt{-1}\,\tau}\xi_2 X^{(2)}, & \dot{a}^{(2)} &= \frac{1}{\Delta} e^{\alpha\sqrt{-1}\,\tau}\xi_1 X^{(2)}, \\ \dot{a}_3^{(2)} &= -\frac{1}{\Delta} e^{-\alpha\sqrt{-1}\,\tau}\xi_2 Y^{(2)}, & \dot{a}_4^{(2)} &= \frac{1}{\Delta} e^{\alpha\sqrt{-1}\,\tau}\xi_1 Y^{(2)}, \end{aligned}\right\} \tag{37}$$

where Δ is the determinant of the fundamental set of solutions (18), and is therefore different from zero. Since $X^{(2)}$, $Y^{(2)}$ contain terms in $e^{\pm\alpha\sqrt{-1}\,\tau}$ multiplied by power series in μ in which the coefficients are sums of cosines of even multiples of τ, the integration of the equations (37) will yield non-periodic terms. We shall be concerned with the explicit form of only the non-periodic terms arising from (37). When the values of $a_i^{(2)}\,(i = 1, \ldots, 4)$ are obtained from (37) and substitued in (36), we have

$$\left.\begin{aligned} x_2 = \tau\,(\varepsilon p^{(1)} + a_5 p^{(2)}) &\left[\left(\frac{a_1}{2} - \frac{a_2}{\Delta}\right) e^{\alpha\sqrt{-1}\,\tau}\xi_1 - \left(\frac{a_1}{2} + \frac{a_2}{\Delta}\right) e^{-\alpha\sqrt{-1}\,\tau}\xi_2\right] \\ &+ \tau\varepsilon p^{(3)}\left[\left(\frac{a_3}{2} - \frac{a_4}{\Delta}\right) e^{\alpha\sqrt{-1}\,\tau}\xi_1 - \left(\frac{a_3}{2} + \frac{a_4}{\Delta}\right) e^{-\alpha\sqrt{-1}\,\tau}\xi_2\right] \\ &+ \varepsilon a_5\,[\text{non-periodic terms}] + \text{periodic terms}, \\ y_2 = \tau\varepsilon p^{(3)} &\left[\left(\frac{a_1}{2} - \frac{a_2}{\Delta}\right) e^{\alpha\sqrt{-1}\,\tau}\xi_1 - \left(\frac{a_1}{2} + \frac{a_2}{\Delta}\right) e^{-\alpha\sqrt{-1}\,\tau}\xi_2\right] \\ + \tau\,(\varepsilon p^{(4)} + a_5 p^{(2)}) &\left[\left(\frac{a_3}{2} - \frac{a_4}{\Delta}\right) e^{\alpha\sqrt{-1}\,\tau}\xi_1 - \left(\frac{a_3}{2} + \frac{a_4}{\Delta}\right) e^{-\alpha\sqrt{-1}\,\tau}\xi_2\right] \\ &+ \varepsilon a_5\,[\text{non-periodic terms}] + \text{periodic terms}, \end{aligned}\right\} \tag{38}$$

where $p^{(i)}\,(i = 1, \ldots, 4)$ are power series in μ with constant coefficients. Hence, the solutions of (16) as power series in ε and $a_i\,(i = 1, \ldots, 5)$ are

$$\left.\begin{aligned} x &= x_1 + x_2 + \ldots, \\ y &= y_1 + y_2 + \ldots, \\ w &= \frac{a_5}{\chi(0)}[\chi + A\tau\phi] + \varepsilon P_3(a_1, \ldots, a_5, \varepsilon;\ \tau), \end{aligned}\right\} \tag{39}$$

where x_1, y_1; x_2, y_2; and P_3 are defined in (35), (38) and (25) respectively.

Now let us impose upon (39) the necessary periodicity conditions of (26). Since x_1, y_1 are periodic when α is an integer, we obtain

$$\left.\begin{aligned} 0 &= -\frac{4\pi a_2}{\Delta}(\varepsilon p^{(1)} + a_5 p^{(2)} - \frac{4\pi a_4}{\Delta}\varepsilon p^{(3)} + \varepsilon a_5 r^{(1)} \\ &\qquad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \ldots, a_5], \\ 0 &= -\pi\Delta a_1(\varepsilon p^{(1)} + a_5 p^{(2)}) - \pi\Delta a_3 \varepsilon p^{(3)} + \varepsilon a_5 r^{(2)} \\ &\qquad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \ldots, a_5], \\ 0 &= -\frac{4\pi a_2}{\Delta}\varepsilon p^{(3)} - \frac{4\pi a_4}{\Delta}(\varepsilon p^{(4)} + a_5 p^{(2)}) + \varepsilon a_5 r^{(3)} \\ &\qquad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \ldots, a_5], \\ 0 &= -\pi\Delta a_1 \varepsilon p^{(3)} - \pi\Delta a_3(\varepsilon p^{(4)} + a_5 p^{(2)}) + \varepsilon a_5 r^{(4)} \\ &\qquad + \varepsilon\,[\text{quadratic and higher degree terms in } \varepsilon, a_1, \ldots, a_5], \\ 0 &= \frac{2\pi A a_5 \dot{\phi}(2\pi)}{\chi(0)} + \varepsilon\,[\text{terms in } \varepsilon, a_1, \ldots, a_5], \end{aligned}\right\} \tag{40}$$

where $r^{(i)}$ $(i = 1, \ldots, 4)$ denote power series in μ with constant coefficients. The coefficient of a_5 in the last equation of (40) is different from zero and therefore this equation can be solved uniquely for a_5 as a power series in ε and a_j $(j = 1, \ldots, 4)$ of the form

$$a_5 = \varepsilon\, p\,(a_1, \ldots, a_4, \varepsilon). \tag{41}$$

When (41) is substituted for a_5 in the first four equations of (40), the factor ε can be divided out and we obtain four equations in ε and a_j $(j = 1, \ldots, 4)$ of the form

$$\left.\begin{aligned} 0 &= -\frac{4\pi q^{(1)} a_2}{\Delta} - \frac{4\pi p^{(3)} a_4}{\Delta} + \varepsilon Q_1(\varepsilon, a_1, \ldots, a_4), \\ 0 &= -\pi\Delta q^{(1)} a_1 - \pi\Delta p^{(3)} a_3 + \varepsilon Q_2(\varepsilon, a_1, \ldots, a_4), \\ 0 &= -\frac{4\pi p^{(3)} a_2}{\Delta} - \frac{4\pi q^{(2)} a_4}{\Delta} + \varepsilon Q_3(\varepsilon, a_1, \ldots, a_4), \\ 0 &= -\pi\Delta p^{(3)} a_1 - \pi\Delta q^{(2)} a_3 + \varepsilon Q_4(\varepsilon, a_1, \ldots, a_4), \end{aligned}\right\} \tag{42}$$

where $q^{(1)}, q^{(2)}$ are power series in μ with constant coefficients, and Q_i $(i = 1, \ldots, 4)$ are power series in ε and a_i $(i = 1, \ldots, 4)$ in which the coefficients

are power series in μ. The determinant of the coefficients of the linear terms in a_i in (42) is

$$16\,\pi^4\,\{q^{(1)}\,q^{(2)} - (p^{(3)})^2\}^2. \tag{43}$$

This determinant is not zero, in general, but it may be possible to choose such values of the constants c_1, c_2 and f_{ijk} (see equation (2)) that (43) shall vanish. We shall exclude such special values of these constants, if any exist, and therefore (43) is not zero. Hence, (42) can be solved uniquely for $a_j\,(j=1, \ldots, 4)$ as power series in ε, vanishing with ε. When these series for a_j are substituted in (41), a_5 is likewise a power series in ε, vanishing with ε. Consequently, when α is an integer or zero, periodic solutions of (16) exist uniquely and are power series in ε of the form

$$x = \sum_{i=1}^{\infty} x^{(i)}\,\varepsilon^i, \qquad y = \sum_{i=1}^{\infty} y^{(i)}\,\varepsilon^i, \qquad w = \sum_{i=1}^{\infty} w^{(i)}\,\varepsilon^i, \tag{44}$$

where each $x^{(i)}$, $y^{(i)}$, $w^{(i)}$ is separately periodic for $|\varepsilon|$ sufficiently small.

Now the solutions (44) include the solutions (34) as the special case when α is zero; and since both the solutions are unique, they are therefore identical. Consequently, in making the practical construction of the periodic solutions of (16), it is not necessary to make a special consideration of the case when α is a real integer.

If the period were chosen to be $2\nu\pi$, ν an integer, then the proof of the existence of periodic solutions of (16) with the period $2\nu\pi$ would be identical with the preceding proof except that, in the preceding, 2π would be replaced by $2\nu\pi$. Then periodic solutions exist having the same form as (34). Since these solutions are unique for every ν, and since the orbits having the period $2\nu\pi$ include those having the period 2π, there are no orbits with the period $2\nu\pi$ which do not have the period 2π also.

If α is a rational fraction N/n, where N and n are real integers relatively prime, the question might be raised whether orbits exist which have the period $2n\pi$ in τ, and not the period 2π. From the preceding paragraph we conclude that such orbits do not exist.

§ 6. *Direct Construction of the Periodic Solutions.*

Let us substitute (34) in (16) and equate the coefficients of the same powers of ε. Since the results are identities in ε, there is obtained a series of

differential equations from which the coefficients in (34) can be determined. The constants of integration arising at each step are to be determined so that the solutions shall be periodic and satisfy the initial condition $\dot{w}=0$, from which it follows that

$$\dot{w}_j(0)=0, \qquad (j=1,\ldots,\infty). \tag{45}$$

The differential equations for the terms in ε are

$$\left.\begin{aligned}\ddot{x}_1+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots]\,x_1&=R_1^{(1)},\\ \ddot{y}_1+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots]\,y_1&=R_2^{(1)},\\ \ddot{w}_1+[b^2+\theta_1^{(2)}\mu+\theta_2^{(2)}\mu^2+\ldots]\,w_1&=R_3^{(1)},\end{aligned}\right\} \tag{46}$$

where the $R_i^{(1)}(i=1,2,3)$ are power series in μ with sums of cosines of even multiples of τ in the coefficients. The complementary functions of (46) are the same as (18) and (23); that is,

$$\left.\begin{aligned}x_1&=a_1^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+a_2^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2,\\ y_1&=a_3^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+a_4^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2,\\ w_1&=a_5^{(1)}\phi+a_6^{(1)}[\chi+A\tau\phi],\end{aligned}\right\} \tag{47}$$

where the $a_i^{(1)}(i=1,\ldots,6)$ are the constants of integration. On using the method of the variation of parameters, we have

$$\left.\begin{aligned}\Delta\dot{a}_1^{(1)}&=-e^{-a\sqrt{-1}\,\tau}\xi_2R_1^{(1)}, & \Delta\dot{a}_2^{(1)}&=e^{a\sqrt{-1}\,\tau}\xi_1R_1^{(1)},\\ \Delta\dot{a}_3^{(1)}&=-e^{-a\sqrt{-1}\,\tau}\xi_2R_2^{(1)}, & \Delta\dot{a}_4^{(1)}&=e^{a\sqrt{-1}\,\tau}\xi_1R_2^{(1)},\\ D\dot{a}_5^{(1)}&=-[\chi+A\tau\phi]R_3^{(1)}, & D\dot{a}_6^{(1)}&=\phi R_3^{(1)},\end{aligned}\right\} \tag{48}$$

where Δ and D are the determinants of fundamental sets of solutions (18) and (23) respectively and therefore are not zero.

When equations (48) are integrated and the resulting values for $a_i^{(1)}$ are substituted in (47), we obtain the general solutions of (36) which are

$$\left.\begin{aligned}x_1&=A_1^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+A_2^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2+C_1^{(1)}(\tau),\\ y_1&=A_3^{(1)}e^{a\sqrt{-1}\,\tau}\xi_1+A_4^{(1)}e^{-a\sqrt{-1}\,\tau}\xi_2+C_2^{(1)}(\tau),\\ w_1&=A_5^{(1)}\phi+A_6^{(1)}[\chi+A\tau\phi]-p_1\tau\phi+C_3^{(1)}(\tau),\end{aligned}\right\} \tag{49}$$

where $A_i^{(1)}(i=1,\ldots,6)$ are the constants of integration; $C_i^{(1)}(\tau)$ $(i=1,2,3)$ are power series in μ with sums of cosines of even multiples of τ in the coefficients; and p_1 is a power series in μ with constant coefficients.

Since we have shown that the periodic solutions of (16) have the same form whether α is an integer or not, the constants $A_i^{(1)}(i=1,\ldots,4)$ must be zero in order that the first two equations of (49) shall be periodic when α is not an integer. If these constants are not put equal to zero at this step, a consideration of the terms in ε^2 will show that they must be zero in order that x_2 and y_2 shall be periodic.

In order that the last equation of (49) shall be periodic, the constant $A_6^{(1)}$ must have the value

$$A_6^{(1)}=\frac{p_1}{A}=\frac{1}{\mu}q_1,$$

where q_1 is a power series in μ with constant coefficients. From the condition (45) it follows that $A_5^{(1)}=0$. When these values of $A_i^{(1)}(i=1,\ldots,6)$ are substituted in (49), the solutions of (46) become

$$x_1=C_1^{(1)}(\tau),\quad y_1=C_2^{(1)}(\tau),\quad w_1=\frac{1}{\mu}q_1\chi+C_3^{(1)}(\tau)=\frac{1}{\mu}\overline{C}_1(\tau),\tag{50}$$

where $\overline{C}_1(\tau)$ has the same form as $C_i^{(1)}(i=1,2,3)$.

It is easy to show that all the succeeding steps of the integration are entirely similar. The differential equations for the coefficients of ε^n are

$$\left.\begin{aligned}\ddot{x}_n+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots\ldots]\,x_n&=\frac{1}{\mu^{n-1}}R_1^{(n)},\\ \ddot{y}_n+[a^2+\theta_1^{(1)}\mu+\theta_2^{(1)}\mu^2+\ldots\ldots]\,y_n&=\frac{1}{\mu^{n-1}}R_2^{(n)},\\ \ddot{w}_n+[b^2+\theta_1^{(2)}\mu+\theta_2^{(2)}\mu^2+\ldots\ldots]\,w_n&=\frac{1}{\mu^{n-1}}R_3^{(n)},\end{aligned}\right\}\tag{51}$$

where $R_i^{(n)}(i=1,2,3)$ are similar in form to $R_i^{(1)}(i=1,2,3)$. Then on forming the equations analogous to (47) and (48), we obtain the general solutions of (51), viz.,

$$\left.\begin{aligned}x_n&=A_1^{(n)}e^{a\sqrt{-1}\,\tau}\xi_1+A_2^{(n)}e^{-a\sqrt{-1}\,\tau}\xi_2+\frac{1}{\mu^{n-1}}C_1^{(n)}(\tau),\\ y_n&=A_3^{(n)}e^{a\sqrt{-1}\,\tau}\xi_1+A_4^{(n)}e^{-a\sqrt{-1}\,\tau}\xi_2+\frac{1}{\mu^{n-1}}C_2^{(n)}(\tau),\\ w_n&=A_5^{(n)}\phi+A_6^{(n)}[\chi+A\tau\phi]-\frac{1}{\mu^{n-1}}p_n\tau\phi+\frac{1}{\mu^{n-1}}C_3^{(n)}(\tau),\end{aligned}\right\}\tag{52}$$

where $A_i^{(n)}(i=1,\ldots,6)$ are the constants of integration; $C_i^{(n)}(\tau)\,(i=1,2,3)$ are periodic functions similar in form to $C_i^{(1)}(i=1,2,3)$; and p_n is a power series in μ with constant coefficients. In order that the solutions (52) shall

satisfy the periodicity and the initial conditions, the constants of integration must have the values

$$A_i^{(n)} = 0 \ (i = 1, \ldots, 5), \quad A_6^{(n)} = \frac{1}{\mu^{n-1}} \frac{p_n}{A} = \frac{1}{\mu^n} q_n,$$

where q_n is a power series in μ with constant coefficients. Hence, the desired solutions of (51) are

$$\left.\begin{aligned} x_n &= \frac{1}{\mu^{n-1}} C_1^{(n)}(\tau), \\ y_n &= \frac{1}{\mu^{n-1}} C_2^{(n)}(\tau), \\ w_n &= \frac{1}{\mu^n} q_n \chi + \frac{1}{\mu^{n-1}} C_3^{(n)}(\tau) = \frac{1}{\mu^n} \overline{C}_n(\tau), \end{aligned}\right\} \tag{53}$$

where $\overline{C}_n(\tau)$ is similar in form to $\overline{C}_1(\tau)$. Thus the general step of the integration is entirely similar to the first step.

When we are dealing with Cases II and III of the solutions of the first two equations of (17), the method of proving the existence and of making the construction of the periodic solutions of (16) is similar to the preceding. In the other two cases, as in Case I, the solutions of (16) are power series in ε. In Case II the coefficients of the various powers of ε are power series in μ similar to those obtained in (53), but they contain additional terms in $\cos 2(aj + k)\tau$, j and k integers. In Case III the coefficients of the various powers of ε are power series in $\sqrt{\mu}$ with coefficients similar to those in (53).

§ 7. *The Character of the Surface.*

The equation of the surface is general except that $|\varepsilon|$ must be taken small in order to insure the convergence of certain series appearing in the preceding. Suppose the periodic orbits are desired on a given surface S. Because $|\varepsilon|$ is small, it may not be possible to choose such values of ε and f_{ijk} in (2) that (1) shall represent S. Let us suppose that certain values of f_{ijk} are taken in (1) and that $\bar{\varepsilon}$ is the largest value $|\varepsilon|$ may take, and let us denote the resulting equation of the surface by S_1. Then by the preceding method we can determine periodic orbits on the surface

$$S_2 \equiv S_1 + 2\varepsilon_1 f_1(x, y, z) = 0,$$

where ε_1 is a new parameter and $f_1(x, y, z)$ has the same form as $f(x, y, z)$. By repeating the same process over and over again, we can determine periodic orbits on a sequence of surfaces

$$S_{k+1} \equiv S_k + 2\varepsilon_k f_k(x, y, z) = 0, \qquad (k = 1, 2, 3, \ldots),$$

where $f_k(x, y, z)$ has the same form as (2) and ε_k is a parameter, *provided, of course, that the solutions obtained do not pass through any singularities.* Thus, in general, the given surface S can be approached by the sequence of surfaces S_{k+1}, and the periodic orbits described on it can be obtained as power series in $\mu, \varepsilon, \varepsilon_1, \ldots, \varepsilon_k, \ldots$

QUEEN'S UNIVERSITY, KINGSTON, CANADA, *September* 23, 1913.

On Foucault's Pendulum.

By William Duncan MacMillan.

§ 1. *Introduction.*

A number of papers have appeared during the past sixty years on the theory of the motion of the Foucault pendulum,* but the theory is still far from being in a satisfactory state. The theory given in the various treatises on mechanics includes only the case of infinitesimal oscillations. For oscillations of this type the equations of motion are completely integrable, and it is found that, if the motion is referred to a horizontal plane rotating in clockwise direction with uniform angular speed of period $\frac{24}{\sin\beta}$ hours, where β is the latitude of the place, the pendulum describes a relatively long, narrow ellipse in which the ratio of the minor to the major axis is the same as the ratio of the period of a single oscillation of the pendulum to the period of the rotating plane. This result is independent of the azimuth of the initial vertical plane of the pendulum, so that the theory for this case is complete.

For finite oscillations, however, only approximate solutions have been given. If ω represents the angular rate of the earth's rotation, and if terms of the order ω^2 and higher are neglected in convenient places, the equations of motion can be integrated by means of elliptic functions.† Since the quantity ω^2 is very small, the results obtained by this process doubtless represent the motion very accurately for a considerable interval of time; but as the results obtained do not satisfy the equations of motion, no inferences can be drawn from them with safety over extended intervals of time.

It is the purpose of the present paper to set forth explicitly two rigorous particular solutions of the equations of motion as they are usually given.‡ In these equations the oblateness of the earth and the terms in ω^2 are neglected. It will be shown that if the pendulum is started from rest in the plane of the meridian, if certain conditions of commensurability are satisfied, and if the

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* References to the literature of this subject are given in the "Encyklopädie der Mathematischen Wissenschaften," IV, 7, S. 6.

† Chessin, American Journal of Mathematics, Vol. XVII, 1895.

‡ Poisson, *Journal Polytechnique*, 1838. Routh, "Advanced Rigid Dynamics," p. 27. Appel, "Traité de Mécanique Rationelle," II, p. 310, edition of 1896.

oscillations of the pendulum are not too large, then the motion of the pendulum is always periodic. Furthermore, it will be shown that the period of the rotating plane, which is 24 hours divided by the sine of latitude for infinitesimal oscillations, increases as the amplitude of the oscillations of the pendulum increases. Similar results are obtained when the pendulum is started into motion with an initial impulse.

§ 2. *The Differential Equations.*

It is assumed that the earth is a sphere and that the xy-plane is tangent to the sphere at the place of observation. The positive end of the x-axis is towards the south, the positive end of the y-axis is towards the east, and the positive end of the z-axis is directed downward. The origin is taken at the point of suspension of the pendulum. We will let ω denote the rate of the earth's rotation, β the latitude, l the length and mT the tension of the suspending wire, and g the acceleration of gravity. The equations of motion are

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{T}{l}x + 2\omega\sin\beta\frac{dy}{dt},\\ \frac{d^2y}{dt^2} &= -\frac{T}{l}y - 2\omega\sin\beta\frac{dx}{dt} + 2\omega\cos\beta\frac{dz}{dt},\\ \frac{d^2z}{dt^2} &= -\frac{T}{l}z + g - 2\omega\cos\beta\frac{dy}{dt}.\end{aligned}\right\}\qquad(1)$$

Since the pendulum is of invariable length l, the following relations are always satisfied:

$$\left.\begin{aligned}&x^2+y^2+z^2=l^2,\\ &x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}=0,\\ &x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}=-\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right];\end{aligned}\right\}\qquad(2)$$

and the energy integral is

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=2gl\left(\frac{z}{l}-c\right),\qquad(3)$$

where c is the constant of integration. Combining (1), (2) and (3), it is found without difficulty that

$$\frac{T}{l}=\frac{g}{l}\left(3\frac{z}{l}-2c\right)+\frac{2\omega}{l^2}\left[\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)\sin\beta+\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)\cos\beta\right].\qquad(4)$$

It will be supposed that the pendulum is initially displaced towards the south through an angle whose sine is μ. Then the initial values are

$$x=l\mu, \quad y=0, \quad z=l\sqrt{1-\mu^2}, \quad \frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=0, \tag{5}$$

from which it follows that $c=\sqrt{1-\mu^2}$.

Let us now make a change of variables by the transformation

$$\left.\begin{aligned} x&=l[\xi \cos nt+\eta \sin nt], \\ y&=l[-\xi \sin nt+\eta \cos nt], \\ z&=l[1-\zeta], \end{aligned}\right\} \tag{6}$$

so that $\xi^2+\eta^2+(1-\zeta)^2=1$, and for brevity let us take $\omega \sin\beta=\sigma$, $\omega\cos\beta=\sigma_1$. Then the motion of the pendulum is referred to a system of rectangular axes rotating with the angular velocity n, where n is an arbitrary at our disposal, and the differential equations are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}+\left[\frac{T}{l}+n^2-2n(n-\sigma)\right]\xi&=-2(n-\sigma)\frac{d\eta}{dt}+2\sigma_1 \sin nt\frac{d\zeta}{dt}, \\ \frac{d^2\eta}{dt^2}+\left[\frac{T}{l}+n^2-2n(n-\sigma)\right]\eta&=+2(n-\sigma)\frac{d\xi}{dt}-2\sigma_1 \cos nt\frac{d\zeta}{dt}, \\ \frac{d^2\zeta}{dt^2}+\frac{T}{l}\zeta&=\frac{T-g}{l}-2\sigma_1\left(n\xi-\frac{d\eta}{dt}\right)\cos nt-2\sigma_1\left(\frac{d\xi}{dt}+n\eta\right)\sin nt. \end{aligned}\right\} \tag{7}$$

From the relation $\zeta=1-\sqrt{1-(\xi^2+\eta^2)}$ it is seen that, if ξ and η are small quantities of the first order, ζ is a small quantity of the second order. Neglecting terms of the second and higher orders and choosing $n=\sigma$, the differential equations become simply

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}+\left[\frac{g}{l}+\sigma^2\right]\xi&=0, \\ \frac{d^2\eta}{dt^2}+\left[\frac{g}{l}+\sigma^2\right]\eta&=0. \end{aligned}\right\} \tag{8}$$

In order to satisfy the initial conditions in the xy-plane we must have at $t=0$, $\xi=\mu$, $\frac{d\xi}{dt}=0$, $\eta=0$, $\frac{d\eta}{dt}=\sigma\mu$, and therefore the solution is

$$\xi=\mu\cos\sqrt{\frac{g}{l}+\sigma^2}\,t, \quad \eta=\frac{\sigma\mu}{\sqrt{\frac{g}{l}+\sigma^2}}\sin\sqrt{\frac{g}{l}+\sigma^2}\,t, \tag{9}$$

which is an ellipse of which the semi-major axis is μ and the semi-minor axis is $\dfrac{\sigma\mu}{\sqrt{\frac{g}{l}+\sigma^2}}$. Hence the theorem of Chevilliet for infinitesimal oscillations,

"*The ratio of the major to the minor axis is equal to the ratio of the period of the rotating plane to the period of oscillation in the ellipse.*"

There is no difficulty in introducing an azimuth constant into this solution if it is so desired.

§ 3. *Development of a Periodic Solution.*

The choice $n=\sigma$ certainly simplifies the differential equations. Nevertheless it does not seem to be convenient to develop a periodic solution in this manner. For finite oscillations we will take $n=\frac{\sigma}{1+\varkappa}$, where $\varkappa$ is an arbitrary constant, the form and value of which will be determined by the initial and periodicity conditions. It will be convenient also to take $l=\frac{l_0}{1+\lambda}$, where l_0 is a fixed constant and λ is a constant of the same type as $\varkappa$. We will transform the independent variable by taking $t=(1+\varkappa)\tau$ so that $nt=\sigma\tau$, and, finally, we will write $\frac{g}{l_0}+\sigma^2=p^2$. Since the length of the pendulum is invariable throughout the motion, we can discard the differential equation in ζ, and there remains

$$\left.\begin{aligned} \xi''+\left[(1+\varkappa)^2\frac{T}{l}+(1+2\varkappa)\sigma^2\right]\xi&=+2\varkappa\sigma\eta'+2(1+\varkappa)\sigma_1\zeta'\sin\sigma\tau,\\ \eta''+\left[(1+\varkappa)^2\frac{T}{l}+(1+2\varkappa)\sigma^2\right]\eta&=-2\varkappa\sigma\xi'-2(1+\varkappa)\sigma_1\zeta'\cos\sigma\tau,\end{aligned}\right\}\qquad(10)$$

where accents denote derivatives with respect to τ, and

$$\begin{aligned}(1+\varkappa)^2\frac{T}{l}=&(1+\varkappa)^2(1+\lambda)\frac{g}{l_0}\left[3\sqrt{1-(\xi^2+\eta^2)}-2\sqrt{1-\mu^2}\right]\\ &+2(1+\varkappa)\sigma(\xi\eta'-\eta\xi')-2\sigma^2(1+\varkappa)(\xi^2+\eta^2)\\ &+2(1+\varkappa)\sigma_1\left[\left((\sigma\xi-\eta')\sqrt{1-(\xi^2+\eta^2)}-\frac{\eta(\xi\xi'+\eta\eta')}{\sqrt{1-(\xi^2+\eta^2)}}\right)\cos\sigma\tau\right.\\ &\left.+\left((\sigma\eta+\xi')\sqrt{1-(\xi^2+\eta^2)}+\frac{\xi(\xi\xi'+\eta\eta')}{\sqrt{1-(\xi^2+\eta^2)}}\right)\sin\sigma\tau\right],\end{aligned}\qquad(11)$$

and $\zeta=1-\sqrt{1-(\xi^2+\eta^2)}$. The integral becomes

$$\begin{aligned}\xi'^2+\eta'^2+\frac{(\xi\xi'+\eta\eta')^2}{1-(\xi^2+\eta^2)}=&2\sigma(\xi\eta'-\eta\xi')-\sigma^2(\xi^2+\eta^2)\\ &+\frac{2g}{l_0}(1+\lambda)(1+\varkappa)^2\left[\sqrt{1-(\xi^2+\eta^2)}-\sqrt{1-\mu^2}\right].\end{aligned}\qquad(12)$$

We propose to integrate these equations as power series in μ, and we therefore assume

$$
\begin{aligned}
\xi &= \xi_1\mu + \xi_2\mu^2 + \xi_3\mu^3 + \ldots, \\
\eta &= \eta_1\mu + \eta_2\mu^2 + \eta_3\mu^3 + \ldots, \\
\varkappa &= \varkappa_1\mu + \varkappa_2\mu^2 + \varkappa_3\mu^3 + \ldots, \\
\lambda &= \lambda_1\mu + \lambda_2\mu^2 + \lambda_3\mu^3 + \ldots.
\end{aligned}
$$

In these series the constants $\varkappa_i$ and λ_i will be chosen so that the coefficients ξ_j η_j shall be periodic functions of τ. As is seen from equations (8), the free period of the differential equations is $\frac{2\pi}{p}$ and the forced period $\frac{2\pi}{\sigma}$. It is necessary to assume, therefore, that the constant l_0 has such a value that $p=\sqrt{\frac{g}{l_0}+\sigma^2}$ is commensurable with σ. This assumption is of importance in the theory of convergence of the series, but, obviously, is of little importance physicallv. It is found readily by a few preliminary computations that $\varkappa_1=\lambda_1=\sigma\eta_1-\xi_1'=\xi_2=0$, and $\xi_1\eta_1'-\eta_1\xi_1'=\sigma$. Utilizing these facts to simplify the expansions, the differential equations become

$$
\begin{aligned}
&\sum_{j=1}^{\infty}[\xi_j''+p^2\xi_j]\mu^j = \\
&\quad[-2\sigma_1(\sigma\eta_1+\xi_1')\xi_1\sin\sigma\tau+\sigma_1(\xi_1^2+\eta_1^2)'\sin\sigma\tau]\mu^2 \\
&\quad+[-((p^2-\sigma^2)\lambda_2+2p^2\varkappa_2+p^2+\sigma^2)\xi_1+2\varkappa_2\sigma\eta_1'+(\tfrac{3}{2}p^2+\tfrac{1}{2}\sigma^2)(\xi_1^2+\eta_1^2)\xi_1 \\
&\qquad+2\sigma_1\xi_1\eta_2'\cos\sigma\tau+(2\sigma_1(\eta_1\eta_2)'-2\sigma\sigma_1\xi_1\eta_2)\sin\sigma\tau]\mu^3 \\
&\quad+[-((p^2-\sigma^2)\lambda_3+2p^2\varkappa_3)\xi_1+2\varkappa_3\sigma\eta_1'+(3p^2+\sigma^2)\xi_1\eta_1\eta_2-2\sigma\xi_1(\xi_1\eta_2'-\xi_2\eta_1')+2\varkappa_2\sigma\eta_2' \\
&\qquad+\{-2\sigma_1\xi_1(\sigma\xi_3-\eta_3')+\sigma_1\xi_1\eta_1(\xi_1^2+\eta_1^2)'\}\cos\sigma\tau+\{-2\sigma_1\varkappa_2\xi_1(\sigma\eta_1+\xi_1') \\
&\qquad\quad-2\sigma_1\xi_1(\sigma\eta_3+\xi_3')+\sigma\xi_1(\sigma\eta_1+\xi_1')(\xi_1^2+\eta_1^2)-\sigma\xi_1^2(\xi_1^2+\eta_1^2)'-2\sigma_1\xi_3(\sigma\eta_1+\xi_1') \\
&\qquad\quad+2\sigma_1(\xi_1\xi_3+\eta_1\eta_3)'+2\sigma_1\eta_2\eta_2'+\tfrac{1}{2}\sigma_1(\xi_1^2+\eta_1^2)(\xi_1^2+\eta_1^2)' \\
&\qquad\quad+\varkappa_2\sigma_1(\xi_1^2+\eta_1^2)'\}\sin\sigma\tau]\mu^4+\ldots,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\sum_{j=1}^{\infty}[\eta_j''+p^2\eta_j]\mu^j = \\
&\quad[-2\sigma_1(\sigma\eta_1+\xi_1')\eta_1\sin\sigma\tau-\sigma_1(\xi_1^2+\eta_1^2)'\cos\sigma\tau]\mu^2 \\
&\quad+[-(p^2-\sigma^2)\lambda_2\eta_1-2\varkappa_2(p^2\eta_1+\sigma\xi_1')-(p^2+\sigma^2)\eta_1+\tfrac{1}{2}(3p^2+\sigma^2)(\xi_1^2+\eta_1^2)\eta_1 \\
&\qquad+2\sigma_1\eta_1(\eta_2\cos\sigma\tau)'-2\sigma_1(\eta_1\eta_2)'\cos\sigma\tau-2\sigma_1(\sigma\eta_1+\xi_1')\eta_2\sin\sigma\tau]\mu^3 \\
&\quad+[-(p^2-\sigma^2)\lambda_3\eta_1-2\varkappa_3(p^2\eta_1+\sigma\xi_1')+(3p^2+\sigma^2)\eta_1^2\eta_2-2\sigma\eta_1(\xi_1\eta_2'-\xi_1'\eta_2) \\
&\qquad-\{(p^2-\sigma^2)\lambda_2+2p^2\varkappa_2+p^2+\sigma^2\}\eta_2+\tfrac{1}{2}(3p^2+\sigma^2)(\xi_1^2+\eta_1^2)\eta_2+2\sigma_1\eta_2(\eta_2\cos\sigma\tau)' \\
&\qquad+\{-2\sigma_1\eta_1(\sigma\xi_3-\eta_3')+\sigma_1\eta_1^2(\xi_1^2+\eta_1^2)'-2\sigma_1(\xi_1\xi_3+\eta_1\eta_3)'-2\sigma_1\eta_2\eta_2' \\
&\qquad\quad-\tfrac{1}{2}\sigma_1(\xi_1^2+\eta_1^2)(\xi_1^2+\eta_1^2)'-\sigma_1\varkappa_2(\xi_1^2+\eta_1^2)'\}\cos\sigma\tau+\{-2\sigma_1\varkappa_2\eta_1(\sigma\eta_1+\xi_1') \\
&\qquad\quad-2\sigma_1\eta_1(\sigma\eta_3+\xi_3')+\sigma_1\eta_1(\sigma\eta_1+\xi_1')(\xi_1^2+\eta_1^2)-\sigma\eta_1\xi_1(\xi_1^2+\eta_1^2)' \\
&\qquad\quad-2\sigma_1\eta_3(\sigma\eta_1+\xi_1')\}\sin\sigma\tau]\mu^4+\ldots.
\end{aligned}
\tag{14}
$$

From the initial conditions (5) and the equations of transformation (6), it is found that the initial values of ξ and η are

$$\xi(0)=\mu, \quad \eta(0)=0; \quad \xi'(0)=0, \quad \eta'(0)=\sigma\mu.$$

Consequently,

$$\left.\begin{array}{llll} \xi_1(0)=1, & \eta_1(0)=0; & \xi_1'(0)=0, & \eta_1'(0)=\sigma; \\ \xi_j(0)=0, & \eta_j(0)=0; & \xi_j'(0)=0, & \eta_j'(0)=0, \quad j=2,\ldots,\infty. \end{array}\right\} \tag{15}$$

Equating the coefficients of the first powers of μ in the left and right members of (13) and (14), we find

$$\left.\begin{array}{l} \xi_1''+p^2\xi_1=0, \\ \eta_1''+p^2\eta_1=0, \end{array}\right\} \tag{16}$$

and the solutions of these equations which satisfy the initial conditions are

$$\left.\xi_1=\cos p\tau, \quad \eta_1=\frac{\sigma}{p}\sin p\tau.\right\} \tag{17}$$

From these expressions it results that

$$\xi_1^2+\eta_1^2=\frac{p^2+\sigma^2}{2p^2}+\frac{p^2-\sigma^2}{2p^2}\cos 2p\tau, \quad \sigma\eta_1+\xi_1'=-\frac{p^2-\sigma^2}{p}\sin p\tau, \quad \sigma\xi_1-\eta_1'=0. \tag{18}$$

From the coefficients of the second power of μ in (13) and (14), and from the values given in (17) and (18), it is found that

$$\left.\begin{array}{l} \xi_2''+p^2\xi_2=0, \\ \eta_2''+p^2\eta_2=\dfrac{(p^2-\sigma^2)\sigma\sigma_1}{p^2}\sin\sigma\tau+\dfrac{(p+\sigma)^2(p-\sigma)\sigma_1}{2p^2}\sin(2p-\sigma)\tau \\ \qquad\qquad\qquad +\dfrac{(p-\sigma)^2(p+\sigma)\sigma_1}{2p^2}\sin(2p+\sigma)\tau, \end{array}\right\} \tag{19}$$

On integrating these equations and imposing the initial conditions, we find

$$\left.\begin{array}{l} \xi_2=0, \\ \eta_2=\dfrac{6(p^2-\sigma^2)\sigma_1}{p(9p^2-\sigma^2)}\sin p\tau+\dfrac{\sigma\sigma_1}{p^2}\sin\sigma\tau-\dfrac{(p-\sigma)^2\sigma_1}{2p^2(3p+\sigma)}\sin(2p+\sigma)\tau \\ \qquad\qquad\qquad -\dfrac{(p+\sigma)^2\sigma_1}{2p^2(3p-\sigma)}\sin(2p-\sigma)\tau. \end{array}\right\} \tag{20}$$

Similarly, the differential equations obtained from the coefficients of the third power of μ in (13) and (14) are

$$
\begin{aligned}
&\xi_3'' + p^2\xi_3 = \\
&\quad\left[-(\lambda_2+2\varkappa_2)+\frac{p^2-\sigma^2}{8p^2}-\frac{(3p^2+\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\right](p^2-\sigma^2)\cos p\tau \\
&\quad+\left[\frac{3p^2+\sigma^2}{8p^2}-\frac{(3p^2-\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\right](p^2-\sigma^2)\cos 3p\tau+\frac{6(p^2-\sigma^2)\sigma_1^2}{9p^2-\sigma^2}\cos\sigma\tau \\
&\quad+\frac{3(p^2-\sigma^2)(p-\sigma)\sigma_1^2}{p(9p^2-\sigma^2)}\cos(2p+\sigma)\tau+\frac{3(p^2-\sigma^2)(p+\sigma)\sigma_1^2}{p(9p^2-\sigma^2)}\cos(2p-\sigma)\tau \\
&\quad+\frac{(-2p^4+p^3\sigma+9p^2\sigma^2+5p\sigma^3-3\sigma^4)\sigma_1^2}{4p^3(3p+\sigma)}\cos(p+2\sigma)\tau \\
&\quad+\frac{(-2p^4-p^3\sigma+9p^2\sigma^2-5p\sigma^3-3\sigma^4)\sigma_1^2}{4p^3(3p-\sigma)}\cos(p-2\sigma)\tau \\
&\quad-\frac{(p-\sigma)^3(2p+\sigma)\sigma_1^2}{4p^3(3p+\sigma)}\cos(3p+2\sigma)\tau-\frac{(p+\sigma)^3(2p-\sigma)\sigma_1^2}{4p^3(3p-\sigma)}\cos(3p-2\sigma)\tau, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&\eta_3''+p^2\eta_3= \\
&\quad\left[-\lambda_2-\frac{5p^2+3\sigma^2}{8p^2}+\frac{(7p^2-3\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\right]\frac{\sigma}{p}(p^2-\sigma^2)\sin p\tau \\
&\quad+\left[\frac{3p^2+\sigma^2}{8p^2}+\frac{(5p^2+\sigma^2)\sigma_1^2}{p(9p^2-\sigma^2)}\right]\frac{\sigma}{p}(p^2-\sigma^2)\sin 3p\tau+\frac{6(p^2-2\sigma^2)(p^2-\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin\sigma\tau \\
&\quad-\frac{3(p^2-\sigma^2)(p+2\sigma)(p-\sigma)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin(2p+\sigma)\tau \\
&\quad+\frac{3(p^2-\sigma^2)(p-2\sigma)(p+\sigma)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin(2p-\sigma)\tau \\
&\quad-\frac{(p^4+3p^3\sigma+9p^2\sigma^2-7p\sigma^3-6\sigma^4)\sigma_1^2}{4p^3(3p+\sigma)}\sin(p+2\sigma)\tau \\
&\quad+\frac{(p^4-3p^3\sigma+9p^2\sigma^2+7p\sigma^3-6\sigma^4)\sigma_1^2}{4p^3(3p-\sigma)}\sin(p-2\sigma)\tau \\
&\quad+\frac{(p-\sigma)^3(p+2\sigma)\sigma_1^2}{4p^3(3p+\sigma)}\sin(3p+2\sigma)\tau-\frac{(p+\sigma)^3(p-2\sigma)\sigma_1^2}{4p^3(3p-\sigma)}\sin(3p-2\sigma)\tau. \qquad (22)
\end{aligned}
$$

In order that the solution of these equations may be periodic, it is necessary and sufficient that the coefficient of $\cos p\tau$ in (21) and the coefficient of $\sin p\tau$ in (22) should vanish. Both of these coefficients carry $(p^2-\sigma^2)$ as a factor, but this factor cannot vanish, since σ is very small as compared with p. The coefficient of $\sin p\tau$ in (22) carries σ/p as a factor, and this can vanish only at the earth's equator. For other places on the earth's surface we must have

$$
\begin{aligned}
\lambda_2+2\varkappa_2 &= +\frac{p^2-\sigma^2}{8p^2}-\frac{(3p^2+\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}, \\
\lambda_2 &= -\frac{5p^2+3\sigma^2}{8p^2}+\frac{(7p^2-3\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}.
\end{aligned}
$$

From these equations it results that

$$\left.\begin{aligned}\lambda_2 &= -\frac{5p^2+3\sigma^2}{8p^2}+\frac{(7p^2-3\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)},\\ \varkappa_2 &= +\frac{3p^2+\sigma^2}{8p^2}-\frac{(5p^2-\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}.\end{aligned}\right\} \tag{23}$$

Since σ and σ_1 are very small as compared with p, it is seen that λ_2 is approximately equal to $-\frac{5}{8}$, and $\varkappa_2$ is approximately equal to $+\frac{3}{8}$.

Using the values of λ_2 and $\varkappa_2$ given in (23), the solutions of (21) and (22) which satisfy the initial conditions are

$$\begin{aligned}\xi_3 = &\left[\frac{(3p^2+\sigma^2)(p^2-\sigma^2)}{64p^4}-\frac{(243p^8-256p^6\sigma^2+208p^4\sigma^4-4p^2\sigma^6+\sigma^8)\sigma_1^2}{8p^4(p^2-\sigma^2)(9p^2-\sigma^2)^2}\right]\cos p\tau\\ &+\left[-\frac{(3p^2+\sigma^2)(p^2-\sigma^2)}{64p^4}+\frac{(3p^2-\sigma^2)(p^2-\sigma^2)\sigma_1^2}{8p^4(9p^2-\sigma^2)}\right]\cos 3p\tau+\frac{6\sigma_1^2}{9p^2-\sigma^2}\cos\sigma\tau\\ &-\frac{3(p^2-\sigma^2)(p-\sigma)\sigma_1^2}{p(p+\sigma)(3p+\sigma)(9p^2-\sigma^2)}\cos(2p+\sigma)\tau\\ &-\frac{3(p^2-\sigma^2)(p+\sigma)\sigma_1^2}{p(p-\sigma)(3p-\sigma)(9p^2-\sigma^2)}\cos(2p-\sigma)\tau\\ &+\frac{(2p^4-p^3\sigma-9p^2\sigma^2-5p\sigma^3+3\sigma^4)\sigma_1^2}{16p^3\sigma(p+\sigma)(3p+\sigma)}\cos(p+2\sigma)\tau\\ &-\frac{(2p^4+p^3\sigma-9p^2\sigma^2+5p\sigma^3+3\sigma^4)\sigma_1^2}{16p^3\sigma(p-\sigma)(3p-\sigma)}\cos(p-2\sigma)\tau\\ &+\frac{(p-\sigma)^3\sigma_1^2}{16p^3(p+\sigma)(3p+\sigma)}\cos(3p+2\sigma)\tau+\frac{(p+\sigma)^3\sigma_1^2}{16p^3(p-\sigma)(3p-\sigma)}\cos(3p-2\sigma)\tau,\end{aligned} \tag{24}$$

$$\begin{aligned}\eta_3 = &\left[\frac{3(3p^2+\sigma^2)(p^2-\sigma^2)\sigma}{64p^5}-\frac{(27p^8+900p^6\sigma^2-572p^4\sigma^4+32p^2\sigma^6-3\sigma^8)\sigma_1^2}{8p^5\sigma(9p^2-\sigma^2)^2}\right]\sin p\tau\\ &-\left[\frac{(3p^2+\sigma^2)(p^2-\sigma^2)\sigma}{64p^5}+\frac{(5p^2+\sigma^2)(p^2-\sigma^2)\sigma\sigma_1^2}{8p^5(9p^2-\sigma^2)}\right]\sin 3p\tau+\frac{6(p^2-2\sigma^2)\sigma_1^2}{p^2(9p^2-\sigma^2)}\sin\sigma\tau\\ &+\frac{3(p^2-\sigma^2)(p-\sigma)(p+2\sigma)\sigma_1^2}{p^2(p+\sigma)(3p+\sigma)(9p^2-\sigma^2)}\sin(2p+\sigma)\tau\\ &-\frac{3(p^2-\sigma^2)(p+\sigma)(p-2\sigma)\sigma_1^2}{p^2(p-\sigma)(3p-\sigma)(9p^2-\sigma^2)}\sin(2p-\sigma)\tau\\ &+\frac{(p^4+3p^3\sigma+9p^2\sigma^2-7p\sigma^3-6\sigma^4)\sigma_1^2}{16p^3\sigma(p+\sigma)(3p+\sigma)}\sin(p+2\sigma)\tau\\ &+\frac{(p^4-3p^3\sigma+9p^2\sigma^2+7p\sigma^3-6\sigma^4)\sigma_1^2}{16p^3\sigma(p-\sigma)(3p-\sigma)}\sin(p-2\sigma)\tau\\ &-\frac{(p-\sigma)^3\sigma_1^2}{16p^3(p+\sigma)(3p+\sigma)}\sin(3p+2\sigma)\tau+\frac{(p+\sigma)^3\sigma_1^2}{16p^3(p-\sigma)(3p-\sigma)}\sin(3p-2\sigma)\tau.\end{aligned} \tag{25}$$

§ 4. *Properties of the Solution.*

It is not necessary to carry the computation any further, though by induction it will be shown that it is possible to carry it as far as is desired, and that the constants λ_{i-1} and $\varkappa_{i-1}$ can always be determined in the coefficient of μ^i so as to keep the solution periodic. So far as it has been computed, ξ is a cosine series and η is a sine series. Let us suppose that this property holds up to and including the coefficients of μ^{i-1}. Then from (11) and from the properties of evenness and oddness it is seen that T/l is a cosine series up to and including the coefficient of μ^{i-1}. Then it follows from (10) that the differential equations for ξ_i and η_i have the form

$$\left.\begin{aligned}\xi_i''+p_2^2\xi_i&=[-(\lambda_{i-1}+2\varkappa_{i-1})(p^2-\sigma^2)+A_i]\cos p\tau+\text{other cosine terms},\\ \eta_i''+p^2\eta_i&=\left[-\lambda_{i-1}\frac{\sigma}{p}(p^2-\sigma^2)+B_i\right]\sin p\tau+\text{other sine terms},\end{aligned}\right\}(26)$$

where A_i and B_i are known constants. In order that the solution may be periodic, we must have

$$\lambda_{i-1}+2\varkappa_{i-1}=\frac{A_i}{p^2-\sigma^2},$$
$$\lambda_{i-1}=\frac{B_ip}{\sigma(p^2-\sigma^2)};$$

and these equations uniquely determine λ_{i-1} and $\varkappa_{i-1}$. The solution of (26) which satisfies the initial conditions is then

$$\xi_i=\text{sum of cosine terms},$$
$$\eta_i=\text{sum of sine terms}.$$

Thus ξ_i and η_i have the properties which were assumed for ξ_{i-1} and η_{i-1}. The property holds, therefore, for every i, and consequently ξ is a periodic cosine series and η is a periodic sine series.

We have therefore formally determined a periodic solution of the equations of motion, and we are assured of its convergence by the general existence theorem of periodic solutions.*

A second property of the solution which has just been obtained is as follows: If $\begin{matrix}\sin\\ \cos\end{matrix}(ip\pm j\sigma)\tau$ is a term in the solution, then $i+j$ is an odd integer. This is certainly true up to and including the coefficients of μ^3. That it is general is readily seen by induction. Bearing in mind the properties of even and odd multiples, it is readily seen from (11) that if $i+j$ is an odd integer in all coeffi-

* MacMillan, "An Existence Theorem for Periodic Solutions," *Transactions of the American Mathematical Society*, Vol. XIII (1912), p. 146.

cients of ξ and η up to and including $\mu^{\kappa-1}$, then the expansion of T/l contains only terms in which $i+j$ is an even integer; and consequently from (10) it is seen that ξ_κ and η_κ contain only terms in which $i+j$ is odd. The property is therefore general.

We have already supposed that p and σ are commensurable numbers. Let us take, then, $p=m\phi$ and $\sigma=n\phi$, where m and n are integers relatively prime. Any term in the solution, $\frac{\cos}{\sin}(ip\pm j\sigma)\tau$, can therefore be written $\frac{\cos}{\sin}(im\pm jn)\phi\tau$. If m and n are both odd, then $(im\pm jn)$ is necessarily odd and the solution contains only odd multiples of $\phi\tau$. The projection of the path described by the pendulum upon the rotating $\xi\eta$-plane is therefore symmetrical with respect to both the ξ-axis and the η-axis. In the fixed xy-plane, however, since

$$x=l[\xi\cos\sigma\tau+\eta\sin\sigma\tau],$$
$$y=l[-\xi\sin\sigma\tau+\eta\cos\sigma\tau],$$

the expressions for x and y will contain only even multiples of $\phi\tau$. Therefore the xy-curve described in the interval $\frac{\pi}{\phi}\leqq\tau\leqq\frac{2\pi}{\phi}$ is identical with that described in the interval $0\leqq\tau\leqq\frac{\pi}{\phi}$, though in the rotating plane they are distinct. Since x is a cosine series and y is a sine series, the orbit is symmetrical with respect to the x-axis, and it has a cusp at the point for which $\tau=0$. The pendulum is again at this cusp when $\tau=\frac{\pi}{\phi}$. Since the orbit is symmetrical with respect to the x-axis, it follows that at $\tau=\frac{\pi}{2\phi}$ the orbit again crosses the x-axis perpendicularly or at a cusp.

If m and n are not both odd (they cannot both be even since they are relatively prime), then the expressions for ξ and η in the rotating plane contain both even and odd multiples of $\phi\tau$, and the orbit, while symmetrical with respect to the ξ-axis, is not symmetrical with respect to the η-axis.

The period of the rotating plane is $\frac{2\pi}{n}=\frac{2\pi}{\sigma}(1+\varkappa)$. It is seen from (23) that, for small values of μ, $\varkappa$ is approximately $+\frac{3}{8}\mu^2$. Consequently the period of the rotating plane increases as the amplitude of the pendulum increases.

The mean period of oscillation of the pendulum is $\frac{2\pi}{p}$. In order to find the actual period, which will differ but little from $\frac{2\pi}{p}$, we will take $\tau=\frac{2n\pi}{p}+\epsilon_n$ and

impose the condition that $r^2=\xi^2+\eta^2$ is a maximum, *i. e.*, $\dfrac{dr^2}{d\tau}=0$. This condition gives the equation

$$0=\left[-\frac{(p^2-\sigma^2)}{p}\sin 2p\epsilon_n\right]\mu^2+\left[\frac{12(p^2-\sigma^2)\sigma\sigma_1}{p(9p^2-\sigma^2)}\sin 2p\epsilon_n\right.$$
$$\left.-\frac{8(3p^2+\sigma^2)\sigma\sigma_1}{9p^2-\sigma^2}\cos 2n\pi\frac{\sigma}{p}+\frac{32\sigma^2\sigma_1}{9p^2-\sigma^2}\sin 2n\pi\frac{\sigma}{p}\right]\mu^3+\ldots,$$

from which is obtained

$$\epsilon_n=\left[\frac{16\sigma^2\sigma_1}{(p^2-\sigma^2)(9p^2-\sigma^2)}\sin 2n\pi\frac{\sigma}{p}\right]\mu+\ldots.$$

Substituting $\tau=\dfrac{2n\pi}{p}+\epsilon_n$ in the expressions for ξ and η, we have the coordinates of the apse, or the end of the oscillation. The expression for the η-coordinate of the apse is

$$\eta_a=\left[\frac{\sigma}{p}\sin\left(\frac{32\sigma^2\sigma_1 p}{(p^2-\sigma^2)(9p^2-\sigma^2)}\sin 2n\pi\frac{\sigma}{p}\right)\mu\right]\mu+\ldots.$$

Since σ and σ_1 each carry ω as a factor, this quantity is of the order ω^4. Thus the line of apsides does not rotate with uniform angular motion, but its departure from uniformity is exceedingly small.

At first thought, it might seem that this solution should reduce at the north pole to that of the simple pendulum referred to rotating axes, and consequently the rotating axes should have the period of 24 hours exactly. On second thought, however, it is seen that at the poles the Foucault pendulum becomes a spherical pendulum referred to rotating axes, for initially the pendulum is at rest with respect to the earth and therefore in motion with respect to fixed axes. Since this motion is counter-clockwise when the pendulum is released, and since in the motion of the spherical pendulum the line of apsides rotates in the same direction as the motion of the pendulum itself, it follows that the line of apsides rotates in the same direction as the earth itself. Therefore it takes the earth more than 24 hours to come back to the same position with respect to the pendulum. The expression for the period of the rotating axes is $P=(1+\varkappa_2\mu^2+\ldots)24^h$, and the value of $\varkappa_2$ at the pole is positive.

§ 5. *Foucault's Pendulum with Initial Impulse.*

If, however, the pendulum is started from relative rest, in which it hangs freely from its point of suspension, with an initial impulse, the solution so derived should reduce to the simple pendulum at the poles. By exactly the

same method as has been used above, a periodic solution can be obtained, provided the initial impulse be directed towards the east or towards the west.

With the same notation as before, let us suppose that the initial conditions are

$$\xi=\xi'=\eta=0, \quad \eta'=p\mu,$$

which implies an initial impulse towards the east. The solution is found to be

$$\xi=[0]\mu+\left[\frac{4p^2\sigma\sigma_1}{(p^2-\sigma^2)(9p^2-\sigma^2)}\cos p\tau+\frac{p\sigma_1}{2(p+\sigma)(3p+\sigma)}\cos(2p+\sigma)\tau\right.$$
$$\left.-\frac{p\sigma_1}{2(p-\sigma)(3p-\sigma)}\cos(2p-\sigma)\tau\right]\mu^2+\ldots\ldots,$$

$$\eta=[\sin p\tau]\mu+\left[\frac{4p\sigma^2\sigma_1}{(p^2-\sigma^2)(9p^2-\sigma^2)}\sin p\tau-\frac{\sigma\sigma_1}{(p^2-\sigma^2)}\sin\sigma\tau\right.$$
$$\left.-\frac{\sigma\sigma_1}{2(p+\sigma)(3p+\sigma)}\sin(2p+\sigma)\tau+\frac{\sigma\sigma_1}{2(p-\sigma)(3p-\sigma)}\sin(2p-\sigma)\tau\right]\mu^2+\ldots\ldots,$$

$$\lambda=\left[\frac{p^2+3\sigma^2}{8(p^2-\sigma^2)}-\frac{(11p^2-3\sigma^2)\sigma_1^2}{(p^2-\sigma^2)(9p^2-\sigma^2)}\right]\mu^2+\ldots\ldots,$$

$$\varkappa=\left[\frac{4p^2\sigma_1^2}{(p^2-\sigma^2)(9p^2-\sigma^2)}\right]\mu^2+\ldots\ldots.$$

The period of the rotating axes is $P=(1+\varkappa_2\mu^2+\ldots\ldots)24^h$; and since $\sigma_1=\omega\cos\beta$, it is seen that this is 24 hours exactly, at the poles, as it should be.

University of Chicago, *February* 20, 1914.

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Invariantive Theory of Plane Cubic Curves Modulo 2.

By L. E. Dickson.

§ 1. *Introduction.*

The ten types of plane cubic curves in ordinary geometry have been characterized by invariants and covariants by Gordan.* The types in modular geometry can be characterized by invariants only, the abundance of invariants making it unnecessary to resort to covariants. The most effective theory of modular invariants is that based upon a separation of the particular cases of the form in question into classes of equivalent forms.

For the present problem of cubic curves modulo 2, this classification is effected in § 3 by means of the real points (*i. e.*, points with integral coördinates) on the cubic, supplemented by a determination of the real inflexion points and the real and imaginary singular points. While we could test directly each real point on the curve, not a singular point, and find whether or not it is an inflexion point, we have completed the geometrical investigation by making a determination of all of the real and imaginary inflexion points on each of the twenty-two types of cubic curves modulo 2. For this purpose we have set up in § 2 a cubic function H, which here plays a rôle analogous to that played by the Hessian in the algebraic theory.

From the geometrical classification of the modular cubics we easily derive in § 4 a fundamental system of modular invariants.

The methods employed in this paper are applicable to other problems of this nature; they indicate the decided advantage to be gained in the theory of modular invariants from modular geometry as developed by Bussey and Veblen, Coble and the writer.

§ 2. *Inflexion Points on Cubic Curves.*

Let there be a single intersection of

$$x_3 = ax_1 + bx_2 \tag{1}$$

with the cubic $u(x_1, x_2, x_3) = 0$. Then

$$u(x_1, x_2, ax_1 + bx_2) = U(x_1, x_2) = \alpha x_1^3 + \beta x_1^2 x_2 + \gamma x_1 x_2^2 + \delta x_2^3 \tag{2}$$

* *Transactions Amer. Math. Soc.*, Vol. I (1900), p. 402.

is the cube, modulo 2, of $\alpha x_1+\delta x_2$, so that $\beta\equiv\gamma\equiv\alpha\delta$. Thus the point on (1) with $x_1=\delta$, $x_2=\alpha$, is an inflexion point. For these values,

$$\tfrac{1}{2}\frac{\partial^2 U}{\partial x_1^2}=3\alpha x_1+\beta x_2,\quad \tfrac{1}{2}\frac{\partial^2 U}{\partial x_1\partial x_2}=\beta x_1+\gamma x_2,\quad \tfrac{1}{2}\frac{\partial^2 U}{\partial x_2^2}=\gamma x_1+3\delta x_2$$

all reduce to zero modulo 2. By (2),

$$\frac{\partial^2 U}{\partial x_1^2}=\frac{\partial^2 u}{\partial x_1^2}+2a\frac{\partial^2 u}{\partial x_1\partial x_3}+a^2\frac{\partial^2 u}{\partial x_3^2},\quad \frac{\partial^2 U}{\partial x_2^2}=\frac{\partial^2 u}{\partial x_2^2}+2b\frac{\partial^2 u}{\partial x_2\partial x_3}+b^2\frac{\partial^2 u}{\partial x_3^2},$$
$$\frac{\partial^2 U}{\partial x_1\partial x_2}=\frac{\partial^2 u}{\partial x_1\partial x_2}+b\frac{\partial^2 u}{\partial x_1\partial x_3}+a\frac{\partial^2 u}{\partial x_2\partial x_3}+ab\frac{\partial^2 u}{\partial x_3^2}.$$

For each derivative in the second members, every term has an even coefficient except the term derived from $x_1x_2x_3$. We set

$$u=v+kx_1x_2x_3,\quad \xi_i=\tfrac{1}{2}\frac{\partial^2 v}{\partial x_i^2},\quad \eta_1=\tfrac{1}{2}\frac{\partial^2 v}{\partial x_2\partial x_3},\quad \ldots\ldots \tag{3}$$

Thus, at our inflexion point,

$$\xi_1+akx_2+a^2\xi_3\equiv 0,\quad \xi_2+bkx_1+b^2\xi_3\equiv 0, \tag{4}$$

$$\eta_3+b\eta_2+a\eta_1+ab\xi_3+kx_3\equiv 0 \pmod 2. \tag{5}$$

Multiply (4_1) by x_1^2, (4_2) by x_2^2, add and apply (1). We get

$$\xi_1x_1^2+\xi_2x_2^2+\xi_3x_3^2+kx_1x_2x_3\equiv 0. \tag{6}$$

This is equivalent modulo 2 to $u\equiv 0$. We therefore drop (4_1).

Eliminating a between (1) and (5), and then b between the resulting congruence and (4_2), we find, after simplifications by use of (6) and a final deletion of the factor x_1^2, that

$$H=\xi_1\xi_2\xi_3+\Sigma\xi_i\eta_i^2+k\Sigma x_i\xi_i\eta_i+k\Sigma x_1\eta_2\eta_3+k^2\Sigma x_1x_2\eta_3+k^3x_1x_2x_3\equiv 0. \tag{7}$$

Evidently H is a covariant of u with respect to any interchange of variables.* Let $x_1=x_1'+x_2'$, $x_2=x_2'$, $x_3=x_3'$ replace u by u', and let c be the coefficient of $x_1^2x_3$ in u. If we form the function H for u' (*i. e.*, for the transformed variables and coefficients), we find that it equals $H+(c^2+kc)u$ modulo 2, formally in the initial variables and coefficients. Hence, *the system of equations* $u=0, H=0$ *is invariant under every linear transformation with integral coefficients modulo* 2.

Thus the inflexion and singular points of $u=0$ are given by its intersections with $H=0$, so that H here plays a rôle analogous to that of the Hessian in the theory of algebraic curves.

---

* Hence it was no restriction to take the coefficient of x_3 in (1) to be not zero.

§ 3. *Cubic Curves Classified Geometrically.*

We shall find that two cubic curves with integral coefficients modulo 2 are equivalent under the group G of all linear transformations with integral coefficients modulo 2 if and only if they have the same number of real points, real inflexion points and real or imaginary singular points. In § 4, we deduce criteria by invariants.

The seven real points modulo 2 shall be designated

$$1=(100),\ 2=(010),\ 3=(001),\ 4=(110),\ 5=(101),\ 6=(011),\ 7=(111).$$

At these points the values of

$$C=ax^3+by^3+cz^3+dx^2y+exy^2+fx^2z+gxz^2+hy^2z+iyz^2+jxyz$$

are respectively

$$a,\ b,\ c,\ a+b+d+e,\ a+c+f+g,\ b+c+h+i,\ s, \tag{8}$$

where s is the sum of the ten coefficients of C. Denote by C_r a form C for which $C\equiv 0 \pmod 2$ contains exactly r real points.

Case $r=0$. Each function (8) is unity modulo 2, whence

$$C=x^3+y^3+z^3+dx^2y+(d+1)xy^2+fx^2z+(f+1)xz^2+hy^2z+(h+1)yz^2+xyz.$$

Replacing x by $x+dy+fz$, we get a C with $d=f=0$. Then replacing y by $y+hz$, we have also $h=0$ and get

$$C_0=x^3+y^3+z^3+xy^2+xz^2+yz^2+xyz, \quad H=C_0. \tag{I}$$

Its singular points are $(1, z^2, z)$, where $z^3+z+1\equiv 0$ is irreducible modulo 2. It can be shown that C_0 is the product of three imaginary linear functions.

Case $r=1$. After applying a transformation of G, we have a C the only real point on which is 1. In fact, $1=(100)$ is transformed into (α, β, γ) by the linear transformation having α, β, γ as the coefficients of x in x', y', z'. Thus $a=0$, while the remaining functions (8) are unity, so that

$$C=y^3+z^3+d(x^2y+xy^2)+f(x^2z+xz^2)+hy^2z+(h+1)yz^2.$$

The transformations leaving point 1 fixed are

$$x'=x+ry+tz, \quad y'=\alpha y+\beta z, \quad z'=\gamma y+\delta z. \tag{9}$$

By the interchange of y and z, d and f are permuted, while h is replaced by $h+1$. For $y'=y+z$, we get $f'=f+d$, $h'=h+1$. Unless $d=f=0$, we may therefore set $d=1$, $f=0$; then, replacing x by $x+hz$, we have also $h=0$ and get

$$C_1=y^3+z^3+x^2y+xy^2+yz^2, \quad H=x^2y+xy^2+x^2z+yz^2+xyz. \tag{II}$$

To eliminate x, note that

$$y(C_1+H)+zC_1 \equiv y^4+y^3z+z^4 \pmod 2.$$

The nine inflexion points* are $1=(100)$ and $(\xi, 1, \zeta)$, where

$$\zeta^4+\zeta+1\equiv 0, \quad \xi^2+\xi+\zeta^3+\zeta^2+1\equiv 0,$$

of which the first is irreducible modulo 2, while the second has no root rational in ζ, since $\xi^{16}+\xi\equiv 1$. The coördinates of the eight imaginary inflexion points are in the Galois field of order 2^8. There is no singular point.

If $d=f=0$, we may set $h=0$, after interchanging y and z when necessary, and get

$$C_1'=y^3+z^3+yz^2, \quad H\equiv 0. \tag{III}$$

This binary form represents three intersecting imaginary lines. The point 1 is the only singular point.

Case $r=2$. Since (9) replaces (010) by (r, α, γ), where α and γ are not both zero, any pair of real points can be transformed into the pair 1, 2. If the latter are the only real points on it, the cubic is

$$C=z^3+dx^2y+(d+1)xy^2+f(x^2z+xz^2)+h(y^2z+yz^2)+xyz.$$

After interchanging x and y if necessary, we may set $d=0$. Then replacing x by $x+hz$, we have also $h=0$, and get

$$C_2=z^3+xy^2+x^2z+xz^2+xyz, \quad H=x^3+xz^2+y^2z+yz^2; \tag{IV}$$

$$C_2'=z^3+xy^2+xyz, \qquad H=xy^2+y^2z+yz^2+xyz. \tag{V}$$

The nine inflexion points of C_2 are (010), $(x, y, 1)$, where

$$x^4+x+1\equiv 0, \quad y^2+y\equiv x^3+x.$$

There is no singular point. Since

$$C_2'+H\equiv z(y^2+yz+z^2),$$

the three inflexion points of C_2' are (010), $(1, y, 1)$, where $y^2+y+1\equiv 0$, while the only singular point is (100).

Case $r=3$. As two of the points we may take 1 and 2. First, let the third real point on C be 4, the only real point collinear with 1 and 2. Then

$$C=z^3+d(x^2y+xy^2)+f(x^2z+xz^2)+h(y^2z+yz^2).$$

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* The imaginary ones are determined for the sake of completeness; but a knowledge of them is not essential for our present purposes.

If $d=1$, replace x by $x+hz$ and y by $y+fz$ to make $f=h=0$:

$$C_3=z^3+x^2y+xy^2, \quad H=(x^2+y^2+xy)z. \tag{VI}$$

There is no singular point. The nine inflexion points are 1, 2, 4, $(k, 1, 1)$, $(1, k, 1)$, $(k+1, k, 1)$, where $k^2+k+1\equiv 0$. If $d=f=h=0$, we have

$$C_3'=z^3, \quad H\equiv 0. \tag{VII}$$

The only singular points are (100), $(\alpha, 1, 0)$. If $d=0$, while f and h are not both zero, we may set $f=1$ after interchanging x and y when necessary. Replacing x by $x+hy$, we have also $h=0$ and get the binary form

$$C_3''=z^3+x^2z+xz^2, \quad H\equiv 0, \tag{VIII}$$

with the single singular point 2.

Second, let the three real points on C be 1, 2 and a point not collinear with them and hence having $z=1$. Since

$$x'=x+cz, \quad y'=y+sz, \quad z'=z \tag{10}$$

leaves 1 and 2 fixed and replaces 3 by $(c, s, 1)$, we can transform the three real points into 1, 2, 3. Thus

$$C=dx^2y+(d+1)xy^2+fx^2z+(f+1)xz^2+hy^2z+(h+1)yz^2. \tag{11}$$

Interchanging x and y if necessary, we may set $d=0$. If $f=h=1$, the substitution (x, y, z) replaces C by a similar C with $d=f=h=0$. If $f=1$, $h=0$, we have

$$K_3=xy^2+x^2z+yz^2, \quad H=x^3+y^3+z^3+xyz. \tag{IX}$$

Eliminating x^3 and then x^2, we get $xy^4=z^5$. We see that the inflexion points are $(z^5, 1, z)$, where $z^9+z^3+1=0$, with the irreducible factors z^3+z^2+1, $z^6+z^5+z^4+z^2+1$. But if $f=0$, we may set $h=0$, after interchanging y and z if necessary, and get

$$K_3'=xy^2+xz^2+yz^2, \quad H=xz^2+xy^2+y^3. \tag{X}$$

They intersect in just two points, the singular point 1 and inflexion point 3.

Case $r=4$. The three real points not on $C=0$ may be taken, as in case $r=3$, to be any three collinear points, as 4, 5, 6; or any three non-collinear points, as 5, 6, 7.

In the first case, the only real points on C are 1, 2, 3, 7, and C is $C'+xyz$, where C' is given by (11). Since the normalizations of C' were effected by

permutations of the variables, we have merely to add xyz to the canonical forms (IX) and (X) to get provisional canonical forms $K_4=K_3+xyz$ and $K_3'+xyz$. But $y'=y+x$, $z'=z+x$ replaces the latter by the former:

$$K_4=xy^2+x^2z+yz^2+xyz,\quad H=x^3+y^3+z^3+xy^2+x^2z+yz^2. \tag{XI}$$

The sum of the two is

$$x^3+y^3+z^3+xyz\equiv(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z),$$

where $\omega^2+\omega+1\equiv0$ (mod 2). If the first factor is zero, we get the three inflexion points $(z+1, 1, z)$, where $z^3+z+1=0$. If the second factor is zero, K_4 becomes, after x is eliminated, $\omega(y+z)^3$. To treat the third factor, we have only to replace ω by ω^2. In each of the latter cases we obtain the unique singular point (111).

In the second main case, the only real points on C are 1, 2, 3, 4. Thus

$$C=d(x^2y+xy^2)+fx^2z+(f+1)xz^2+hy^2z+(h+1)yz^2+xyz.$$

For $f=h=0$, C is transformed into a like C with $f=1$, $h=0$, by replacing y by $x+y$. The new C is transformed into one with $f=0$, $h=1$ by (xy). Hence we may set $f=h$ and obtain the four types

$$C_4=xz^2+yz^2+xyz,\quad H=C_4+z^3; \tag{XII}$$

$$C_4'=C_4+x^2y+xy^2,\quad H=x^3+y^3+z^3+x^2y+xy^2+xyz; \tag{XIII}$$

$$C_4''=x^2z+y^2z+xyz,\quad H\equiv0; \tag{XIV}$$

$$C_4'''=C_4''+x^2y+xy^2,\quad H=x^2y+xy^2. \tag{XV}$$

The only singular points of C_4 are 1 and 2. There is no singular point on C_4'; for $z=0$, H vanishes only at (110), an inflexion point; for $z\not\equiv0$, we obtain the eight imaginary inflexion points $(x, y, 1)$, where

$$y^8+y^7+y^6+y^5+y^2+y+1\equiv0,\quad x=\frac{y^5+y}{y^4+y^2+1},$$

the congruence in y being irreducible modulo 2. The singular points on C_4'' are 3 and $(1, k, 0)$, $k^2+k+1\equiv0$ (mod 2). The only singular point on C_4''' is 3; the only inflexion points are 1, 2, 4.

Case $r=5$. The real points not on C may be taken to be 1, 2. Then

$$C=x^3+y^3+d(x^2y+xy^2)+fx^2z+(f+1)xz^2+hy^2z+(h+1)yz^2.$$

It is transformed by (10) into one with

$$f'=f+c+ds, \quad h'=h+s+dc.$$

If $d=0$, we take $c=f$, $s=h$, and have $f'=h'=0$. Then

$$C=x^3+y^3+xz^2+yz^2.$$

Replacing x by $x+y$ and z by $x+y+z$, we get

$$C_5=x(xy+z^2), \quad H=x^3. \tag{XVI}$$

The only singular point is 2. Next, let $d=1$; we take $s=0$, $c=f$ and have $f'=0$. If $h=0$, $C=(x+y)(x^2+y^2+z^2)$; replacing x by $x+y$ and z by $z+x$, we get

$$C_5'=xz^2, \quad H\equiv 0, \tag{XVII}$$

and see that any point with $z=0$ is singular. If $h=1$,

$$C=(x+y)^3+xz^2+y^2z.$$

Replacing x by $x+y$, we get

$$C_5''=x^3+(x+y)z^2+y^2z, \quad H=z^3+xy^2+xz^2+x^2z+xyz. \tag{XVIII}$$

Since $xC_5''+zH=x^4+xz^3+z^4$, the nine inflexion points are (010) and $(x, y, 1)$, where $x^4+x+1=0$, $y^2+y=x^3+x$. From the square of the latter we get $x=y^4+y$, whence $y^8+y^6+y^5+y^4+y^3+y+1\equiv 0$, which is irreducible modulo 2.

Case $r=6$. Let 7 be the real point not on C. Then

$$C=d(x^2y+xy^2)+f(x^2z+xz^2)+h(y^2z+yz^2)+xyz. \tag{12}$$

Replacing x by $x+y+z$, we get a C with $h'=h+d+f+1$. Again, any two of the coefficients d, f, h are interchanged by a suitably chosen interchange of the variables. Hence, if two of the coefficients are zero, we may set $d=f=0$; then either h itself is zero or else h' is zero, and we get

$$C_6=xyz=H, \tag{XIX}$$

with the singular points 1, 2, 3.

If every coefficient is unity, then $h'=0$. Thus there remains only the case in which just one coefficient is zero. We may then take $d=0$, $f=h=1$; replacing x by $x+z$, we get

$$C_6'=z(x^2+xz+y^2+xy), \quad H=z^3. \tag{XX}$$

The singular points are $(1, k, 0)$, where $k^2+k+1\equiv 0$.

Case $r=7$. We have (12) without the term xyz. If the cubic is not identically zero, we may set $f=1$. Replacing x by $x+hy$, we have $f=1$, $h=0$. Replacing z by $z+dy$, we get

$$C_7=x^2z+xz^2, \quad H\equiv 0. \tag{XXI}$$

The only singular point is 2.

§ 4. *Classification by Invariants.*

Since the transformations of our group G permute the real points $1, \ldots, 7$, they permute the values (8). Hence any symmetric function of $a, \ldots, s$ is a formal invariant of C modulo 2. In particular, their elementary symmetric function E_k of degree k is an invariant. The value of E_1 is the coefficient of xyz.

Denote by C_r a form C for which the cubic curve $C\equiv 0 \pmod 2$ contains exactly r real points. For a C_r, exactly $7-r$ of the values $a, \ldots, s$ are unity and the others are zero. Thus $E_k=1$ or 0, according as $k=7-r$ or $k>7-r$. We may therefore employ the rational integral invariants E_k to show that a C_r and a C_r' are not equivalent under the group G if $r\neq r'$.

For a real point (x, y, z), the product

$$P=\left(1+\frac{\partial C}{\partial x}\right)\left(1+\frac{\partial C}{\partial y}\right)\left(1+\frac{\partial C}{\partial z}\right)$$

has the value 1 or 0 modulo 2, according as the point is or is not a singular point. Let ϵ_k be the elementary symmetric function of order k of the values of P at the seven real points. For a cubic with s real singular points, s of the values of P are unity and the other values are zero, so that $\epsilon_s=1$, $\epsilon_k=0\,(k>s)$, for this cubic. We may therefore use the rational integral invariants ϵ_k to show that two cubics are not equivalent under G if the numbers of their real singular points are different.

To account for the total number of real singular and inflexion points, we may make a similar use of the symmetric functions of the values of $(1+C)(1+H)$ at the seven real points.

Corresponding formal invariants of C may be found by employing, in place of the real points, the points whose coördinates are in the Galois field of a given order 2^n.

The twenty-two classes of cubic curves under G, including the cubic which is identically zero, can therefore be characterized by rational integral invariants of the preceding types. In view of a general theorem (Madison Colloquium Lectures, 1914, p. 14), these invariants form a fundamental system of modular invariants of C modulo 2.

§ 5. *Classes of Cubic Curves under Imaginary Transformations.*

In what follows, let $a^2+a+1=0$, $b^2+b+1=0$. Then C_0 has the three factors $x+ay+a^2z$ and is equivalent to C_6. The latter is true of C_4'', with the factors z, $x+by$, $x+b^2y$. From C_7 written in capitals, we obtain C_8'' by setting $X=x+bz$, $Z=x+b^2z$, whence $X+Z=z$; while we obtain C_1' by setting $X=ay+a^2z$, $Z=a^2y+a^4z$, whence $X+Z=a^4y+az$.

We pass to the curves C for which C and H are linearly independent. The two real types having only two singular points are C_4 and C_6'. The latter written in capitals becomes C_4 if $X=x+y$, $Y=kx+k^2y$, $Z=z$. Of the five real types with one singular point, C_5 has no inflexion point (being a conic and a tangent), K_8' has a single inflexion, while C_2', K_4 and C_4'''' have three inflexions. Now C_2' in capitals becomes C_4'''' if $X=x+z$, $Y=bx+y$, $Z=x$, and K_4 in capitals becomes C_4'''' if

$$X=ax+a^2y+z,\ Y=a^2x+(a+a^2)y+z,\ Z=(a+a^2)x+ay+z.$$

There remain six real types with nine inflexions; of these, C_1, C_3, K_8 and C_5'' have no xyz term and are equivalent to $\sigma=X^3+Y^3+Z^3$ (for example, C_3 becomes σ if $x=Y+Z$, $y=X+Y$, $z=X+Y+Z$); while C_2 and C_4' have such a term and are equivalent to $\sigma+tXYZ$, where $t\neq0$. Note that the coefficient of xyz in the general cubic C is a relative invariant of C modulo 2.

The resulting ten types under imaginary transformations are seen to be in complete accord with the ten of Gordan, after his coefficient 6 is deleted. Our last two types are his C_1, C_2. His C_3 is X^3+Y^3+XYZ and becomes our

C_2' for $X=y$, $Y=y+z$, $Z=x+z$. His C_4 is Z^3+XYZ and becomes our C_4 for $X=x+z$, $Y=y+z$, $Z=z$. His C_6 is XY^2+Z^3 and becomes our K_3' for $X=x+y$, $Y=y+z$, $Z=y$. His C_8 is equivalent to $(z+x)^3+x^3=C_3''$. His C_5, C_7, C_9, C_{10} are our C_6, C_5, C_5', C_3', respectively.

Although we might derive this list of ten types independently of the longer list of real types, it would be difficult to deduce therefrom the classes of real cubics under real transformation.

UNIVERSITY OF CHICAGO, *July*, 1914.

On the Projective Differential Classification of N-Dimensional Spreads Generated by ∞^1 Flats.

BY ARTHUR RANUM.

INTRODUCTION.

This paper is a continuation of the investigation begun in my former paper in the *Annali di Matematica Pura ed Applicata*, Vol. XIX (1912), pp. 205–249, "On the Projective Differential Geometry of N-Dimensional Spreads Generated by ∞^1 Flats." The chief goal aimed at here is the classification of m-spreads generated by a single infinity of $(m-1)$-flats (linear spreads) in n-dimensional space from the projective differential standpoint.

The methods developed and apparatus set up in the earlier paper, and more particularly those in Part I, are here applied to the problem of classifying these m-spreads by means of their most fundamental projective differential properties. So far as four and five dimensions are concerned, this classification is carried out with considerable detail in §§ 39–73. For the higher spaces the broader outlines alone are given (§§ 33–38). The first half of the paper (§§ 1–32) contains a variety of somewhat disconnected developments of a general nature, but all have a more or less direct bearing on the main subject that follows and are necessary as preparing the way for it. For the sake of brevity the earlier paper will be referred to as "P. D. G."

Very recently a paper by E. Bompiani has appeared in the *Rendiconti di Palermo*, Vol. XXXVII (1914), pp. 305–331, entitled "Alcune proprietà proiettivo-differenziali dei sistemi di rette negli iperspazi," which has rather close contact with the part of my paper dealing with ruled surfaces in F_4. He has there introduced two very useful numbers called the *indices of developability*, the second of which corresponds exactly to my classes (a), (b) and (c). He has not, however, carried out the classification so completely as I have.*

* In the *Transactions of the American Mathematical Society*, Vol. XVI (1915), pp. 89–110, I have made a further contribution to the theory of ruled surfaces and planar three-spreads in four-space, based on §§ 42–59 of this paper, and have made an application of the results to the sphere-geometry of cyclic surfaces in three-space.

TABLE OF CONTENTS.

I. ENCLOSED SPREADS.

1. Enclosed spreads were defined in "P. D. G.", § 17; about the only cases considered in that paper were those in which one spread encloses another belonging to the same tree. We now consider the general case.

Let $S = S^r_m$ enclose $\bar{S} = S^{\bar{r}}_{\bar{m}}$. Then the tangent spread $S^1 = S_{m+r}$ encloses $\bar{S}^1 = S_{\bar{m}+\bar{r}}$, and the focal spread $S_{1,0} = S_{m-r}$ encloses $\bar{S}_{1,0} = S_{\bar{m}-\bar{r}}$. Hence, the conditions

$$\left.\begin{array}{c} m \geqq r, \qquad \bar{m} \geqq \bar{r}, \qquad m \geqq \bar{m}, \\ m + r \geqq \bar{m} + \bar{r}, \qquad m - r \geqq \bar{m} - \bar{r}, \end{array}\right\} \tag{1}$$

and therefore

$$m - \bar{m} \geqq r - \bar{r} \geqq -(m - \bar{m}), \tag{2}$$

must be satisfied. It follows that *while the range of a spread may be greater or less than that of an enclosed spread, the difference between their ranges can not be greater than the difference between their dimensionalities.* It is easy to show by concrete examples that $r - \bar{r}$ can have any integral value from $m - \bar{m}$ to $-(m - \bar{m})$.

If $r - \bar{r} = m - \bar{m}$ and S^r_m encloses $S^{\bar{r}}_{\bar{m}}$, their focal spreads coincide.

If $r - \bar{r} = -(m - \bar{m})$ and S^r_m encloses $S^{\bar{r}}_{\bar{m}}$, their tangent spreads coincide.

2. If S is given and the conditions (1) are satisfied, there does not necessarily exist any spread $\bar{S}$ enclosed in it. For instance, if a spread S^2_4 expressed in normal form ("P. D. G.," § 44) is $[A_1, A_2;\ A'_1, A'_2]$, it encloses an infinite number of spreads S^2_3 and S^2_2, but no S^1_3 and no S^0_2.

But it is easy to see that

if the focal spread of a given S_m^r is of range s, and if $r - \bar{r} = m - \bar{m}$ and $r \geqq \bar{r} \geqq s$, there exists an $S_{\bar{m}}^{\bar{r}}$ enclosed in S_m^r. In particular, putting $\bar{r} = s$, we find that *an S_m^r whose focal spread is of range s has just one enclosed spread of dimensionality $m - r + s$ and range s*, and that is its special enclosed spread.*

More generally, an S_m^r whose first i focal spreads are of ranges $r_1, \ldots, r_i$, respectively, has only one enclosed spread of dimensionality $m - r - r_1 - \ldots - r_{i-1} + ir_i$ and range r_i, namely its i-th special enclosed spread.

if the tangent spread of a given S_m^r is of range t, and if $r - \bar{r} = \bar{m} - m$ and $r \geqq \bar{r} \geqq t$, there exists an $S_{\bar{m}}^{\bar{r}}$ enclosing S_m^r. In particular, putting $\bar{r} = t$, we find that *an S_m^r whose tangent spread is of range t has just one enclosing spread of dimensionality $m + r - t$ and range t*, and that is its special enclosing spread.*

More generally, an S_m^r whose first i tangent spreads are of ranges $r_1, \ldots,$ r_i, respectively, has only one enclosing spread of dimensionality $m + r + r_1 + \ldots + r_{i-1} - ir_i$ and range r_i, namely its i-th special enclosing spread.

Common Enclosed Spreads.

3. We define the *common enclosed spread* of two given spreads as the spread of highest dimensionality enclosed in them. It is the locus of the flats of intersection of their corresponding generators. Its directrices are those which are common to the two given spreads.

We define the *common enclosing spread* of two given spreads as the spread of lowest dimensionality enclosing them. It is the locus of the connecting flats of their corresponding generators. Its directrices are those curves which are linearly dependent on the directrices of the two given spreads.

Notice that the common enclosed spread is not the locus of all the common points of the two given spreads, but only of those which are common to the pairs of corresponding generators. It depends not merely on the given spreads themselves, but also on the form of their equations; that is, on the correspondence that has been established between their generators by the choice of the parameter ω in both cases. If the equations of one of the given spreads are altered by a transformation to a new parameter,

$$\bar{\omega} = f(\omega),$$

a new correspondence is set up, and a new common enclosed (enclosing) spread

* See "P. D. G.," § 31. Here "special enclosed (enclosing) spread" means what is there called "*first* special enclosed (enclosing) spread."

is obtained. In the later applications (§§ 8–12, 42–73) to the case where the given spreads belong to the same tree, this element of arbitrariness is not present.

*The sum of the dimensionalities of the common enclosed spread and the common enclosing spread is obviously equal to the sum of the dimensionalities of the given spreads.**

In view of "P. D. G.," § 29, we see that

if U is the common enclosed spread of S and T, and if $\overline{U}$, $\overline{S}$ and $\overline{T}$ are corresponding spreads of their respective trees, then $\overline{U}$ is enclosed in $\overline{S}$ and in $\overline{T}$, and therefore also in their common enclosed spread.

if V is the common enclosing spread of S and T, and if $\overline{V}$, $\overline{S}$ and $\overline{T}$ are corresponding spreads of their respective trees, then $\overline{V}$ encloses $\overline{S}$ and $\overline{T}$, and therefore also their common enclosing spread.

4. But we can say more than this. Let

$$U = [A_1, \ldots, A_a];$$

then we can put

$$S = [A_1, \ldots, A_a;\ B_1, \ldots, B_b],$$
$$T = [A_1, \ldots, A_a;\ C_1, \ldots, C_c],$$

and

$$V = [A_1, \ldots, A_a;\ B_1, \ldots, B_b;\ C_1, \ldots, C_c],$$

choosing the fundamental directrices in such a way that they are independent in all four cases. The fundamental directrices of the tangent spread

$$U^1 = [A_1, \ldots, A_a;\ A'_1, \ldots, A'_a]$$

are not, in general, independent; let a' be the number of independent linear relations connecting them; then, $0 \leqq a' \leqq a$. S^1, T^1 and V^1 will be affected by these same relations and also, in general, by others. Let $a' + b'$ be the number of independent relations connecting the fundamental directrices of S^1, and $a' + c'$ the number connecting those of T^1; then,

$$0 \leqq b' \leqq 2b, \quad a' + b' \leqq a + b,$$
$$0 \leqq c' \leqq 2c, \quad a' + c' \leqq a + c.$$

The fundamental directrices of V^1 will be connected by the $a' + b' + c'$ independent relations already mentioned, and also, in general, by d' others that are independent of one another and of the former relations. Evidently,

$$a' + b' + c' + d' \leqq a + b + c.$$

These latter d' relations are of the form

$$\phi(\ldots, A'_i, A_i, \ldots) + \chi(\ldots, B'_i, B_i, \ldots) + \psi(\ldots, C'_i, C_i, \ldots) = 0, \quad (3)$$

where neither the χ-function nor the ψ-function is identically zero.

* This holds even when one or more of the spreads is an $S_m{}^0 = F_{m-1}$, provided the word "dimensionality" be here taken to stand for the number m. See "P. D. G.," § 9.

The ranges of the four spreads U, S, T, V are equal to

$$a-a', \quad a-a'+b-b', \quad a-a'+c-c', \quad a-a'+b-b'+c-c'-d',$$

respectively; and the sum of the ranges of S and T is exactly d' units greater than the sum of the ranges of U and V. Hence, *the sum of the ranges of two spreads can not be less than the sum of the ranges of their common enclosed spread and their common enclosing spread.*

5. V^1 is obviously the common enclosing spread of S^1 and T^1; and ascending higher, we see that V^2 is the common enclosing spread of S^2 and T^2, etc. This theorem and its correlative may be stated as follows:

If U is the common enclosed spread of S and T, and if U_i, S_i and T_i are their respective i-th focal spreads, then U_i *coincides with the common enclosed spread of* S_i *and* T_i.

If V is the common enclosing spread of S and T, and if V^i, S^i and T^i are their respective i-th tangent spreads, then V^i *coincides with the common enclosing spread of* S^i *and* T^i.

6. On the other hand, the common enclosed spread of S^1 and T^1 encloses U^1, but does not, in general, coincide with it. By reason of the relations (3) it also encloses precisely d' directrices that are independent of each other and of those of U^1; and its dimensionality is d' units greater than that of U^1.

By the principle of "P. D. G.," § 16, every identical relation affecting S^1 gives rise to a directrix of the focal spread S_1 and *vice versa*. Hence, the dimensionalities of U_1; S_1, T_1, V_1 are equal to a', $a'+b'$, $a'+c'$, $a'+b'+c'+d'$, respectively; and the dimensionality of the common enclosing spread of S_1 and T_1 is $a'+b'+c'$, which is d' units less than that of V_1.

Comparing the last two paragraphs, we see

that the tangent spread U^1 of the common enclosed spread of S and T is enclosed in the common enclosed spread $\overline{U}$ of their tangent spreads,

that the focal spread V_1 of the common enclosing spread of S and T encloses the common enclosing spread $\overline{V}$ of their focal spreads,

and that *the difference between the dimensionalities of* $\overline{U}$ *and* U^1 *is equal to the difference between the dimensionalities of* $\overline{V}$ *and* V_1. Hence, if U^1 coincides with $\overline{U}$, V_1 coincides with $\overline{V}$, and conversely. A simple example in which U^1 does not coincide with $\overline{U}$ occurs in the discussion of ruled surfaces in F_4 (§§ 45–48), and is obtained by taking the spreads S_2^2 and S_2^1 for S and T, respectively.

The Extended Tree.

7. If two spreads belong to the same tree, their common enclosed (enclosing) spread may or may not belong to the tree. If not, it determines a *secondary tree* which may, however, have some spreads in common with the

primary tree. The totality of the spreads of a tree together with those of all its secondary trees we shall call an *extended tree.*

Secondary Trees Determined by Common Enclosing Spreads.

8. Following the notation of "P. D. G.," §§ 39–46, let us consider a spread S expressed in its normal form. By merely inspecting the array (24), of § 44, we can easily learn the structure of a large number of the spreads of the extended tree determined by S. Thus the two spreads $S_{\alpha\beta}$ and $S_{\lambda\mu}$,* for which

$$\alpha \geqq \beta \text{ and } \lambda \geqq \mu, \tag{4}$$

belong to the respective ascending series determined by two of the focal spreads of S, and are both enclosed in S. The fundamental directrices of each of these two spreads are those situated in a rectangle in the lower left-hand corner of the array (24). Their common enclosed spread is therefore another spread of the same kind, and belongs to the primary tree. Their common enclosing spread, however, will give us something new, provided neither spread encloses the other. To ensure this we first put

$$\lambda > \alpha, \quad 0 < r_{t-\lambda} < r_{t-\alpha}, \tag{5}$$

so that the two spreads may belong to distinct ascending series; $S_{\lambda\mu}$ can not then enclose $S_{\alpha\beta}$. Moreover, if

$$\alpha - \beta > \lambda - \mu, \tag{6}$$

$S_{\alpha\beta}$ can not enclose $S_{\lambda\mu}$.†

Hence, if the conditions (4), (5) and (6) are satisfied, the common enclosing spread T of $S_{\alpha\beta}$ and $S_{\lambda\mu}$ will belong to the extended tree determined by S without belonging to the primary tree. T is evidently of range $r_{t-\alpha}$. By varying $\alpha, \beta, \lambda, \mu$ in all possible ways, we obtain a fourfold system W of such spreads.

9. Consider the secondary tree determined by T. By the right-hand theorem of § 5, T^j is the common enclosing spread of $(S_{\alpha\beta})^j = S_{\alpha, \beta+j}$ and $(S_{\lambda\mu})^j = S_{\lambda, \mu+j}$. Hence, for $j = 1, \ldots, \lambda - \mu$, T^j belongs to the system W, and is of range $r_{t-\alpha}$. For higher values of j, a complete discussion of T^j would involve a consideration of the identical linear relations connecting its fundamental directrices, and we shall not go into it.

It is also clear that for $i = 1, \ldots, \beta$, T_i is the common enclosing spread of $(S_{\alpha\beta})_i = S_{\alpha, \beta-i}$ and $(S_{\lambda\mu})_i = S_{\lambda, \mu-i}$, and is of range $r_{t-\alpha}$ and belongs to W. Hence, the first $\lambda - \mu$ successive tangent spreads of T and its first β successive focal spreads, together with T itself, form a regular series of $\beta + \lambda - \mu + 1$ spreads of range $r_{t-\alpha}$.

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* See "P. D. G.," § 29.

† Cf. "P. D. G.," § 30.

Continuing the descending series, we find that for $i = \beta + 1, \ldots,$ $\beta + \lambda - \alpha - 1$, T_i is the common enclosing spread of $(S_{\alpha\beta})_i = S_{\alpha-\beta+i,0}$ and $(S_{\lambda\mu})_i = S_{\lambda,\mu-i}$, and belongs to W, if its range $r_{t-\alpha+\beta-i} > r_{t-\lambda}$. Otherwise it coincides with $S_{\lambda,\mu-i}$ and belongs to the primary tree. Further on, for $i = \beta + \lambda - \alpha, \ldots, \mu$, T_i surely coincides with $S_{\lambda,\mu-i}$, and is of range $r_{t-\lambda}$. In the final stage of the descent, for which $i = \mu + 1, \ldots, t - \lambda + \mu$, $T_i = S_{\lambda-\mu+i}$ and is of range $r_{t-i-\lambda+\mu}$. That is, the descending series determined by T here coalesces with that determined by S, although the corresponding terms do not coincide. The last spread of the series is $T_{t-\lambda+\mu} = S_t$, which is, naturally, of range $r_0 = 0$.

We see, therefore, that the descending series determined by T, consisting of $t - \lambda + \mu + 1$ distinct spreads, comprises *four radically different parts.* The first part is a regular series of $\beta + 1$ terms; the second part has $\lambda - \alpha - 1$ terms and is irregular; the third part is again regular, with $\alpha - \beta - \lambda + \mu + 1$ terms; and the fourth part is again irregular, with $t - \lambda$ terms.

10. Going back to a spread T_i belonging to the second part of this series, where $\beta < i < \beta + \lambda - \alpha$, we see that its j-th tangent spread T_{ij}, for all values of j, is the common enclosing spread of $(S_{\alpha\beta})_{ij} = S_{\alpha-\beta+i,j}$ and $(S_{\lambda\mu})_{ij} = S_{\lambda,\mu-i+j}$. For values of j not greater than $\lambda - \mu + i$, it is of range $r_{t-\alpha+\beta-i}$ and belongs, in general, to W.

It is now clear that all those spreads of W for which λ and $\alpha - \beta - \lambda + \mu \, (= \varkappa)$ are constant, and form, therefore, a twofold system, *belong to the same secondary tree*, and that this secondary tree is determined by the common enclosing spread of $S_{\varkappa,0}$ and $S_{\lambda\lambda}$, where

$$0 < \varkappa < \lambda \quad \text{and} \quad 0 < r_{t-\lambda} < r_{t-\varkappa}. \tag{7}$$

By varying λ and $\varkappa$ in all possible ways, subject to the condition (7), we obtain a twofold system of secondary trees determined by S.

Secondary Trees Determined by Common Enclosed Spreads.

11. By merely applying the principle of duality, we can discuss in a similar manner an entirely different part of the extended tree determined by a spread S. The results are briefly as follows: Let the spreads $S^{\alpha\beta}$ and $S^{\lambda\mu}$ satisfy the conditions (4), (6) and

$$\lambda > \alpha, \quad 0 < r_{t+\lambda} < r_{t+\alpha}.^* \tag{5'}$$

Their common enclosed spread T is of range $r_{t+\alpha}$ and does not belong to the primary tree determined by S. T_j is the common enclosed spread of $S^{\alpha,\beta+j}$

* As in "P. D. G.," § 68, $r_{t+\lambda}$ denotes the range of S^λ.

and $S^{\lambda,\mu+j}$, and, for $j=1,\ldots,\lambda-\mu$, is of range $r_{t+\alpha}$ and belongs to the system W of spreads T. T^i $(i=1,\ldots,\beta)$ is the common enclosed spread of $S^{\alpha,\beta-i}$ and $S^{\lambda,\mu-i}$, and is of range $r_{t+\alpha}$ and belongs to W. Hence, the first $\lambda-\mu$ successive focal spreads and the first β successive tangent spreads of T, together with T itself, form a regular series of range $r_{t+\alpha}$.

Continuing the ascending series, we see that for $i=\beta+1,\ldots,\beta+\lambda-\alpha-1$, T^i is the common enclosed spread of $S^{\alpha-\beta+i,0}$ and $S^{\lambda,\mu-i}$ and is of range $r_{t+\alpha-\beta+i}$; for $i=\beta+\lambda-\alpha,\ldots,\mu$, T^i coincides with $S^{\lambda,\mu-i}$ and is of range $r_{t+\lambda}$; and for $i=\mu+1,\ldots,u-\lambda+\mu$, $T^i=S^{\lambda-\mu+i}$ and is of range $r_{t+i+\lambda-\mu}$. The last spread of the series, $T^{u-\lambda+\mu}=S^u$, is of range $r_{t+u}=0$. Hence, the ascending series determined by T consists of four parts, containing $\beta+1$, $\lambda-\alpha-1$, $\alpha-\beta-\lambda+\mu+1$ and $u-\lambda$ spreads, respectively; *the first and third of these parts are regular, and the second and fourth irregular.*

If $i=\beta+1,\ldots,\beta+\lambda-\alpha-1$, and $j=1,\ldots,\lambda-\mu+i$, T^{ij} is the common enclosed spread of $S^{\alpha-\beta+i,j}$ and $S^{\lambda,\mu-i+j}$, and is of range $r_{t+\alpha-\beta+i}$ and belongs to W. Those spreads of the fourfold system W for which λ and $\alpha-\beta+\lambda-\mu$ $(=\kappa)$ are constant belong to the same secondary tree, which is determined by the common enclosed spread of $S^{\kappa,0}$ and $S^{\lambda\lambda}$, where

$$0<\kappa<\lambda \quad \text{and} \quad 0<r_{t+\lambda}<r_{t+\kappa}. \tag{7'}$$

By varying λ and κ, we obtain a twofold system of secondary trees determined by S.

12. Incidentally, it is clear that

if T is the common enclosing spread of $S_{\alpha\beta}$ and $S_{\lambda\mu}$, satisfying (4), T_{ij} is the common enclosing spread of $(S_{\alpha\beta})_{ij}$ and $(S_{\lambda\mu})_{ij}$, for all values of i and j.	if T is the common enclosed spread of $S^{\alpha\beta}$ and $S^{\lambda\mu}$, satisfying (4), T^{ij} is the common enclosed spread of $(S^{\alpha\beta})^{ij}$ and $(S^{\lambda\mu})^{ij}$, for all values of i and j.

II. THE ORDER OF AN ALGEBRAIC SPREAD.

13. If an m-spread S_m generated by ∞^1 $(m-1)$-flats in space of $n-1$ dimensions is algebraic, and if its range is >0, its *order* is the number of points it has in common, in general, with a fixed $(n-m-1)$-flat F_{n-m-1}. This $(n-m-1)$-flat must and can always be so chosen as to meet no generator in more than one point. Hence, the order of S_m is simply the number of its generators which meet F_{n-m-1}.

This definition is somewhat broader than the usual one, because it includes the case $m=n-1$, in which S_m, being generated by $(n-2)$-flats, is not a spread

in the ordinary sense of the word. In this case its order is the number of generators that pass through a fixed point F_0.

In the simple case in which S_1 is a plane curve and S_2 its tangent spread (the locus of its tangents), their respective orders, as so defined, agree with what are ordinarily called the order and class of the curve. If S_1 is a curve immersed in a three-flat, S_2 its tangent surface and S_3 its second tangent spread (the locus of its osculating planes), their respective orders are what Salmon calls the order, rank and class of the entire system.* It is evident that, in the theory of the algebraic spread, not only its order but the orders of its tangent and focal spreads, and indeed of all the spreads of its tree, must be taken into account.

The Order of a Rational Spread.

14. Let us now confine ourselves to the case in which $S_m = [A_1, \ldots, A_m]$ is a rational spread. Then the equations of every directrix A_i can be written in the form

$$X_1 = A_{i1}(\omega), \ldots, X_n = A_{in}(\omega),$$

where $A_{i1}(\omega), \ldots, A_{in}(\omega)$ are rational integral functions of the parameter ω having no common factor in ω. Let the fundamental directrices $A_1, \ldots, A_m$ be independent; then the m-rowed determinants of the matrix

$$\| A_{ij}(\omega) \| \qquad (i = 1, \ldots, m;\ j = 1, \ldots, n) \tag{8}$$

can not all vanish identically. Let $F_{n-m-1} = [C_1, \ldots, C_{n-m}]$ be the fixed $(n-m-1)$-flat referred to in § 13, and let the constant coordinates of C_i be $C_{i1}, \ldots, C_{in}$. A necessary and sufficient condition that F_{n-m-1} meet a generator of S_m is

$$\begin{vmatrix} A_{11}(\omega) & \ldots & A_{1n}(\omega) \\ \cdots & \cdots & \cdots \\ A_{m1}(\omega) & \ldots & A_{mn}(\omega) \\ C_{11} & \ldots & C_{1n} \\ \cdots & \cdots & \cdots \\ C_{n-m,1} & \ldots & C_{n-m,n} \end{vmatrix} = 0.$$

Using Laplace's expansion in terms of the first m rows, we obtain an equation whose degree in ω gives the number of generators of S_m that meet F_{n-m-1}. We must, however, exclude those indeterminate generators which result from the vanishing of a possible common factor of all the m-rowed determinants of (8); for those particular values of ω the corresponding points of the directrices A_i are not independent. Hence, *the order of S_m is equal to the highest degree in ω of the expanded m-rowed determinants of the matrix* (8), *after the removal of their common factors.*

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* "Analytic Geometry of Three Dimensions," Vol. I (1912), Art. 325, p. 336.

III. PROJECTION AND SECTION.

Restricted and Conical Spreads.

15. If an m-spread in $(n-1)$-dimensional space is enclosed in a q-flat but not in a $(q-1)$-flat, it is said to be *immersed* in the q-flat. If $q < n-1$, it is a *restricted* spread.

If a spread S encloses a p-flat F_p but no $(p+1)$-flat, we shall call it a *conical spread* (or *cone*) whose *vertex* is F_p. If $p = 0, 1$ or 2, S is a point-cone, line-cone or plane-cone, respectively. A non-conical spread may be included by putting $p = -1$.

If S is immersed in F_q, all the spreads of its tree lie in F_q. Its tangent spreads and all the spreads S^{ij} $(j \leq i)$ are themselves immersed in F_q, while its focal spreads and the other spreads of its tree may be restricted to smaller flats lying in F_q.

Dually, if S is a cone whose vertex is F_p, all the spreads of its tree are conical and enclose F_p. Its focal spreads and all the spreads S_{ij} $(j \leq i)$ have the same vertex F_p, while its tangent spreads and the other spreads of its tree may have larger vertices enclosing F_p.

The Effect of Projection and Section on the Structure of a Spread.

16. Let us consider the effect of projection and section on the structure of a spread. Two spreads, or the primary trees determined by them, will have the same *structure*, let us say, if these trees consist of the same number of spreads, with the same arrangement and distribution of ascending and descending series, and with equal ranges for corresponding spreads. The dimensionalities of corresponding spreads must, therefore, either be equal or differ by a constant.

Let S be an m-spread immersed in F_q; if S is conical, let its vertex be F_p.

If we project S from a fixed point P, the projecting spread V may be regarded as the common enclosing spread of S and $P = S_1^0$.

The spread of intersection U of S and a fixed hyperplane* Q may be regarded as the common enclosed spread of S and $Q = S_{n-1}^0$.

We first consider two extreme cases.

If S is a restricted spread and the point P lies outside of its enclosing flat F_q, V is a conical $(m+1)$-spread of the same structure as S. Its vertex is the connecting flat of P and F_p; it (V) is immersed in the connecting flat of P and F_q.

If S is a conical spread and the hyperplane Q does not enclose its vertex F_p, U is a restricted $(m-1)$-spread of the same structure as S. It is immersed in the flat of intersection of Q and F_q; its vertex (when it is conical) is the flat of intersection of Q and F_p.

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* A *hyperplane* is an $(n-2)$-flat.

This case includes the operations of projection and section in the ordinary or narrow sense of the words.

If S is conical and P lies in or coincides with F_p, V coincides with S.

If S is restricted and Q passes through or coincides with F_q, U coincides with S.

This trivial case is included merely for the sake of completeness. These are obviously the only two cases in which the structure of a spread is unaltered by projection from a point or section by a hyperplane.

17. In every other case

P is enclosed in F_q but not in F_p. P is then not enclosed in S, but must be enclosed in one of its tangent spreads S^{i+1}. This can happen in two ways. Either S^{i+1} is the highest tangent spread of the tree, of range zero, and coincides with F_q; or S^{i+1} is conical, with a vertex $F_{p'}$ $(p' > p)$ enclosing F_p, and P lies in $F_{p'}$ (but outside of F_p).

Q encloses F_p but not F_q. Q does not then enclose S, but must enclose one of its focal spreads S_{i+1}. This can happen in two ways. Either S_{i+1} is the lowest focal spread of the tree, of range zero, and coincides with F_p; or S_{i+1} is restricted to a flat $F_{q'}$ $(q' < q)$ enclosed in F_q, and Q encloses $F_{q'}$ (without enclosing F_q).

18. Let the point P be enclosed in S^{i+1}, in either one of these two ways, but not in S^i, and therefore not in $S^{i-1}, \ldots, S$. The range r_{i+1} of S^{i+1} must then be less than the range r_i of S^i. By the right-hand theorem of § 5, V^j is the common enclosing spread of S^j and P, for all values of j. Hence, for $j > i$, $V^j = S^j$; and for $j \leq i$, V^j is of dimensionality one unit higher than S^j. The range of V^i is $r_i - 1$, which is less than that of V^{i-1}. Consequently, the spread $V^{i\beta\gamma\delta\ldots}$, although enclosing the common enclosing spread of $S^{i\beta\gamma\delta\ldots}$ and P,* does not, in general, coincide with it; and this part of the tree determined by V has a different structure from the corresponding part of the tree determined by S.

But it is not difficult to see that every spread $V^{\alpha\beta}(0 \leq \alpha < i)$ coincides with the common enclosing spread of $S^{\alpha\beta}$ and P, has the same range as $S^{\alpha\beta}$, and has a dimensionality one unit higher. The same relation also holds between their tangent spreads $V^{\alpha\beta\gamma}$ and $S^{\alpha\beta\gamma}$, provided $S^{\alpha\beta\gamma}$ does not enclose P. There exists, however, a value j of γ, such that P is enclosed in $S^{\alpha\beta,j+1}$, but not in $S^{\alpha\beta j}$. Hence, $S^{\alpha\beta,j+1}$ can not be enclosed in S^i; and "P. D. G.," § 32, shows that $\alpha - \beta + j \geq i$.

* See §§ 3–6.

The same line of argument before applied to S^{i+1} and S^{i} can now be applied here. Thus, if $\gamma > j$, $V^{\alpha\beta\gamma} = S^{\alpha\beta\gamma}$; if $\gamma = j$, $V^{\alpha\beta\gamma}$ is of range one unit less, and dimensionality one unit greater, than $S^{\alpha\beta\gamma}$; and if $\gamma < j$, $V^{\alpha\beta\gamma}$ is of the same range as $S^{\alpha\beta\gamma}$ and dimensionality one unit greater. Starting with the latter spreads, in which $\gamma < j$, and taking their focal spreads $V^{\alpha\beta\gamma\delta}$, we can continue in this way, until we have exhausted the spreads of the entire tree.

We see, then, that the primary tree determined by the projecting spread V consists, in general, of *three essentially different parts*: (1) a part which coincides with the corresponding part of the tree determined by S; (2) a part which has the same structure as the corresponding part of the tree determined by S, but in which the dimensionalities are one unit higher; and (3) a part which has a radically different structure from the corresponding part of the tree determined by S. Part (1) consists of the partial trees determined by V^{i+1}, the various spreads $V^{\alpha\beta, j+1}$ $(\alpha < i)$, etc. Part (2) consists of those spreads $V^{\alpha\beta\gamma\ldots}$ for which $\alpha < i$, $\gamma < j$, etc. Part (3) consists of those portions of the trees determined by V^{i}, the various spreads $V^{\alpha\beta j}$ $(\alpha < i)$, etc., which are not included in part (1).

19. It may happen, however, that one or more subdivisions of part (3) coincide with portions of the tree determined by S, though not with the corresponding portions. For instance, if $r_{i+1} = r_i - 1$, V^{i} has the same range as its tangent spread $V^{i+1} = S^{i+1}$, and so coincides with the focal spread of the latter. Similarly, if the range of $S^{\alpha\beta, j+1}$ is one unit less than that of $S^{\alpha\beta j}$, $V^{\alpha\beta j} = S^{\alpha\beta, j+1, 1}$. If this holds for all the subdivisions of part (3), the entire tree determined by V has the same structure as that portion of the tree determined by S which remains after the excision of S^{i}, the various spreads $S^{\alpha\beta j}$, etc., and certain spreads determined by them.

Part (2) will evidently be absent, if, and only if, $i = 0$; in this case all the subdivisions of parts (1) and (3) except the first will also be absent.

20. Of the correlative discussion we shall merely state the principal results. Let the hyperplane Q enclose S_{i+1}, but not $S, \ldots,$ or S_i; also let Q enclose $S_{\alpha\beta, j+1}$ $(\alpha < i)$, but not $S_{\alpha\beta j}$; then $\alpha - \beta + j \geq i$.

The tree determined by the spread of intersection U of S and Q consists of three parts. Part (1) coincides with the corresponding part of the tree determined by S, and is made up of the partial trees determined by U_{i+1}, the spreads $U_{\alpha\beta, j+1}$ $(\alpha < i)$, etc. Part (2), consisting of those spreads $U_{\alpha\beta\gamma\ldots}$ for which $\alpha < i$, $\gamma < j$, etc., has the same structure as the corresponding part of the tree determined by S, but the dimensionalities of its spreads are one unit lower. Part (3), consisting of certain portions of the trees determined by U_i, the spreads $U_{\alpha\beta j}$ $(\alpha < i)$, etc., has a different structure from the corresponding part of the tree determined by S.

21. The following pairs of correlative theorems are easily deduced from what we have said.

If a spread S is projected from a point P by a spread V, and if in the tree determined by S every spread not enclosing P, whose tangent spread does enclose P, has a range exactly one unit greater than its tangent spread, then and only then does the tree determined by V have the same structure as a part, or the whole, of the tree determined by S.

If U is the spread of intersection of a given spread S and a hyperplane Q, and if in the tree determined by S every spread not enclosed in Q, whose focal spread is enclosed in Q, has a range exactly one unit greater than its focal spread, then and only then does the tree determined by U have the same structure as a part, or the whole, of the tree determined by S.

22. If P is enclosed in the immersing flat F_q of S, but not in any of the other spreads of the tree determined by S, then part (1) of the tree determined by V will reduce to the single flat F_q, and conversely.

If Q encloses the vertex F_p of S (when S is conical), but does not enclose any of the other spreads of the tree determined by S, then part (1) of the tree determined by U will reduce to the single flat F_p, and conversely.

If P is enclosed in F_q but not in any of the other spreads of the tree determined by S, and if every spread of this tree whose tangent spread is F_q is of range 1, then and only then does the entire tree determined by V have the same structure as a part of the tree determined by S, the dimensionalities of all its spreads being one unit higher, with the single exception of the immersing flat, which is F_q.

If Q encloses F_p but does not enclose any of the other spreads of the tree determined by S, and if every spread of this tree whose focal spread is F_p is of range 1, then and only then does the entire tree determined by U have the same structure as a part of the tree determined by S, the dimensionalities of all its spreads being one unit lower, with the single exception of the vertex, which is F_p.

23. If P is enclosed in S^1 but not in S, the tree determined by V consists of two parts, of which the first, determined by V^1, coincides with the corresponding part determined by S^1, and the second, determined by V, has a different structure from the corresponding part determined by S.

If Q encloses S_1 but not S, the tree determined by U consists of two parts, of which the first, determined by U_1, coincides with the corresponding part determined by S_1, and the second, determined by U, has a different structure from the corresponding part determined by S.

A necessary and sufficient condition that the entire tree determined by V coincide with a part of the tree determined by S, is that P be enclosed in S^1 but not in S, and that the range of S^1 be one unit less than that of S.

A necessary and sufficient condition that the entire tree determined by U coincide with a part of the tree determined by S, is that Q enclose S_1 but not S, and that the range of S_1 be one unit less than that of S.

IV. ENUMERATION OF ARBITRARY FUNCTIONS.

24. We shall now take up certain questions concerning the degree of generality of m-spreads generated by ∞^1 flats in F_{n-1}; this is determined by counting the number of independent and essential functions of the parameter ω which enter into their equations.

For instance, the general curve or one-spread in F_{n-1}, whose equations are

$$X_j = A_j(\omega), \quad j = i, \ldots, n,$$

depends apparently on n functions; but by means of the transformations (a) and (c), of "P. D. G.," § 2, two of these functions can be reduced to the values 1 and ω, respectively; the other $n-2$ functions are independent and essential. Using the notation of Kasner,* we may say that the number of curves in F_{n-1} is $\infty^{(n-2)f_1}$.

Similarly, the general m-spread

$$X_j = \sum_{i=1}^{m} a_i A_{ij}(\omega), \quad j = 1, \ldots, n,$$

involves the mn functions $A_{ij}(\omega)$; but m^2 of these functions can be reduced to constants by the transformation (b), of "P. D. G.," § 2, and one other function can be chosen as a new parameter ω by means of (a), of the same section; so we see that *the general m-spread S_m in F_{n-1} depends essentially on $m(n-m)-1$ arbitrary functions.* By arbitrary functions we mean functions which are continuous and possess a certain finite number of successive derivatives, but otherwise are arbitrary.

Geometrically, we obtain the same result by making use of the well-known† fact that the choice of every generating $(m-1)$-flat of S_m depends on $m(n-m)$ arbitrary constants; hence, a continuous system of ∞^1 such generators depends on $m(n-m)$ arbitrary functions of ω, one of which, again, can be chosen to replace ω.

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* *Bulletin of the American Mathematical Society*, Vol. XIX (October, 1912), p. 14.

† Cf. Schoute, "Mehrdimensionale Geometrie," erster Teil (1902), p. 253.

The range of the general m-spread in F_{n-1} is obviously the smaller of the two integers m and $n-m$. If n is even, the number $m(n-m)-1$ will be a maximum, when $m=\frac{n}{2}$; and if n is odd, it will be a maximum, when $m=\frac{n-1}{2}$ or $\frac{n+1}{2}$. Hence, the most general spread in F_{n-1}, if n is even, is $S^{n/2}_{n/2}$, which depends on $\frac{n^2-4}{4}$ arbitrary functions; and if n is odd, it is either $S^{(n-1)/2}_{(n-1)/2}$ or $S^{(n-1)/2}_{(n+1)/2}$, each of which depends on $\frac{n^2-5}{4}$ arbitrary functions.

25. If a spread S is of higher range than both its tangent spread S^1 and its focal spread S_1, it remains to a certain extent arbitrary, even when S^1 and S_1 are known; that is, when all the spreads of its primary tree except itself are known. For, putting $S=S^r_m$, $S^{1,0}=S^{r_1}_{m+r}$, $S_{1,0}=S^{r_{-1}}_{m-r}$,* we see that S can be chosen to be any spread enclosed in $S^{1,1}=S_{m+r-r_1}$* and enclosing $S_{1,1}=S_{m-r+r_{-1}}$, provided that the common enclosing spread of S and $S^{1,2}$ be precisely $S^{1,1}$ and not a lower spread enclosed in it, and provided that the common enclosed spread of S and $S_{1,2}$ be precisely $S_{1,1}$ and not a higher spread enclosing it. These provisos, however, are in general satisfied. In other words, the generators of S can in general be chosen to be any $(m-1)$-flats lying in the corresponding $(m+r-r_1-1)$-flats of $S^{1,1}$ and passing through the corresponding $(m-r+r_{-1}-1)$-flats of $S_{1,1}$.

Hence † the choice of each generator of S depends on $(r-r_1)(r-r_{-1})$ arbitrary constants. That is, *if there exists* ‡ *a spread of range r, having a given tangent spread of range r_1 and a given focal spread of range r_{-1}, there exist exactly $\infty^{(r-r_1)(r-r_{-1})f_1}$ such spreads.*

26. In a similar manner we can easily prove the following dual theorems, which may be regarded as an extension of "P. D. G.," § 51.

If a spread S^r_m, in F_{n-1}, has a given tangent spread of range r_1, the determination of S^r_m depends on $(r-r_1)m$ arbitrary functions.

If a spread S^r_m, in F_{n-1}, has a given focal spread of range r_{-1}, the determination of S^r_m depends on $(r-r_{-1})(n-m)$ arbitrary functions.

27. Let us now determine the degree of generality of S^r_m *per se*; that is, when neither its tangent spread nor its focal spread is known. In the first place, the general case is the one in which the ascending and descending series determined by S^r_m are both regular so far as the given values of m, r and n

* For the notation, see "P. D. G.," § 17. † Cf. Schoute, *loc. cit.* ‡ See "P. D. G.," §§ 51, 52.

will allow. That is, the spreads of the ascending (descending) series are all of range r except the last one, which is of range zero, and the next to the last one, which may be of range $<r$. For, any additional irregularity is due to the existence of additional linear relations which diminish the generality of the spread. Moreover, the different spreads of a regular series all have the same degree of generality.

Let $m=rk+r_1$, where k is an integer and $0\leqq r_1<r$. The $(k-1)$-st focal spread of S^r_m will be an $S^r_{r+r_1}$, in the general case, and the k-th focal spread an $S^{r_1}_{r_1}$. S^r_m will be uniquely determined by first choosing $S^{r_1}_{r_1}$, which, by the theorem of § 24, depends on $r_1(n-r_1)-1$ arbitrary functions, and then choosing $S^r_{r+r_1}$, which, by the right-hand theorem of § 26, depends on $(r-r_1)\,(n-r-r_1)$ additional arbitrary functions. Hence, the total number of arbitrary functions, all independent, on which S^r_m depends, is $r\,(n-r)-1$.

That is, *the general m-spread of range r in* F_{n-1} *is determined by* $r\,(n-r)-1$ *arbitrary functions.* Since this number is independent of r_1 and of m, it follows that the spreads $S^r_r, S^r_{r+1}, \ldots, S^r_{n-r}$ all have the same degree of generality.

28. We proceed to the consideration of spreads whose ascending or descending series is irregular. We first take the case where the descending series is irregular, and the ascending series regular (so far as possible).

THEOREM. *In* F_{n-1} *the most general spread* S^r_m $(r=r_0)$, *whose successive focal spreads are of ranges* $r_{-1}, r_{-2}, \ldots, r_{-(t-1)}, 0$, *whose t-th focal spread is a flat* S^0_{m-t}, *and whose tangent spreads are entirely arbitrary, is determined by* μ *arbitrary functions, where*

$$\mu=r\,(n-m_{-t}-r)-\sum_{i=-1}^{-(t-2)} r_{i-1}\,(r-r_i)-1=r\,(n-m)+\sum_{i=0}^{-(t-2)} r_i r_{i-1}-1. \qquad (9)$$

As to the proof, it is sufficient to remark that $S^r_m=S$ can be determined by first choosing $S_{t,0}$, then $S_{t-1,0}$, etc., and finally S itself. Since $S_{t,0}$ is a flat S^0_{-t}, its choice depends on $m_{-t}(n-m_{-t})$ arbitrary constants, and not on any arbitrary functions. When $S_{i+1,0}$ has been chosen, $S_{i,0}$, of dimensionality $m-\sum_{a=0}^{-(i-1)} r_a$, is restricted only by the fact that it must enclose $S_{i+1,1}$ and be enclosed in F_{n-1}; hence, if $t-1>i>0$, its choice depends on

$$(r_{-i}-r_{-(i+1)})\,(n-m+\sum_{a=0}^{-(i-1)} r_a)$$

arbitrary functions. When so chosen, it will, in general, have a regular series of tangent spreads $S_{ij}\,(j=1,\ldots,i)$, and will therefore satisfy the condition of "P. D. G.," § 52.

29. Dually, it is clear that *the most general spread S^r_m $(r = r_0)$, whose successive tangent spreads are of ranges $r_1, r_2, \ldots, r_{u-1}$, 0, whose u-th tangent spread is a flat $S^0_{m_u}$, and whose focal spreads are entirely arbitrary, is determined by μ' arbitrary functions, where*

$$\mu' = r(m_u - r) - \sum_{i=1}^{u-2} r_{i+1}(r - r_i) - 1 = rm + \sum_{i=0}^{u-2} r_i r_{i+1} - 1. \qquad (9')$$

30. We next proceed to prove that *if there exists a spread of range r, whose first tangent and focal spreads are of ranges r_1 and r_{-1}, respectively, and whose second tangent and focal spreads are given spreads of ranges r_2 and r_{-2}, respectively, then there exist just ∞^{k/t_1} such spreads, where*

$$\begin{aligned} k &= (r-r_1)(r-r_{-1}) + 2(r-r_1)(r_{-1}-r_{-2}) + 2(r-r_{-1})(r_1-r_2) + 3(r_1-r_2)(r_{-1}-r_{-2}) \\ &= r^2 + r(r_1+r_{-1}) - 2r(r_2+r_{-2}) - (r_1 r_{-2} + r_2 r_{-1}) + 3 r_2 r_{-2}. \end{aligned} \qquad (10)$$

Proof. The required spread $S = S^r_m$ can be determined by first choosing $S^{1,0}$, then $S_{1,0}$, and finally S itself. $S^{1,0}$ is the most general spread S_{m+r} enclosed in $S^{2,1} = S_{m+r+r_1-r_2}$ and enclosing $S_{2,3} = S_{m-r-r_{-1}+3r_{-2}}$, and so depends on $(r_1 - r_2)(2r + r_{-1} - 3r_{-2})$ arbitrary functions. Then $S_{1,0}$ is the most general spread S_{m-r} enclosing $S_{2,1} = S_{m-r-r_{-1}+r_{-2}}$ and enclosed in $S^{1\,2} = S_{m+r-2r_1}$, and so depends on $(r_{-1} - r_{-2})(2r - 2r_1)$ additional functions. Finally, S, by the theorem of § 25, depends on $(r - r_1)(r - r_{-1})$ additional functions. The sum of these numbers gives (10), and proves the theorem.

31. We are now prepared to take up the case where the ascending and descending series are both irregular.

Theorem. *In F_{n-1} the most general spread S^r_m $(r = r_0)$, whose successive tangent spreads are of ranges $r_1, r_2, \ldots, r_{u-1}$, 0, the last one being an $S^0_{m_u}$, and whose successive focal spreads are of ranges $r_{-1}, r_{-2}, \ldots, r_{-(t-1)}$, 0, the last one being an $S^0_{m_{-t}}$, is determined by λ arbitrary functions, where*

$$\begin{aligned} \lambda &= r(m_u - m_{-t} - r) - \sum_{i=1}^{u-2} r_{i+1}(r - r_i) - \sum_{i=-1}^{-(t-2)} r_{i-1}(r - r_i) - 1 \\ &= r^2 + \sum_{i=0}^{u-2} r_i r_{i+1} + \sum_{i=0}^{-(t-2)} r_i r_{i-1} - 1. \end{aligned} \qquad (11)$$

32. The method of proof will be sufficiently clear, if carried out for the special case in which $t = u = 3$, $m_{-t} = 0$, $m_u = n$; so that

$$n = 2r + (r_1 + r_2) + (r_{-1} + r_{-2}), \qquad (12)$$

where

$$r \geqq r_1 \geqq r_2 \text{ and } r \gtreqless r_{-1} \gtreqless r_{-2}. \qquad (13)$$

We first find the number of arbitrary functions that determine $S_2 = S^{r_{-2}}_{r_{-2}}$ and $S^2 = S^{r_2}_{n-r_2}$. The number of additional functions that determine S^1, S_1 and S will then be given by (10), § 30. When S_2 is chosen in the most general manner possible for the given values of the ranges, its fourth tangent spread $S_{2,4}$, which is enclosed in S^2, will be of dimensionality $5\,r_{-2}$, provided

$$n \geqq r_2 + 5\,r_{-2}. \tag{14}$$

In this case S_2 depends on $r_{-2}(n-r_{-2})-1$ functions, and S^2 on $r_2(n-r_2-5r_{-2})$ additional functions; so that together they depend on

$$r_2(n-r_2)+r_{-2}(n-r_{-2})-5\,r_2\,r_{-2}-1=(2\,r+r_1+r_{-1})\,(r_2+r_{-2})-3\,r_2\,r_{-2}-1 \tag{15}$$

arbitrary functions.

If (14) is not satisfied, this method fails; but we can proceed to choose S^2 first and then S_2. For, $S^{2,4}$ encloses S_2 and can be chosen to be of dimensionality $n-5\,r_2$, provided

$$n \geqq r_{-2} + 5\,r_2. \tag{14$'$}$$

In this case S^2 depends on $r_2(n-r_2)-1$ functions, and S_2 on $r_{-2}(n-r_{-2}-5\,r_2)$ additional functions; so that the number of functions on which both depend is given by the formula (15), as before. But one of the two conditions (14), (14′) must in every case be satisfied. For if not, we should have, by addition,

$$n < 3\,(r_2 + r_{-2}),$$

which would contradict (12) and (13).

Finally, by (15) and (10), we find that the total number of arbitrary functions on which the original spread S depends is

$$r(n-r)-r_2(r-r_1)-r_{-2}(r-r_{-1})-1 = r^2+(r\,r_1+r_1\,r_2)+(r\,r_{-1}+r_{-1}\,r_{-2})-1,$$

which proves formula (11) for the special case under consideration.

It is clear from (11) that the maximum value of λ for given values of n and r is $r\,(n-r)-1$, and that this value is obtained by putting $m_u = n$, $m_{-t}=0$ and $r_{u-2} = \ldots\ldots = r_1 = r = r_{-1} = \ldots\ldots = r_{-(t-2)}$. This verifies the theorem of § 27 from another point of view.

V. Classification.

33. In classifying spreads generated by ∞^1 flats from the projective differential standpoint, we shall employ three different kinds of classification: first, a broad classification, depending only on the structure of the two principal

series, one ascending and the other descending, determined by the given spread; second, a narrower and more detailed classification, taking account of the branch series as well, and depending on the structure of the entire primary tree determined by the spread; and third, a still more detailed classification, taking account of the common enclosed and enclosing spreads of the various spreads of the tree, and depending, therefore, on the structure of the extended tree determined by the spread.

If two spreads S_m and $T_{m'}$, of different dimensionalities $(m < m')$, have the same structure or belong to the same class according to any one of these three principles of classification, it is evident that by means of one or more projections or sections of the ordinary kind (see § 16), we can obtain from either one of the spreads a new spread of the same dimensionality and class as the other. Thus, if $m' = m + \mu$, then by μ successive projections we obtain from S_m a spread $S_{m'}$ of the same class as $T_{m'}$, and by μ successive sections we obtain from $T_{m'}$ a spread T_m of the same class as S_m.

34. For instance, in four-dimensional space F_4 all plane curves S_1^1 (not including straight lines), all cones T_2^1 immersed in three-flats, and all line-cones U_3^1 immersed in F_4, belong to the same class K_1. Their respective trees are

$$\left.\begin{array}{ccc} S_3^0 & T_4^0 & U_5^0 \\ S_2^1 & T_3^1 & U_4^1 \\ S_1^1 & T_2^1 & U_3^1 \\ S_0^0 & T_1^0 & U_2^0 \end{array}\right\}$$

where the order from the bottom up is in each case: the first focal spread, the given spread, the first tangent spread and the second tangent spread. T_1^0 is the point-vertex of T_2^1, T_3^1 is the system of tangent planes to T_2^1, T_4^0 is the three-flat in which T_2^1 is immersed, etc. Every S_1^1 projects into a T_2^1, and every T_2^1 into a U_3^1. Moreover, the spreads S_2^1, T_3^1 and U_4^1 belong to a class K_2 different from K_1; for although the tree determined by S_2^1 is the same as that determined by S_1^1, the corresponding spreads are not of the same range.

Since the principle of duality holds between the pairs of spreads S_1^1 and U_4^1, S_2^1 and U_3^1, T_2^1 and T_3^1, we can speak of K_1 and K_2 as dual, or correlative, classes of spreads.

The ranges 0, 1, 1, 0 of the spreads of these trees serve to characterize the classes K_1 and K_2; hence, we write $K_1 = (0\bar{1}10)$ and $K_2 = (01\bar{1}0)$, indicating the range of the original spread by a bar. The symbol (0110), without the bar, will then denote the two classes taken together.

Developable Spreads.

35. Returning to the general discussion, we shall now consider systematically the different classes of spreads existing in F_{n-1}, in the order of their simplicity.

First of all, there is the trivial class (0), consisting of the spreads of range zero, or the fixed points, lines, planes, etc.

Then come the various classes (011....110) of spreads of range 1, or *developable spreads.* Since their trees have no branch series, the three methods of classification here yield precisely the same results. The simplest class of developables is $(0\bar{1}0)$, which includes the ranges of points, the pencils of lines, the pencils of planes, etc. Notice that the straight line considered as generated by its points belongs here, while considered as a whole it is of range 0. Next in order are the two classes (0110) considered (for $n=5$) in the preceding section. After them come the three classes (01110), of which the simplest (non-conical) representatives are: for $(0\bar{1}110)$, a curve immersed in a three-flat; for $(01\bar{1}10)$, its tangent surface; and for $(011\bar{1}0)$, its system of osculating planes. The classes $(0\bar{1}110)$ and $(011\bar{1}0)$ are dual, while $(01\bar{1}10)$ is self-dual.

Obviously, the total number of classes of developables in F_{n-1} is $1+2+\ldots.+(n-1)=\frac{1}{2}n(n-1)$, of which $n-1$ are classes of *proper* spreads in F_{n-1}, in the sense that they have no representatives in a space of lower dimensionality than $n-1$. A proper spread, therefore, is a *non-conical* spread *immersed* in F_{n-1}.

Spreads of Range 2.

36. Spreads of range 2 are of four general types,

$$(0,2\ldots.2,0),\quad (0,2\ldots.2,1\ldots.1,0),\quad (0,1\ldots.1,2\ldots.2,0),$$
$$\text{and}\quad (0,1\ldots.1,2\ldots.2,1\ldots.1,0).$$

Those of the first type may be called *regular* spreads, since the entire tree, exclusive of the first and last spreads, consists of a single regular series. All developable spreads are, therefore, also regular. Those of the second type possess a descending branch series, of range 1, not indicated in the symbol; similarly, those of the third type have an ascending branch series, and those of the fourth type have two branch series, one ascending and the other descending.

The simplest class of regular spreads of range 2 is $(0\bar{2}0)$, which is represented in F_3 by a skew ruled surface S_2^2. If n is even, there are evidently $\frac{1}{8}n(n-2)$ classes of regular spreads of range 2 in F_{n-1}, of which $\frac{1}{2}(n-2)$

are proper; if n is odd, there are $\frac{1}{8}(n-1)(n-3)$ classes, but not one is proper.

37. When we come to the non-regular spreads, the number of classes depends on the particular method of classification. We shall for the present use only the first and broadest method, reserving a minuter classification of a few of the simpler spreads for sections 42–73.

In the case of the second type of spreads of range 2, the total number of classes in F_{n-1}, if n is even, is easily computed to be $\frac{1}{24}n(n-2)(n-4)$; and the number of proper classes is $\frac{1}{8}(n-2)(n-4)$. If n is odd, the total number of classes is $\frac{1}{24}(n-1)(n-2)(n-3)$; and the number of proper classes is $\frac{1}{8}(n-1)(n-3)$. The same figures hold for the third type.

In the case of the fourth and last type, the total number of classes in F_{n-1}, if n is even, is $\frac{1}{96}n(n-2)(n-4)^2$; and $\frac{1}{24}(n-2)(n-3)(n-4)$ of them are proper classes. The total number, if n is odd, is $\frac{1}{96}(n-1)^2(n-3)(n-5)$; and $\frac{1}{24}(n-1)(n-3)(n-5)$ of them are proper.

Spreads of Range Higher than 2.

38. As the range increases, the number and complexity of the different types increases rapidly. For instance, if the range of S is 3, the principal ascending series determined by S may include spreads of ranges 1 or 2, or both or neither, and the same is true of the principal descending series. Hence, there are $4 \cdot 4 = 16$ types, of which the simplest is the regular type

$$(0, 3 \ldots . 3, 0),$$

and the most complicated is

$$(0, 1 \ldots . 1, 2 \ldots . 2, 3 \ldots . 3, 2 \ldots . 2, 1 \ldots . 1, 0).$$

Spreads of the latter type have two principal branch series of range 1, ascending and descending respectively, and two principal branch series of range 2, ascending and descending; moreover, each of the latter may determine a secondary branch series of range 1 and possibly a connecting series.

The simplest class, however, in which a connecting series exists, is obviously $(02\bar{3}20)$; and it is represented in F_9 by a spread

$$S_5^3 = [A_1, A_2, A_1^1, A_2^1, A_3^1],$$

whose fundamental directrices satisfy a differential equation of the form*

$$\sum_{\beta=0}^{3} \sum_{\lambda=1}^{2} h_{\beta\lambda} A_\lambda^{3-\beta} = 0.$$

Proper Spreads in F_4 and in F_5.

39. From this point on we shall confine our attention to four and five dimensions, and to the classes of proper, non-developable spreads.

In F_4 there are just two such classes, $(0\bar{2}10)$ and $(01\bar{2}0)$, and they are correlatives. In view of the theorem of § 27, each of these classes contains $\infty^{5/1}$ spreads. The first is the class of skew† ruled surfaces S_2^2 immersed in F_4, and the second is the class of non-conical skew planar‡ spreads S_3^2. The fact that the first tangent spread of S_2^2 is an S_4^1, and that the second tangent spread is $S_5^0 = F_4$, may be expressed by saying that two consecutive generating lines of S_2^2 are connected by a three-flat and that the connecting flat of three consecutive generators is F_4 itself. Dually, we may say that two consecutive generating planes of S_3^2 meet in a point, and three consecutive generators do not meet.

40. In F_5 there are six classes of proper, non-developable spreads:

$(0\bar{2}20)$, $(02\bar{2}0)$, $(0\bar{2}110)$, $(011\bar{2}0)$, $(01\bar{2}10)$ and $(0\bar{3}0)$,

of which the first two and the last are regular.

The class $(0\bar{2}20)$ consists of $\infty^{7/1}$ spreads, namely the skew ruled surfaces S_2^2 whose tangent spreads are of range 2; these tangent spreads S_4^2 constitute the dual class $(02\bar{2}0)$; their focal spreads are then the original ruled surfaces S_2^2. In other words, two consecutive generating lines of S_2^2 are connected by a generating three-flat of S_4^2; and two consecutive generating three-flats of S_4^2 intersect in a generating line of S_2^2. Three consecutive generators of S_2^2 are connected by F_5, and three consecutive generators of S_4^2 do not meet.

On the other hand, the class $(0\bar{2}110)$ consists of the skew ruled surfaces S_2^2 immersed in F_5, whose tangent four-spreads are developable; two consecutive generators of S_2^2 are connected, as in the preceding case, by a three-flat, but three consecutive generators are now connected by a four-flat. The dual class $(011\bar{2}0)$ consists of the non-conical skew four-spreads S_4^2 whose focal spreads are developable surfaces. Two consecutive generators of S_4^2 meet in

* See "P. D. G.," §§ 56 and 69.

† A non-developable spread may be called a *skew* spread.

‡ A three-spread S_3, being generated by ∞^1 planes, may be called a *planar* spread.

a line and three consecutive generators meet in a point. Each of these classes, in view of §§ 28, 29, contains $\infty^{6/_1}$ spreads.

The self-dual class ($0\bar{3}0$) includes the planar three-spreads of range 3, S_3^3, the number of which is $\infty^{8/_1}$. They have no tangent or focal spreads, properly speaking; that is, two consecutive generating planes of S_3^3 do not meet, and their connecting flat is F_5.

Finally, the self-dual class ($01\bar{2}10$) includes the proper planar three-spreads of range 2, S_3^2, the number of which is $\infty^{7/_1}$ (cf. §§ 27 and 31). Their tangent five-spreads and focal curves are necessarily developable. Two consecutive generating planes of S_3^2 meet in a point and are connected by a four-flat; three consecutive generators do not meet and are connected by F_5.

41. The difference between planar three-spreads of ranges 3, 2 and 1 in F_5 may be expressed by saying that two *consecutive* planes of S_3^3 do not meet, those of S_3^2 meet in a point, and those of S_3^1 meet in a line. This must be understood to mean merely that the focal spreads in the three cases are of dimensionalities 0, 1 and 2, respectively. Two *neighboring* generators of S_3^2 and of S_3^1 will not, in general, meet at all. The relative position of two neighboring generators can be determined by introducing the metric concepts of distance and angle, and recalling that two planes in F_5 have a distance α, a maximum angle β and a minimum angle γ.* Then it can easily be shown that in the case of two neighboring planes of S_3^3 the three quantities α, β, γ are infinitesimals of the same order; in the case of S_3^2 two of them are infinitesimals of the same order, while the third is of higher order; and in the case of S_3^1 one of them is of lower order than the other two.

In sub-classifying these spreads in F_4 and F_5 with respect to the nature of their branch series and extended trees, only the irregular spreads need be taken into account. Thus we have two classes to consider in F_4 and three classes in F_5.

Skew Ruled Surfaces Immersed in F_4.

42. Let $S = S_2^2$ be a skew ruled surface immersed in F_4. Its homogeneous parametric equations are

$$X_j = a_1 A_{1j}(\omega) + a_2 A_{2j}(\omega),$$

where j runs from 1 to 5, and the parameters are ω and $a_1 : a_2$. Using the notation and methods of "P. D. G.," §§ 29, 39–45, 55–62 and 69,† we write

* Cf. Schoute, *loc. cit.*, pp. 53–72.

† According to the more usual custom, however, we denote the second derivative of A_1 with respect to ω by A_1'' instead of A_1^2 (as in "P. D. G.") and reserve the latter expression for the square of A_1.

$$S = S_2^2 = [A_1, A_2],$$
$$S^{1,0} = S_4^1 = [A_1, A_2, A_1', A_2'],$$

and

$$S^{2,0} = S_5^0 = F_4 = [A_1, A_2, A_1', A_2', A_1'', A_2''].$$

The elements of the array

$$\left.\begin{array}{cc} A_1''' & A_2''' \\ A_1'' & A_2'' \\ A_1' & A_2' \\ A_1 & A_2 \end{array}\right\}$$

are connected by a linear homogeneous relation

$$\Sigma g_{0i} A_i''' + \Sigma g_{1i} A_i'' + \Sigma g_{2i} A_i' + \Sigma g_{3i} A_i = 0, \tag{16}$$

and the elements of the three bottom rows by a relation

$$\Sigma h_{0i} A_i'' + \Sigma h_{1i} A_i' + \Sigma h_{2i} A_i = 0, \tag{16'}$$

where

$$\begin{vmatrix} g_{01} & g_{02} \\ h_{01} & h_{02} \end{vmatrix} \neq 0.$$

All the quantities involved in these equations are functions of ω.

By means of a transformation to new directrices, of the form (b), § 2, "P. D. G.," together with a transformation of the coefficients, of the form (37) and (38), § 57, "P. D. G.," the differential equations (16) and (16′) can easily be reduced to the canonical form

$$A_2''' + (k_1 A_2)'' + \Sigma (k_{2i} A_i)' + \Sigma k_{3i} A_i = 0 \tag{17}$$

and

$$A_1'' + (l_1 A_2)' + l_2 A_2 = 0. \tag{17'}$$

In all that follows, therefore, we shall assume (17) and (17′) to be the fundamental differential equations of the spread. It is clear that

$$S^{1,1} = S_3^1 = [A_1, A_2, A_1'] \quad \text{and} \quad S^{1,2} = S_2 = [A_1, A_1' + l_1 A_2].$$

43. $S^{1,1}$ is a planar three-spread, whose generating planes contain the generating lines of S; and it is the only such three-spread that is developable, as can easily be seen, either by direct calculation, or by duality from the theorem of § 53. Hence, *every skew ruled surface S immersed in four-space determines one and only one developable three-spread whose generating planes contain the generating lines of S, namely its special enclosing spread $S^{1,1}$.*

44. The focal spread $S^{1,2}$ of $S^{1,1}$ may be, as we shall see, either a developable surface or a fixed line m; but its generators (or generator) will meet the corresponding generators of S in the points of a curve (or range on m)

$$T = T_1^1 = [A_1],$$

which is therefore the common enclosed spread or curve of intersection of S and $S^{1,2}$.

Let $A_1(\omega)$ denote the point on T whose coordinates are $A_{11}(\omega), \ldots, A_{15}(\omega)$. The tangent plane* to S at the point $A_1(\omega)$ is evidently determined by the three points $A_1(\omega)$, $A_2(\omega)$ and $A_1'(\omega)$. But this plane is precisely the corresponding generator of $S^{1,1}$. Hence, the planes of $S^{1,1}$ touch S, and the locus of their points of contact is T. It follows that on a *skew ruled surface S in F_4 there exists (apart from the generators themselves) a unique curve (or straight line) T,† such that the tangent planes to S at the points of T form a developable three-spread; T is the curve of intersection of S and $S^{1,2}$.*

The fact that T turns out to be one of the fundamental directrices $[A_1]$ of S is a consequence of choosing the fundamental differential equations in their canonical form (17), (17′).

We are now prepared to classify ruled surfaces in F_4, by means of the values of the coefficients l_1 and l_2 in (17′), as follows:

Class (a), $l_1 \neq 0$,
Class (b), $l_1 = 0$, $l_2 \neq 0$,
Class (c), $l_1 = l_2 = 0$.

Class (a).

45. This is the general case, and depends on five arbitrary functions. $S^{1,2}$ is of range $1 (= S_2^1)$, and its focal spread $S^{1,3}$ is easily found, by the method of "P. D. G.," §§ 16, 19–21, to be

$$S^{1,3} = S_1 = \left[A_1' + l_1 A_2 - \frac{l_2}{l_1} A_1 \right].$$

$S^{1,3}$ is in general a curve (of range 1), but *will be a fixed point* (of range 0), *if and only if l_2/l_1 satisfies the Riccati equation*

$$\frac{d}{d\omega}\left(\frac{l_2}{l_1}\right) + \left(\frac{l_2}{l_1}\right)^2 = 0.$$

That is, $S^{1,1}$ is in general non-conical, but may be a point-cone.

---

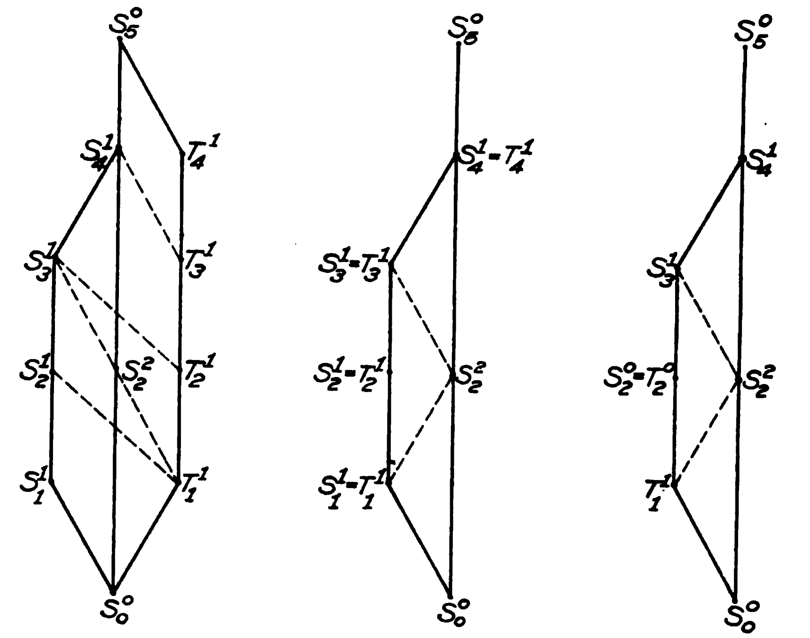
* Note the distinction between the tangent *plane at a point* of S and the tangent *three-flat along a generator* of S; the latter, as defined in "P. D. G.," § 4, is a generator of $S^{1,0}$.

† In F_4, as in F_3, a ruled surface S obviously has a line of striction; but in F_3 there is no purely projective curve on S analogous to T in F_4.

The successive tangent spreads of T are

$$\begin{aligned}
T^{1,0} &= T^1_2 = [A_1,\ A'_1],\\
T^{2,0} &= T^1_3 = [A_1,\ A'_1,\ (l_1 A_2)' + l_2 A_2],\\
T^{3,0} &= T^1_4 = [A_1,\ A'_1,\ (l_1 A_2)' + l_2 A_2,\ (l_1 A_2)'' + (l_2 A_2)'].
\end{aligned}$$

T^1_1, T^1_2 and T^1_3 are enclosed in S^1_2, S^1_3 and S^1_4, respectively; but T^1_2 cannot coincide with S^1_2. $T^{3,0}$ is in general a family of ∞^1 three-flats (of range 1), but *will be a fixed three-flat* (of range 0), *if and only if* $(l_1 A_2)''' + (l_2 A_2)''$ *is expressible, by virtue of* (17), *as a linear homogeneous function of the funda-*



CLASS (a) SUB CLASS (a_{11}) CLASS (b) CLASS (c)

—FIG. 1—

mental directrices of $T^{3,0}$. Hence, T^1_1 is in general a four-space curve, but may be a three-space curve.

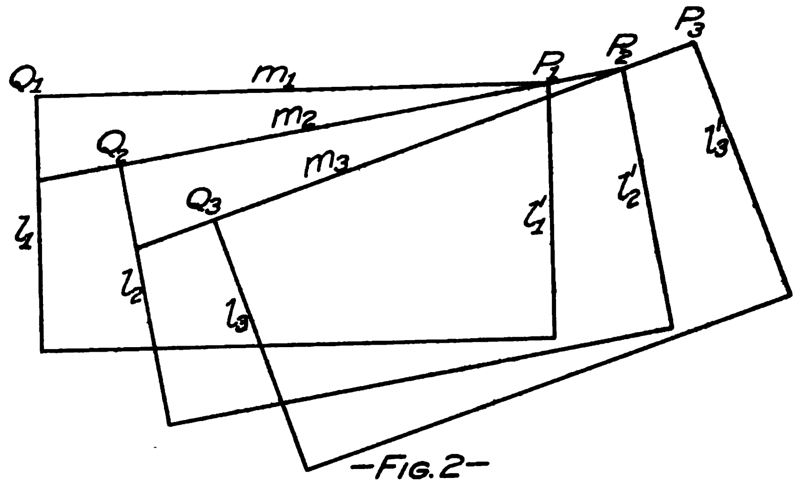
Let r and r' be the ranges of $S^{1,3}$ and $T^{3,0}$, respectively. Their values determine four subclasses into which class (a) is divided, namely:

$$\begin{aligned}
&\text{subclass } (a_{11}),\ r = 1,\ r' = 1; \quad \text{subclass } (a_{12}),\ r = 1,\ r' = 0;\\
&\text{subclass } (a_{21}),\ r = 0,\ r' = 1; \quad \text{subclass } (a_{22}),\ r = 0,\ r' = 0.
\end{aligned}$$

We have completed the enumeration of the spreads of the extended tree determined by a surface S of class (a). For the subclass (a_{11}) this tree is indicated in Fig. 1, where every spread is connected with its tangent and focal

spreads by full lines, vertical or oblique, and with its other enclosed and enclosing spreads by dotted lines.

46. Fig. 2 gives a schematic representation of the surface S_2^2 itself, when it belongs to the subclass (a_{11}) or (a_{12}). The lines l_1, l_2, l_3 are three neighboring generators of S_2^2; the three-flats (l_1, l_2) and (l_2, l_3) intersect in the plane $\pi_2 = (l_2, m_2)$, the limiting position of which is therefore a generator of S_3^1. Let $\pi_1 = (l_1, m_1)$ and $\pi_3 = (l_3, m_3)$ be two other generators of S_3^1; π_1 and π_2 intersect in the line m_2, π_2 and π_3 in the line m_3; m_1, m_2, m_3 may be regarded as three neighboring generators of the developable S_2^1, and P_1, P_2, P_3 as three neighboring points on its edge of regression S_1^1. When S_2^2 belongs to



—FIG. 2—

the subclass (a_{21}) or (a_{22}), the points P_1, P_2, P_3 coincide, and S_2^1 is a cone whose vertex S_1^0 is P_1. Q_1, Q_2 and Q_3, being the points of intersection of l_1 and m_1, l_2 and m_2, l_3 and m_3, respectively, are generators of the curve T_1^1; the tangent planes to S_2^2 at the points Q_1, Q_2, Q_3 are π_1, π_2, π_3. The lines Q_1Q_2 and Q_2Q_3, lying in the planes π_1 and π_2, respectively, may therefore be regarded as generators of T_2^1, the tangent developable of T_1^1; and the plane $Q_1Q_2Q_3$, lying in the three-flat (l_1, l_2), may be regarded as an osculating plane of T_1^1; that is, a generator of T_3^1.

If we start with five consecutive generators of S_2^2, they determine four consecutive three-flats of S_4^1, three consecutive planes of S_3^1, two consecutive lines of S_2^1 and one point of S_1^1.

It is to be observed that of the two fundamental curves T_1^1 and S_1^1, associated with the surface S, the first lies on the surface, while the second does not.

47. In order to obtain the most general skew ruled surface S_2^2 of class (a), we can *first choose at random a developable S_2^1 immersed in F_4*, which, if non-

conical, depends on three arbitrary functions, and if conical, on two arbitrary functions. We can *then choose any curve* T_1^1 *lying on* S_2^1, *except its edge of regression* S_1; T_1^1 will be either immersed in F_4 or restricted to an F_3. If it is immersed in F_4, it depends on one additional arbitrary function, but if in F_3, it is simply the curve of intersection of S_2^1 and F_3, and therefore involves no additional arbitrary functions, but merely four arbitrary constants, since F_3 can be any three-flat not passing through the vertex (if any) of S_2^1. Finally, we can *choose* S_2^2 *itself to be any ruled surface whose generators lie in the planes of* S_3^1 (the tangent spread of S_2^1) *and pass through the points of* T_1^1, *with two exceptions, namely,* S_2^1 *itself and the tangent surface* T_2^1 *of* T_1^1; this choice depends on one additional arbitrary function. Consequently, the number ν of arbitrary functions on which S_2^2 depends, for each of the four subclasses of (a), is as follows: (a_{11}), $\nu = 5$; (a_{12}), $\nu = 4$; (a_{21}), $\nu = 4$; (a_{22}), $\nu = 3$.

Among all the ruled surfaces enclosed in S_3^1 *and enclosing* T_1^1, *just two are developable, namely,* S_2^1 *and* T_2^1, *while all the rest,* ∞^{f_1} *in number, are skew.*

Another method of constructing S_2^2 is to choose T_1^1 first, then S_2^1 as a developable enclosing T_1^1, and finally S_2^2. If S_2^1 is a cone whose vertex is $S_1^0 = F_0$, it is obtained by simply projecting T_1^1 from F_0.

48. As an illustration let us write the equations of a simple rational surface of subclass (a_{22}). Let S_2^1 be a cone whose vertex S_1^0 is $C = (1, 0, 0, 0, 0)$, and whose directrix $T_1^1 = [A_1]$ is a curve immersed in the three-flat $x_1 = 0$. Let the equations of T_1^1 be

$$x_1 : x_2 : x_3 : x_4 : x_5 = 0 : 1 : \omega : \omega^2 : \omega^3;$$

more briefly, we shall write:

$$T_1^1 = [A_1] = (0, 1, \omega, \omega^2, \omega^3).$$

Evidently, $S_2^1 = [C, A_1]$, $S_3^1 = [C, A_1, A_1']$ and $T_2^1 = [A_1, A_1']$, where

$$A_1' = (0, 0, 1, 2\omega, 3\omega^2).$$

The second fundamental directrix A_2 of $S_2^2 = [A_1, A_2]$ can be chosen to be any curve of S_3^1, not situated on S_2^1 or on T_2^1. That is, $[A_2] = [a_1 C + a_2 A_1 + a_3 A_1']$, where a_1, a_2, a_3 are functions of ω or constants, such that $a_1 \neq 0$ and $a_3 \neq 0$. The simplest choice is $a_1 = a_3 = 1, a_2 = 0$, so that $[A_2]$ is the conic

$$[C + A_1'] = (1, 0, 1, 2\omega, 3\omega^2).$$

Then

$$S_2^2 = [A_1, A_2] = (\lambda, 1, \omega + \lambda, \omega^2 + 2\lambda\omega, \omega^3 + 3\lambda\omega^2),$$

where ω and λ are the parameters. By reference to the theorem of § 14, we see that S_2^2 is of order 4, S_2^1 and T_1^1 are of order 3, S_3^1 and T_2^1 are of order 4, and S_4^1 and T_3^1 are of order 3.

Class (b).

49. This case depends on four arbitrary functions. Evidently,

$$S^{1,2} = S_2^1 = [A_1, A_1'] \quad \text{and} \quad S^{1,3} = S_1^1 = [A_1],$$

which therefore coincides with T_1^1; that is, $T_1^1 = S_1^1$, $T_2^1 = S_2^1$, $T_3^1 = S_3^1$ and $T_4^1 = S_4^1$. The tree determined by S (see Fig. 1) is now somewhat simpler than for class (a).

Fig. 2 will give us a schematic representation of a ruled surface S of class (b), if we let l_1', l_2', l_3', instead of l_1, l_2, l_3, be three neighboring generators of the surface; Q_1, Q_2, Q_3 will then coincide with P_1, P_2, P_3, respectively. The description given in § 46 for class (a) will apply without change, except that P_1, P_2 and P_3 can not now coincide with one another; that is, S_2^1 can not be a cone.

Hence, in order to construct the most general S_2^2 of class (b), we *first choose any curve $S_1^1 (= T_1^1)$ immersed in F_4*, which depends on three arbitrary functions. We *then choose S_2^2 to be any ruled surface whose generators pass through the points of S_1^1 and lie in its corresponding osculating planes* (generators of S_3^1), *with the single exception of the tangent surface S_2^1 of S_1^1*. This choice involves one additional arbitrary function, making four in all.

To illustrate class (b), let S_1^1 be the rational quartic

$$[A_1] = (1, \omega, \omega^2, \omega^3, \omega^4).$$

Then $S_3^1 = [A_1, A_1', A_1'']$, where

$$[A_1''] = (0, 0, 1, 3\,\omega, 6\,\omega^2).$$

Choosing $A_2 = A_1''$, we have $S_2^2 = [A_1, A_1'']$, which is a quintic surface. The orders of S_1^1, S_2^1, S_3^1 and S_4^1 are 4, 6, 6 and 4, respectively.

Class (c).

50. This case involves only three arbitrary functions. As in class (b), $S^{1,2}$ coincides with $T^{1,0}$, but is now a fixed line m (of range 0). That is,

$$S^{1,2} = S_2^0 = [A_1, A_1'] = T_2^0 = T^{1,0}.$$

S_2^2 may be described briefly as *a ruled surface immersed in F_4, whose generators all meet a fixed line m without passing through a fixed point.* In the extended tree (Fig. 1) T_1^1 no longer, as in class (b), belongs to the descending series determined by S_4^1; it is simply a point-range lying on m. S_3^1 is a line-cone or conical three-spread whose vertex is the line m.

In order to construct S_2^2 of class (c), we first choose any S_3^1 whose generating planes pass through a fixed line m; this involves one arbitrary function.

We then establish a one-to-one continuous correspondence between the planes of S_3^1 and the points of m; this involves one more arbitrary function. Finally, in each plane of S_3^1 we choose a line l passing through the corresponding point on m (and not coinciding with m), in such a way that the lines l form a continuous system; this involves still another arbitrary function, making three in all.

As an example of class (c), let

$$T_1^1 = [A_1] = (1, \omega, 0, 0, 0) \quad \text{and} \quad [A_2] = (0, 0, 1, \omega, \omega^2).$$

Then

$$S_2^2 = [A_1, A_2] = (1, \omega, \lambda, \lambda\omega, \lambda\omega^2),$$

which is a cubic surface; its generators connect the points of the conic $[A_2]$ with the corresponding points of the projective range $[A_1]$. Eliminating λ and ω, we find that S_2^2 is the partial intersection of the quadric three-spreads $x_1 x_5 = x_2 x_4$ and $x_1 x_4 = x_2 x_3$, which also intersect in the plane $x_1 = x_2 = 0$.

51. ***The three fundamental classes of skew ruled surfaces S in F_4 may be distinguished by the fact that for class* (a) *every tangent to the curve T_1^1 at a point Q is distinct, in general, from the characteristic* (*focal line*) *of the tangent plane to S at Q, whereas for class* (b) *these two lines always coincide and are variable, and for class* (c) *they coincide and are fixed.***

We have shown that the tangent spread $S^{1,0}$ of an S_2^2 may be non-conical or may be a point-cone or line-cone, but *can not be a plane-cone*; that is, a pencil of three-flats. Moreover, T_1^1 may be a four-space curve, a three-space curve, or a range (straight line), but *can not be a plane curve.*

The projection of an S_2^2 from a point P of F_4 (see §§ 15–23) is in general a conical S_3^2 of type $(0\bar{2}0)$, illustrating the theorem of § 22. But if S_2^2 is of class (a_{21}) or (a_{22}) and P coincides with $S^{1,3} = S_1^0$, or if S_2^2 is of class (c) and P lies on the line $S^{1,2} = S_2^0$, then S_2^2 projects into $S^{1,1} = S_3^1$, illustrating the theorem of § 23.*

Non-Conical Skew Planar Three-Spreads in F_4.

52. The projective differential geometry of planar three-spreads could be obtained from that of ruled surfaces by the principle of duality. But it will be necessary for our purpose to build it on an independent foundation.

Let $S = S_3^2$ be a non-conical skew planar three-spread in F_4. Since its focal spread $S_{1,0}$ is of range 1, we see that a normal system of fundamental directrices ("P. D. G.," §§ 39–46) of S_3^2 is furnished by the two bottom rows of the array

* For additional properties of S_2^2, see Ranum, *Trans. Amer. Math. Soc.*, Vol. XVI, p. 89, §§ 1-9.

$$\left.\begin{matrix} A_1''' & A_2''' \\ A_1'' & A_2'' \\ A_1' & A_2' \\ A_1 & \end{matrix}\right\} \tag{18}$$

That is,

$$S = S_3^2 = [A_1, A_1', A_2'],^*$$

from which we derive

$$\begin{aligned} S^{1,0} &= S_5^0 = F_4 = [A_1, A_1', A_2', A_1'', A_2''], \\ S_{1,0} &= S_1^1 = [A_1], \\ S_{1,1} &= S_2^1 = [A_1, A_1'], \end{aligned}$$

and

$$S_{1,2} = S_3 = [A_1, A_1', A_1''].$$

The most general transformation to new directrices, which preserves the normal character of the directrices, will be a special case of (b), § 2, "P. D. G."; namely:

$$\bar{A}_1 = \alpha A_1, \qquad \bar{A}_2' = \beta A_2' + \gamma A_1' + \delta A_1, \tag{19}$$

where $\alpha \neq 0$ and $\beta \neq 0$. By means of this transformation it is easy to see that the fundamental differential equations of the spread can be reduced to the canonical form

$$A_1''' + (l_1 A_2')' + l_2 A_2' + l_3 A_1 = 0, \tag{20}$$

$$A_2''' + \Sigma\,(k_{1i} A_i')' + \Sigma\, k_{2i} A_i' + k_3 A_1 = 0. \tag{20'}$$

We assume that this has been done.

53. Of all the ruled surfaces enclosed in S_3^2, the only developable is S_2^1; for A_1 is the only directrix of S_3^2 whose derivative is also a directrix of S_3^2. Hence, *every non-conical skew three-spread S in F_4 determines a unique developable surface whose generating lines lie in the generating planes of S, namely, its special enclosed spread $S_{1,1}$.*

54. The tangent three-flat to S_3^2 at a point

$$P = a A_1(\omega) + b A_1'(\omega) + c A_2'(\omega)$$

of the generator ω is the flat connecting the plane ω and the point

$$b A_1''(\omega) + c A_2''(\omega).$$

Being independent of the coefficient a, it is the same at all the points of the line joining P and the focal point $A_1(\omega)$ in the generator ω.

Defining a *focal line* of S_3^2 as any straight line lying in a generating plane and passing through the corresponding focal point, we see that *the tangent three-flats to S_3^2 at all the points of a focal line coincide*; hence, we may speak

* The reason for writing A_2' with a prime is given in "P. D. G.," § 41.

of the tangent three-flats to S_3^2 *along its focal lines*;* there are ∞^2 focal lines and ∞^2 tangent three-flats. Obviously, the focal lines of an S_3^2 and the tangent planes to an S_2^2 are correlative in F_4.

Now consider those particular focal lines ($c = 0$) which are generators of S_2^1, or tangents to the focal curve S_1^1; the corresponding tangent three-flats to S_3^2 are generators of the spread

$$T = T_4^1 = [A_1,\, A_1',\, A_1'',\, A_2'],$$

which is the common enclosing spread of S and $S_{1,2}$; that is, they are the flats connecting the generating planes of S and the corresponding osculating planes of S_1^1. Hence, *among the ∞^2 tangent three-flats to a non-conical $S_3^2 = S$ there is a unique system T_4^1 of ∞^1 three-flats, such that the focal lines along which they are tangent to S_3^2 form a developable surface; T_4^1 is the common enclosing spread of S and $S_{1,2}$.*

By means of (20) we classify the spreads S_3^2 as follows:

Class (a): $l_1 \neq 0$; $\infty^{5/1}$ spreads.
Class (b): $l_1 = 0$, $l_2 \neq 0$; $\infty^{4/1}$ spreads.
Class (c): $l_1 = l_2 = 0$; $\infty^{3/1}$ spreads.

Class (a).

55. $S_{1,2}$ is of range 1 ($= S_3^1$), and

$$S_{1,3} = S_4 = [A_1,\, A_1',\, A_1'',\, (l_1 A_2')' + l_2 A_2'],$$

which is of range 1 or 0, depending on (20′). Moreover,

$$T_{1,0} = T_3^1 = [A_1,\, A_1',\, A_1'' + l_1 A_2'],$$

$$T_{2,0} = T_2^1 = \left[A_1,\, A_1'' + l_1 A_2' - \frac{l_2}{l_1} A_1'\right],$$

and

$$T_{3,0} = T_1 = \left[A_1'' + l_1 A_2' - \frac{l_2}{l_1} A_1' + \left(\left(\frac{l_2}{l_1}\right)' + \left(\frac{l_2}{l_1}\right)^2\right) A_1\right].$$

$T_{3,0}$ is of range 1 or 0, depending on (20). T_4^1, T_3^1 and T_2^1 enclose S_3^1, S_2^1 and S_1^1, respectively; but T_2^1 cannot coincide with S_2^1. Letting r and r' be the ranges of $S_{1,3}$ and $T_{3,0}$ respectively, we have the four subclasses:

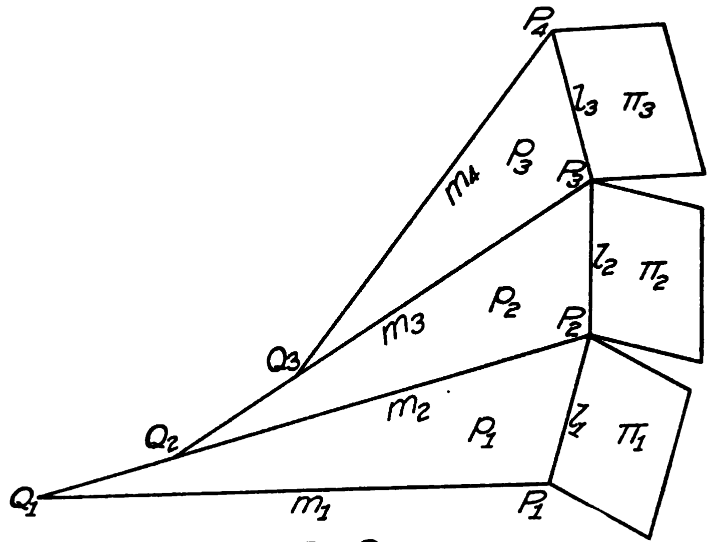
(a_{11}), $r = 1$, $r' = 1$; (a_{12}), $r = 1$, $r' = 0$;
(a_{21}), $r = 0$, $r' = 1$; (a_{22}), $r = 0$, $r' = 0$.

Fig. 3 gives a schematic representation of S_3^2, when it belongs to the subclass (a_{11}) or (a_{21}). The planes π_1, π_2, π_3 are neighboring generators of S_3^2; P_1, P_2, P_3, P_4 are points of S_1^1; l_1, l_2, l_3 are generators of S_2^1; m_1, m_2, m_3, m_4

* The tangent *four-flats* to S_3^2 along its *generators* coincide with F_4 itself.

are generators of T_2^1; the planes $p_1 = (l_1, m_1)$, $p_2 = (l_2, m_2)$, $p_3 = (l_3, m_3)$ are generators of T_3^1; and Q_1, Q_2, Q_3 are points of T_1^1.

56. In constructing the general S_3^2 of class (a), we first choose any planar T_3^1 immersed in F_4, either non-conical or a point-cone, and then any developable surface S_2^1 enclosed in T_3^1, except its focal surface T_2^1. S_2^1 will be either immersed in F_4 or restricted to an F_3. Finally, we choose S_3^2 itself to be *any three-spread whose generating planes contain the generators of S_2^1 and lie in the tangent three-flats of T_3^1, with two exceptions, namely T_3^1 itself and the tangent spread S_3^1 of S_2^1.* These two are the only developable three-spreads enclosing S_2^1 and enclosed in T_4^1.



—FIG. 3—

As an illustration of subclass (a_{11}), let

$$T_1^1 = [A] = (1, \omega, \omega^2, \omega^3, \omega^4),$$
$$S_1^1 = [A_1] = [A + \omega A'] = (1, 2\omega, 3\omega^2, 4\omega^3, 5\omega^4),$$

and

$$S_3^2 = [A_1, A_1', A_2'],$$

where $[A_1'] = (0, 1, 3\omega, 6\omega^2, 10\omega^3)$ and $[A_2'] = [A'''] = (0, 0, 0, 1, 4\omega)$. S_3^2 is easily seen to be of order 5.

Class (b).

57. $S_{1,2}$ is again of range 1, and

$$S_{1,3} = S_4^1 = [A_1, A_1', A_1'', A_2'],$$

which therefore coincides with T_4^1. Fig. 3 can be made to represent this case

by letting the generating planes π_1, π_2, π_3 of S_3^2 pass through the lines m_2, m_3, m_4 instead of l_1, l_2, l_3; they lie in the tangent three-flats of T_3^1, as before.

In constructing the general S_3^2 of class (b), we choose a proper developable T_3^1, and for generators of S_3^2 take *any continuous system of planes containing the focal lines of T_3^1 and lying in the corresponding tangent three-flats, with the exception of T_3^1 itself.*

Class (c).

58. $S_{1,2}$ is now of range 0 (a fixed plane π), and coincides with $T_{1,0}$; that is,

$$S_{1,2} = S_3^0 = [A_1, A_1', A_1''] = T_3^0 = T_{1,0}.$$

S_1^1 is a plane curve lying in π, and T_4^1 is a pencil of three-flats whose vertex is π. S_3^2 may be described as *a non-conical three-spread immersed in F_4, whose generating planes intersect a fixed plane π in straight lines* (instead of points). These straight lines are the tangents to the focal curve S_1^1.

59. Classes (a), (b) and (c) may be characterized briefly as follows: Class (a): $S_{1,2} = S_3^1$ does not coincide with $T_{1,0} = T_3^1$. Class (b): S_3^1 coincides with T_3^1. Class (c): $S_{1,2} = S_3^0$ coincides with $T_{1,0} = T_3^0$.

We have shown that the focal curve S_1^1 of an S_3^2 may be a plane curve, a three-space curve, or a four-space curve, but *can not be a straight line* (point-range); and that T_4^1 may be non-conical, or may be a point-cone or plane-cone (pencil of three-flats), but *can not be a line-cone.*

A three-flat Q of F_4 (see §§ 15–23) will intersect an S_3^2 in a ruled surface U, which is in general an S_2^2 (§ 22). But if S_3^2 is of class (a_{21}) or (a_{22}) and Q coincides with $S_{1,3} = S_4^0$, or if S_3^2 is of class (c) and Q contains the plane $S_{1,2} = S_3^0$, then U coincides with $S_{1,1} = S_2^1$ (§ 23).*

Irregular Ruled Surfaces Immersed in F_5.

60. Coming now to five-dimensional space, we have three irregular classes of spreads to consider, namely ($0\bar{2}110$), ($011\bar{2}0$) and ($01\bar{2}10$), of which the first comprises the skew ruled surfaces whose tangent four-spreads are developable.

Let $S = S_2^2 = [A_1, A_2]$ be any such spread. Then

$$S^{1,0} = S_4^1 = [A_1, A_2, A_1', A_2'].$$

The fundamental differential equations can be chosen in the canonical form

$$A_1'' + (l_1 A_2)' + l_2 A_2 = 0,$$
$$A_2^{(4)} + (k_1 A_2)''' + (k_2 A_2)'' + \Sigma (k_{3i} A_i)' + \Sigma k_{4i} A_i = 0.$$

* For additional properties of S_2^2, see Ranum, *loc. cit.*, §§ 10–14.

Hence,

$$S^{2,0} = S_5^1 = [A_1, A_2, A'_1, A'_2, A''_2]$$

and

$$S^{3,0} = S_6^0 = F_6 = [A_1, A_2, A'_1, A'_2, A''_2, A'''_2].$$

Exactly as in the case of ruled surfaces in F_4 (§ 42) we have $S^{1,1} = S_3^1 = [A_1, A_2, A'_1]$ and $S^{1,2} = S_2 = [A_1, A'_1 + l_1 A_2]$; and we define $T_1^1 = [A_1]$ as the curve of intersection of S and $S^{1,2}$.

The classification of these spreads in F_5 is precisely the same as in F_4, the number of arbitrary functions for each class being one greater than for the corresponding class in F_4. In the description of each class the only changes are the following: In class (a) S_3^1 is immersed in F_5, and T_1^1 is immersed either in F_5 or in an F_4; in class (b) $T_1^1 (= S_1^1)$ is immersed in F_5; in class (c) S_3^1 is a line-cone, as before, but is now immersed in F_5 instead of F_4.

Non-conical, Irregular Four-spreads in F_5.

61. In F_5 the four-spreads generated by ∞^1 three-flats are dual to the ruled surfaces. We now consider the class $(011\bar{2}0)$ of four-spreads S whose focal surfaces are developable (and non-conical). They are $\infty^{6/1}$ in number. Expressing S in terms of normal directrices, we have

$$S = S_4^2 = [A_1, A'_1, A''_1, A''_2].$$

Then

$$S_{1,0} = S_2^1 = [A_1, A'_1],$$
$$S_{1,1} = S_3^1 = [A_1, A'_1, A''_1],$$

and

$$S_{1,2} = S_4 = [A_1, A'_1, A''_1, A'''_1].$$

We define T as the common enclosing spread of S and $S_{1,2}$; that is,

$$T = T_5^1 = [A_1, A'_1, A''_1, A'''_1, A''_2].$$

The three classes of spreads may be described briefly as follows:

Class (a): T_5^1 is non-conical or a point-cone, and $S_{1,2} = S_4^1$ is immersed in F_5 or in an F_4 and is enclosed in T_5^1, but does not coincide with $T_{1,0} = T_4^1$. The generating three-flats of S_4^2 pass through the generating planes of S_3^1 and lie in the generating four-flats of T_5^1.

Class (b): T_5^1 is non-conical. The generating three-flats of S_4^2 lie in generating four-flats of T_5^1 and pass through the generating planes of $T_{2,0} = T_3^1 = S_3^1$.

Class (c): T_5^1 is a pencil of four-flats, whose vertex is a three-flat F. $S_{2,0} = S_1^1$ is a curve immersed in F. The generating three-flats of S_4^2 pass through the osculating planes of S_1^1 and lie in the corresponding four-flats of T_5^1. S_4^2 may be described as a non-conical four-spread whose generating three-flats meet a fixed three-flat in planes.

Proper Planar Three-Spreads of Range 2 *in* F_5.

62. We now come to the last and most interesting class of spreads in F_5, namely the self-dual class $(01\bar{2}10)$, which comprises the non-conical, irregular planar three-spreads immersed in F_5. This is the simplest class of m-spreads in F_n having two branch series, one ascending and one descending. The array (18), § 52, is again fundamental, and we have

$$S = S_3^2 = [A_1, A_1', A_2'].$$

Then

$$\begin{aligned} S_{1,0} &= T_1^1 = [A_1], \\ S_{1,1} &= T_2^1 = [A_1, A_1'], \\ S_{1,2} &= T_3 = [A_1, A_1', A_1''], \end{aligned}$$

and

$$S^{1,0} = R_5^1 = [A_1, A_1', A_2', A_1'', A_2''].$$

The fundamental differential equations are

$$\left. \begin{aligned} G &\equiv \Sigma g_{0i} A_i''' + \Sigma g_{1i} A_i'' + \Sigma g_{2i} A_i' + g_{31} A_1 = 0, \\ H &\equiv \Sigma h_{0i} A_i^{(4)} + \Sigma h_{1i} A_i''' + \Sigma h_{2i} A_i'' + \Sigma h_{3i} A_i' + h_{41} A_1 = 0, \end{aligned} \right\} \tag{21}$$

where

$$\begin{vmatrix} g_{01} & g_{02} \\ h_{01} & h_{02} \end{vmatrix} \neq 0.$$

The most general transformation to new normal directrices is again (19), and the most general transformation of the equations (21) themselves is

$$\bar{G} = \lambda G, \qquad \bar{H} = \mu G' + \nu H,$$

where $\lambda \neq 0$ and $\nu \neq 0$. When (21) is simplified by means of these transformations, the result depends essentially on whether g_{02} is equal to zero or not. If $g_{02} \neq 0$, we obtain class (α); and if $g_{02} = 0$, classes (β), (γ) and (δ).

Class (α).

63. This is the general class, containing ∞^{7h} spreads. The differential equations (21) are reducible to the canonical form

$$A_2''' + (l_1 A_1)' + l_2 A_2' + l_3 A_1 = 0 \tag{22}$$

and

$$A_1^{(4)} + \Sigma (k_{2i} A_i')' + \Sigma k_{3i} A_i' + k_{41} A_1 = 0. \tag{22'}$$

It follows that

$$\begin{aligned} S_{1,2} &= T_3^1, \\ S_{1,3} &= T_4 = [A_1, A_1', A_1'', A_1'''], \\ S^{1,1} &= R_4^1 = [A_1, A_1', A_2', A_2''], \\ S^{1,2} &= R_3^1 = [A_1, A_2', A_2''], \end{aligned}$$

and

$$S^{1,3} = R_2 = [A_2', A_2'' + l_1 A_1].$$

Geometrically, class (α) is distinguished from the other classes by the fact that $S_{1,1}(=T_2^1)$, which is in every case enclosed in $S^{1,1}=R_4^1$, is not in this case enclosed in $S^{1,2}=R_3^1$ (see Fig. 4). Hence, in order to construct the most general S_3^2 of class (α), we can first choose any developable three-spread R_3^1 immersed in F_5, either non-conical, a point-cone or a line-cone; we can then choose any curve T_1^1 situated on R_3^1, but not on its focal surface (or vertex) R_2; T_1^1 will be a five-space, four-space or three-space curve. Finally, we can choose S_3^2 itself to be *any three-spread* (*without exception*) *whose generating planes pass through the tangents to* T_1^1 *and lie in the corresponding tangent three-flats to* R_3^1.

64. Coming to the extended tree, we define R as the common enclosed spread, or surface of intersection of S_3^2 and R_3^1. Then $R=R_2^2\doteq[A_1, A_2']$, $R^{1,0}=R_4^1=S^{1,1}$, $R^{1,1}=R_3^1=S^{1,2}$, etc. R is a non-regular ruled surface. As we have seen (§ 60), its structure depends essentially on that of the curve $U=U_1^1=[A_2']$, in which it is intersected by $R^{1,2}=R_2$. *R can belong to any one of the three classes* (*six subclasses*) *of such surfaces.* Thus it belongs to class (a), if in equation (22) $l_1\neq 0$; to class (b), if $l_1=0$ and $l_3\neq 0$; to class (c), if $l_1=l_3=0$.

Dually, we define T as the common enclosing spread of S_3^2 and T_3; that is,

$$T=T_4^2=[A_1, A_1', A_1'', A_2'],\quad T_{1,0}=T_2^1=S_{1,1},\quad T_{1,1}=T_3^1=S_{1,2}, \text{ etc.}$$

T is a non-conical, irregular four-spread, and its structure (see § 61) depends on that of the five-spread

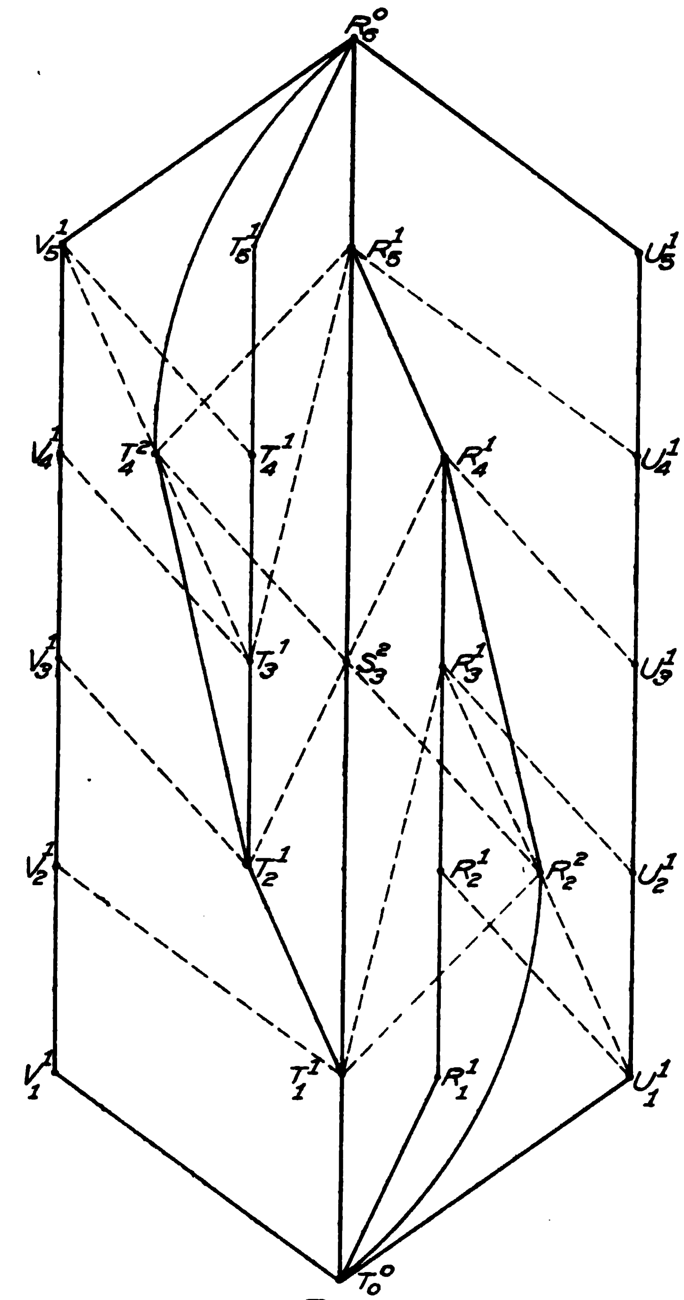
$$V=V_5^1=[A_1, A_1', A_1'', A_1''', A_2'],$$

which is the common enclosing spread of T and $T_{1,2}=T_4$. *T can belong to any one of the three classes* (*six subclasses*) *of such spreads.* If in equation (22′) $k_{22}\neq 0$, T belongs to class (a); if $k_{22}=0$ and $k_{32}\neq 0$, to class (b); and if $k_{22}=k_{32}=0$, to class (c).

The class to which T_4^2 belongs is obviously independent of that to which R_2^2 belongs; this independence can easily be shown to extend to the subclasses (a_{11}), etc., as well. The spreads S_3^2 of class (α) can therefore be divided into $6\cdot 6=36$ subclasses, depending on the subclasses to which R_2^2 and T_4^2 belong. If R_2^2 belongs to subclass (a_{11}) and T_4^2 to subclass (b), we shall say that S_3^2 belongs to the subclass ($\alpha_{a_{11}b}$); and similarly for the other combinations.

It is not difficult to verify that the entire extended tree determined by S_3^2 is made up of the spreads already mentioned, together with their successive tangent and focal spreads. In the case of the most general subclass of all, namely ($\alpha_{a_{11}a_{11}}$), the extended tree is indicated in Fig. 4, where the full and dotted lines have the same significance as in Fig. 1.

65. By means of the extended tree we can construct the most general spread S_3^2 of class (α), for which R_2^2 belongs to any prescribed subclass, as follows. We first choose a developable three-spread R_3^1 immersed in F_5, either



—FIG. 4—

non-conical, a point-cone or a line-cone, according as R_2^2 is to belong to one of the subclasses (a_{11}), (a_{12}), (b), to one of the subclasses (a_{21}), (a_{22}), or to (c); this choice depends on four, three or two arbitrary functions, respectively.

We next choose on its focal surface (or fixed line) R_2 a five-space or four-space curve (or point-range) U_1^1, which depends on either one additional arbitrary function or none, according as R_2^2 belongs to one of the subclasses (a_{11}), (a_{21}), (c) or not.

The third and final step is to choose *a curve* T_1^1, *situated on* R_3^1, *but not on its focal spread* R_2 *or on the tangent spread* U_2^1 *of* U_1^1. Any further specialization in the choice of T_1^1 has no effect on the subclass to which R_2^2 belongs, but does affect the subclass to which T_4^2 belongs. Thus if T_1^1 is a five-space curve and depends, therefore, on two more arbitrary functions, T_4^2 will belong to one of the subclasses (a_{11}), (a_{12}), or (b); if T_1^1 is a four-space curve and involves one more arbitrary function, T_4^2 will belong to (a_{21}) or (a_{22}); finally, if T_1^1 is a three-space curve, involving no more arbitrary functions, T_4^2 will belong to (c).

The curves U_1^1 *and* T_1^1, *as so chosen, determine completely the original spread* S_3^2 and the entire extended tree. For R_2^2 is generated by the lines joining corresponding points of U_1^1 and T_1^1, and S_3^2 is generated by the planes joining the points of U_1^1 and the corresponding tangents to T_1^1. Moreover, T_4^2 is generated by the three-flats joining the points of U_1^1 and the corresponding osculating planes of T_1^1, and V_5^1 is generated by the four-flats joining the points of U_1^1 and the corresponding osculating three-flats of T_1^1.

66. Dually, it is clear that by properly choosing T_3^1, V_5^1 and R_5^1, we can construct the most general S_3^2 for which T_4^2 belongs to any prescribed subclass. The four-flats of V_5^1 intersect the corresponding four-flats of R_5^1 in the three-flats of T_4^2; and they intersect the three-flats of R_4^1 in the planes of S_3^2, the planes of R_3^1 in the lines of R_2^2, and the lines of R_2 in the points of U_1^1.

67. The degree of generality of each of the 36 subclasses of (α) is now easy to compute; the number of arbitrary functions varies from seven to three. At one extreme is $(\alpha_{a_{11}a_{11}})$, which contains $\infty^{7/1}$ spreads; while at the other extreme are $(\alpha_{a_{22}a_{22}})$, $(\alpha_{a_{22}c})$, $(\alpha_{ca_{22}})$ and (α_{cc}), each of which contains $\infty^{3/1}$ spreads.

S_3^2 is obviously determined not merely by U_1^1 and T_1^1, or by V_5^1 and R_5^1, but also by R_2^2 and T_2^1, by T_4^2 and R_4^1, or self-dually by R_2^2 and T_4^2. This is the first case we have met of *an irregular spread that is completely determined by the other spreads of its extended tree.*

It is to be noticed, however, that if R_2^2 is chosen arbitrarily, it is in general impossible to find an S_3^2, for which T_4^2 shall belong to one of the subclasses (a_{22}) or (c), having the lowest degree of generality. This is easily seen, for instance, in the case where R_2^2 belongs to (a_{11}) and T_4^2 to (c). For then R_2^2 involves six arbitrary functions, whereas S_3^2 has only five functions at its dis-

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posal. Geometrically, also, we see that when T_4^2 belongs to (c), R_2^2 can not be the most general spread of class (a_{11}); for T_1^1 is then immersed in an F_3 and is therefore the curve of intersection of F_3 and R_2^2. But if R_2^2 is chosen arbitrarily, its generators will not meet any F_3.

Classes (β), (γ) *and* (δ).

68. Returning to the differential equations (21), we now take up the case in which $g_{02}=0$. They are no longer reducible to the form (22), (22′), but instead to the form

$$A_1''' + (l_1 A_2')' + l_2 A_2' + l_3 A_1 = 0, \tag{23}$$

$$A_2^{(4)} + \Sigma\,(k_{2i} A_i')' + \Sigma\, k_{3i} A_i' + k_{41} A_1 = 0. \tag{23'}$$

It follows that

$$S^{1,1} = R_4^1 = [A_1, A_1', A_1'', A_2'],$$

and

$$S^{1,2} = R_3 = [A_1, A_1', A_1'' + l_1 A_2'].$$

The further classification depends on the values of l_1 and l_2 in (23), as follows: Class (β), $l_1 \neq 0$; class (γ), $l_1 = 0$, $l_2 \neq 0$; class (δ), $l_1 = l_2 = 0$. It is evident that these three classes correspond exactly to classes (a), (b) and (c) of the skew planar three-spreads in F_4, as given in §§ 55–59. The only essential difference is that the rôle of the spread T_4^1 is here played by $S^{1,1} = R_4^1$. The number of arbitrary functions for each class is one greater than for the corresponding class in F_4, and is therefore the same as for the corresponding class of ruled surface of type $(0\bar{2}110)$ in F_5, or of four-spreads of type $(011\bar{2}0)$, as given in §§ 60, 61.

69. Classes (β), (γ) and (δ) may be described briefly as follows:

Class (β), four subclasses: $S^{1,2} = R_3^1$ is non-conical or a point-cone. $S_{1,2} = T_3^1$ is immersed in F_5 or in an F_4. R_3^1 encloses T_2^1, but does not coincide with T_3^1. The number of arbitrary functions varies from six to four. The subclass (β_{11}) or (β_{21}) is represented in Fig. 3.

Class (γ): R_3^1 is a non-conical spread immersed in F_5 and coincides with T_3^1. There are five arbitrary functions involved.

Class (δ): $S^{1,2} = R_3^0$ coincides with $S_{1,2} = T_3^0$. T_1^1 is a plane curve; R_4^1 is a plane-cone. S_3^2 may be described as *a proper three-spread whose generating planes meet a fixed plane in lines.* There are four arbitrary functions involved.

The spreads R and T, as defined in § 64 for the class (α), become coincident, for the classes (β), (γ) and (δ), with T_2^1 and R_4^1, respectively. Hence, the extended tree, Fig. 4, reduces to the primary tree determined by S_3^2.

70. The following general theorems concerning three-spreads S_3^2 of type $(01\bar{2}10)$ follow almost immediately from what has been proved.

Among the ruled surfaces, ∞^{2f_1} in number, which are enclosed in S_3^2, those whose generators are not focal lines of S_3^2 are all skew surfaces.*

Among those whose generators *are* focal lines of S_3^2, ∞^{f_1} in number, there is *one exceptional surface that is developable*, viz., its special enclosed spread T_2^1.

Among the latter there is *also an exceptional surface R having the property that the tangent three-flats to S_3^2 along the generators of R form a developable four-spread R_4^1.*

Among the four-spreads, ∞^{2f_1} in number, which enclose S_3^2, those whose generating three-flats are not tangent to S_3^2 are all skew four-spreads.

Among those whose generating three-flats *are* tangent to S_3^2, ∞^{f_1} in number, there is *one exceptional four-spread that is developable*, viz., its special enclosing spread R_4^1.

Among the latter there is *also an exceptional four-spread T having the property that the focal lines of S_3^2 along which the generators of T are tangent to S_3^2 form a developable surface T_2^1.*

If S_3^2 belongs to class (β), (γ) or (δ), R coincides with T_2^1 and T with R_4^1.

71. *Among the ruled surfaces generated by focal lines of S_3^2, T_2^1 and R are the only ones which are not of type* $(0\bar{2}20)$. If S_3^2 is of class (α), R is of type $(0\bar{2}110)$; its tangent spread R_4^1 is developable.

Among the four-spreads whose generators are tangent to S_3^2, R_4^1 and T are the only ones which are not of type $(02\bar{2}0)$. If S_3^2 is of class (α), T is of type $(011\bar{2}0)$; its focal surface T_2^1 is developable.

72. *If S_2^2 is a skew ruled surface generated by focal lines of S_3^2, the tangent three-flats to S_3^2 along the generators of S_2^2 are also the tangent three-flats to S_2^2 itself*, and therefore generate its tangent spread.

If S_4^2 is a skew four-spread whose generators are tangent to S_3^2, the focal lines of S_3^2 along which the generators of S_4^2 are tangent to S_3^2 are also the focal lines of S_4^2 itself, and therefore generate its focal surface.

If S_3^2 is of class (α), there exist ∞^{f_1} three-spreads enclosed in T_4^2 and enclosing R_2^2, *all of which are of range* 3 *except S_3^2 itself.*

73. The spreads S_3^2 of class (α), for which R_2^2 is of subclass (c), are precisely those proper three-spreads of range 2 whose generating planes meet a fixed line R_2^0, but do not meet a fixed plane in lines.

The spreads S_3^2 of class (α), for which T_4^2 is of subclass (c), are precisely those proper three-spreads of range 2 whose generating planes meet a fixed three-flat T_4^0 in lines, but do not meet a fixed plane in lines.

* See § 54.

The subclass (α_{cc}) consists of the proper three-spreads of range 2 whose *generating planes meet a fixed line R_2^0 and also meet a fixed three-flat T_4^0 in lines.* R_2^0 and T_4^0 will necessarily be non-intersecting.

To illustrate this case, let

$$T_1^1 = [A_1] = (0, 0, 1, \omega, \omega^2, \omega^3)$$

and

$$U_1^1 = [A_2'] = (1, \omega, 0, 0, 0, 0).$$

Then, R_2^0 is the line $x_3 = x_4 = x_5 = x_6 = 0$, and T_4^0 is the three-flat $x_1 = x_2 = 0$. T_1^1 is a curve immersed in T_4^0. R_3^1 is a line-cone generated by the planes connecting R_2^0 with the points of T_1^1. T_1^1 and R_3^1 are of order 3, T_2^1 and R_4^1 are of order 4, T_3^1 and R_5^1 are of order 3. The lines of R_2^2 connect the points of the range U_1^1 with the corresponding points of T_1^1, the planes of S_3^2 connect the points of U_1^1 with the corresponding tangents to T_1^1, and the three-flats of T_4^2 connect the points of U_1^1 with the corresponding osculating planes of T_1^1. The orders of R_2^2, S_3^2 and T_4^2 are 4, 5 and 4, respectively.

CORNELL UNIVERSITY, *January*, 1914.

On the Order of a Restricted System of Equations.

By F. F. Decker.

I. *Introduction.*

§ 1. The conditions under which two rational integral equations of degree μ and ν respectively will have k common roots may be found as follows:* Take

$$f_1(x) \equiv a_0 x^\mu + a_1 x^{\mu-1} + \ldots + a_r x^{\mu-r} + \ldots + a_\mu = 0$$

and

$$f_2(x) \equiv b_0 x^\nu + b_1 x^{\nu-1} + \ldots + b_r x^{\nu-r} + \ldots + b_\nu = 0,$$

having k common roots $\alpha_1, \alpha_2, \ldots, \alpha_k$. They may be written

$$f_1(x) \equiv c_0 (x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_k) \phi_1(x)$$

and

$$f_2(x) \equiv c_0 (x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_k) \phi_2(x),$$

where

$$\phi_1(x) \equiv A_0 x^{\mu-k} + A_1 x^{\mu-k-1} + \ldots + A_r x^{\mu-k-r} + \ldots + A_{\mu-k}$$

and

$$\phi_2(x) \equiv B_0 x^{\nu-k} + B_1 x^{\nu-k-1} + \ldots + B_r x^{\nu-k-r} + \ldots + B_{\nu-k}.$$

Consequently,

$$f_1(x) \cdot \phi_2(x) \equiv f_2(x) \cdot \phi_1(x).$$

By equating the coefficients of like powers of x in the two members of this equation, there result $l-k+1$ equations linear and homogeneous in the $l-2(k-1)$ quantities $A_0, A_1, \ldots, A_{\mu-k}, B_0, B_1, \ldots, B_{\nu-k}$, $(l=\mu+\nu)$, all of which must be satisfied by the coefficients $a_0, a_1, \ldots, a_\mu, b_0, b_1, \ldots, b_\nu$, as follows:

$$\begin{array}{lll}
a_0 B_0 & -b_0 A_0 & =0, \\
a_1 B_0 + a_0 B_1 & -b_1 A_0 - b_0 A_1 & =0, \\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & , \\
a_h B_0 + a_{h-1} B_1 + \ldots + a_{h-\nu+k} B_{\nu-k} & -b_h A_0 - b_{h-1} A_1 - \ldots - b_{h-\mu+k} A_{\mu-k} & =0, \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & , \\
a_\mu B_0 + a_{\mu-1} B_1 + \ldots + a_{\mu-\nu+k} B_{\nu-k} & \ldots\ldots\ldots\ldots & , \\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & , \\
a_\mu B_{\nu-k} & \ldots - b_\nu A_{\mu-k} & =0.
\end{array}$$

The conditions resulting from eliminating the quantities $A_0, A_1, \ldots, A_{\mu-k}$,

* Euler, "Berlin Memoirs," 1764, p. 96; "Histoire de l'Acad. de Paris," 1764, p. 298.

$B_0, B_1, \ldots, B_{\nu-k}$ from each set of $l-2(k-1)$ equations in turn may evidently be written in the form

$$\left\|\begin{array}{cccccc} a_0\, a_1 \ldots a_\mu & 0 & 0 & \ldots & & 0 \\ 0 \quad a_0 \ldots & a_\mu & 0 & \ldots & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 \quad 0 \ldots & & & 0 & a_0 \ldots & a_\mu \\ b_0\, b_1 \ldots & & b_\nu & 0 \ldots & & 0 \\ 0 \quad b_0 \ldots & & & b_\nu & 0 \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 \quad 0 \ldots b_0 & & & & \ldots & b_\nu \end{array}\right\| = 0,$$

or more briefly written (using a notation to be presently explained)

$$\|u_{mn}\|^{(m)}=0,$$

where $m=l-2(k-1)$ and $n=l-k+1$.

§ 2. The number of conditions is ${}_nC_m$, but it seems that not all are independent; in fact, the earlier part of the present work (III–IV) will concern itself with the number of linearly independent determinants in the set.

Subsequently each element u_{rs} of the matrix $\|u_{mn}\|^{(m)}$ will be viewed as a function of degree $a_s+\alpha_r$ of a set of variables.* The degrees of the elements are thus considered to be equidifferent with respect both to rows and to columns. The order of such a matrix will be determined (V). Finally, the geometric significance of the result will be pointed out (VI).

§ 3. A treatment of the number of solutions of the restricted system of equations obtained by equating to zero all the determinants of the m-th order of the matrix $\|u_{mn}\|$ may be found in Salmon's "Modern Higher Algebra"† (fourth edition, pp. 283–313).

After treating the special cases $(m=1, n=1)$, $(m=2, n=2)$, $(m=1, n=2)$, $(m=2, n=3)$, and $(m=1, n=3)$, Salmon announces without proof for the general case the number of solutions of the system of equations herein considered as

$$\begin{aligned} &\Sigma a_1 a_2 \ldots a_{n-m+1} + \Sigma a_1 a_2 \ldots a_{n-m} \cdot \Sigma \alpha_1 \\ &+ \Sigma a_1 a_2 \ldots a_{n-m-1} (\Sigma \alpha_1^2 + \Sigma \alpha_1 \alpha_2) + \Sigma a_1 a_2 \ldots a_{n-m-2} (\Sigma \alpha_1^3 + \Sigma \alpha_1^2 \alpha_2 + \Sigma \alpha_1 \alpha_2 \alpha_3) \\ &+ \ldots, \end{aligned}$$

or

$$\sum_{k=0}^{k=n-m+1} C_{n-m+1-k} \cdot H_k,$$

where C_k is the sum of all the elementary products of weight k involving the n a's, and H_k is the complete symmetric function of weight k involving the m α's.

---

* Compare Segre, "Gli ordini delle varietà che annullano dei diversi gradi estratti da una data matrice," *Rendic. R. Accad. Dei Lincei*, series 5, Vol. IX, session of October 21, 1900.

† This treatment may also be found in Salmon's "Analytic Geometry of Space" (German translation), third edition, Vol. II, pp. 585–607.

Stuyvaert* arrives at the same problem from geometric considerations, but likewise contents himself with the results for certain cases.

§ 4. The object of this investigation is to offer a theory for the general treatment of the restricted system of equations $\|u_{mn}\|^{(m)}=0$.

The method of treatment is in the first place based on the theory of the linear independence of $n-m+1$ of the determinants of $\|u_{mn}\|^{(m)}$. The method of showing the number of such linearly independent determinants indicates also a method for their selection.

Next it is shown how to express the remaining determinants linearly in terms of the linearly independent ones (IV). In that we begin with the apparent order of the system of linearly independent determinants, it is possible, by means of these linear expressions, to discover another system by whose order the apparent order of the given system must be reduced to obtain its actual order. This second system is obtained by combining with a set of equations, less in number than the number of variables and arising from the scheme of linear expressions, a number of equations equal to the deficiency and such that the new system determines values of the variables satisfying the linearly independent determinants without satisfying the complete system. The existence of such additional equations is shown generally, and a method for their derivation is developed (V).

The calculation of the order of the second system is reduced to the calculation of the order of a third system of lower order, by whose order the apparent order of the second system must be reduced. This process is continued until there is reached a simple system whose actual order can be obtained without further reduction (§ 26).

The method gives the result surmised by Salmon, as already stated.

II. *Notation and Notational Relations.*

$$\left\|\begin{matrix} u_{11} \dots\dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots\dots u_{mn} \end{matrix}\right\|^{(p)} \quad \text{or} \quad \|u_{mn}\|^{(p)}$$

denotes the aggregate of all determinants of the p-th order that can be formed from the matrix by suppressing $m-p$ rows and $n-p$ columns. That all these determinants vanish will be denoted by

$$\left\|\begin{matrix} u_{11} \dots\dots u_{1n} \\ \dots\dots\dots \\ u_{m1} \dots\dots u_{mn} \end{matrix}\right\|^{(p)} =0 \quad \text{or} \quad \|u_{mn}\|^{(p)}=0;$$

* M. Stuyvaert, "Cinq études de géométrie analytique," *Memoires de la Société royale des Sciences de Liége*, 3rd series, Vol. VII, 1907; see also M. Giambelli, "Le varietà rappresentate per mezzo di una matrice generica di forme," etc., *Rendic. R. Accad. Dei Lincei*, series 5, Vol. XIV, sessions of December 3 and 17, 1905.

and the order of such a system of equations by

$$O_{\|u_{mn}\|^{(p)}}.$$

$$\|_{\overline{h_1, h_2, \ldots, h_r}, k_1, k_2, \ldots, k_s} u_{mn}\|$$

signifies a matrix formed from the matrix

$$\|u_{mn}\|$$

by suppressing the columns $h_1, h_2, \ldots, h_r$ and inserting the columns $k_1, k_2, \ldots, k_s$.

Certain matrices formed from a given matrix by the suppression or addition of columns, or both, will here be defined.

$$\begin{aligned}
&G_{kA} \equiv \|_{\overline{1, 2, \ldots, k-1}} u_{mn}\|^{(m)}, \\
&G_{kB} \equiv \|_{\overline{1, 2, \ldots, k-2}} u_{mm}\|^{(m-k+2)}, && n-m+3>k>1, \\
&G_{kC} \equiv \|_{\overline{1, 2, \ldots, k-2}, h_1, \ldots, h_{k-2}} u_{mm}\|^{(m)}, && k>1, \\
&G_{kD} \equiv \|_{\overline{1, 2, \ldots, k-2}, h_1, \ldots, h_{k-3}} u_{mm}\|^{(m-1)}, && k>2,
\end{aligned}$$

where $h_1, h_2, \ldots, h_k$ are different integers of the series $m+1, m+2, \ldots, n$. $G_k \equiv G_{kA} + G_{kB}$; that is, the system G_k is composed of all the determinants of the two systems G_{kA} and G_{kB}. Likewise $G'_k \equiv G_{kA} + G_{kC}$. It is to be noticed that $G_1 \equiv G'_1 \equiv G_{1A}$, since $G_{1B} \equiv G_{1C} \equiv 0$; that is, there is no system G_{1B} or G_{1C}.

A second set of matrices will be defined as follows:

$$\begin{aligned}
&H_1 \equiv \|u_{m, m+1}\|^{(m)}, \\
&H_{2A} \equiv \|_{\overline{2}} u_{m, m+1}\|^{(m)}, \\
&H_{2B} \equiv \|_{\overline{1}} u_{m, m+1}\|^{(m)}, \\
&H_{3C} \equiv \|_{\overline{1, 2}} u_{m, m+1}\|^{(m-1)},
\end{aligned}$$

$$H_{3A} \equiv \left\| \begin{matrix} u_{13} \ldots\ldots u_{1\, m+1} \\ u_{33} \ldots\ldots u_{3\, m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\, m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{3B} \equiv \left\| \begin{matrix} u_{23} \ldots\ldots u_{2\, m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\, m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{4C} \equiv \left\| \begin{matrix} u_{33} \ldots\ldots u_{3\, m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\, m+1} \end{matrix} \right\|^{(m-2)},$$

$$H_{4A} \equiv \left\| \begin{matrix} u_{33}\, u_{35} \ldots\ldots u_{3\, m+1} \\ u_{43}\, u_{45} \ldots\ldots u_{4\, m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,3}\, u_{m\,4} \ldots\ldots u_{m\, m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{4B} \equiv \left\| \begin{matrix} u_{34} \ldots\ldots u_{3\, m+1} \\ u_{44} \ldots\ldots u_{4\, m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,4} \ldots\ldots u_{m\, m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{5C} \equiv \left\| \begin{matrix} u_{35} \ldots\ldots u_{3\, m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,5} \ldots\ldots u_{m\, m+1} \end{matrix} \right\|^{(m-3)},$$

. . . .

$H_{m+1A} \equiv u_{m\,m}$ or $u_{m-1, m+1}$, according as m is odd or even, $H_{m+1B} \equiv u_{m, m+1}$, $H_{m+1C} \equiv \|u_{m\, m-1}\, u_{m\, m}\|^{(1)}$ or $u_{m-1, m+1}$, according as m is odd or even. $H_k \equiv H_{kA} + H_{kB}$.

O_R represents the order of the system R.

§ 6. Certain symmetric functions also will be defined.

$$
\begin{array}{lll}
{}_kd_l = H_l(a_{k+1}, a_{k+2}, \ldots, a_m), & d_l = {}_0d_l, & {}_kd_0 = 1, \\
{}_k\delta_l = H_l(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n), & \delta_l = {}_0\delta_l, & {}_k\delta_0 = 1, \\
{}_kC_l = \sum\limits_{k+1}^{n} \prod\limits_{i=k+1}^{i=k+l} a_i, & C_l = {}_0C_l, & {}_kC_0 = 1, \\
{}_k\gamma_l = \sum\limits_{k+1}^{m} \prod\limits_{i=k+1}^{i=k+l} \alpha_i, & \gamma_l = {}_0\gamma_l, & {}_k\gamma_0 = 1. \\
{}_kK_l = \sum\limits_{i=0}^{i=l} {}_kC_i\,\delta_{l-i}, & K_l = {}_0K_l, & {}_kK_0 = 1, \\
{}_kJ_l = \sum\limits_{i=0}^{i=l} {}_kd_i\,\gamma_{l-i}, & J_l = {}_0J_l, & {}_kJ_0 = 1.
\end{array}
$$

§ 7. In order to facilitate later calculation, certain relations will be shown to exist among the symmetric functions just defined. First it will be shown that

$$\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j = 0\,^* \tag{1}$$

by calculating T, the coefficient of $\alpha_{i_1}^{k_1}\cdot\alpha_{i_2}^{k_2}\cdot\ldots\ldots\cdot\alpha_{i_q}^{k_q}$, where $\sum\limits_1^q k_1 = k$, $k_r \neq 0$ and $q \leq k$. From the function $\delta_{k-j}\gamma_j$, $j \leq q$, $\alpha_{i_1}^{k_1}\cdot\alpha_{i_2}^{k_2}\cdot\ldots\ldots\cdot\alpha_{i_q}^{k_q}$ can be formed by selecting terms containing any j of the quantities $\alpha_{i_1}, \ldots, \alpha_{i_q}$ from γ_j and multiplying by the proper factor from δ_{k-j}. Hence its coefficient is ${}_qC_j$. If $j > q$, $\alpha_{i_1}^{k_1}\cdot\alpha_{i_2}^{k_2}\cdot\ldots\ldots\cdot\alpha_{i_q}^{k_q}$ can not be formed at all, as every term of γ_j contains j α's.

$$\therefore T = \sum_{j=0}^{j=q}(-1)^j{}_qC_j = 0,$$

and
$$\therefore \sum_{j=0}^{j=k}\delta_{k-j}\gamma_j = \sum_{q=1}^{q=k}\sum_{j=0}^{j=q}(-1)^j{}_qC_j\,\alpha_1^{k_1}\;\alpha_2^{k_2}\;\ldots\ldots\cdot\alpha_q^{k_q} = 0. \tag{2}$$

§ 8. Next it will be shown that

$$\sum_{j=0}^{j=l}(-1)^jK_{l-j}J_j = 0 \tag{3}$$

by calculating S, the sum of the terms of degree $l-k$ in the a's and k in the α's. The sum of the terms of the proposed degree in $K_{l-j}J_j$ is

$C_{l-k}\cdot\delta_{k-j}\gamma_j + C_{l-k-1}\cdot d_1\cdot\delta_{k-j+1}\cdot\gamma_{j-1} + \ldots + C_{l-k-h}\cdot d_h\,\delta_{k-j+h}\,\gamma_{j-h} + \ldots + C_{l-k-j}\,d_j\,\delta_k.$ Performing the summation for each of these terms and noting that $\delta_{k-j} = 1$ when $j = k$, and that $\delta_{k-j} = 0$ when $j > k$,

$$
\begin{aligned}
S = {} & C_{l-k}\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j - C_{l-k-1}\,d_1\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j + \ldots \\
& + (-1)^h C_{l-k-h}\,d_h\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\,\gamma_j + \ldots + (-1)^{l-k}d_{l-k}\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j \\
= {} & \sum_{i=0}^{=l-k}(-1)^iC_{l-k-i}\cdot d_i\cdot\sum_{j=0}^{j=k}(-1)^j\delta_{k-j}\gamma_j,
\end{aligned}
$$

* See Burnside and Panton, "Theory of Equations," fourth edition, Vol. I, p. 179.

and from relation (1) the second Σ vanishes. Therefore

$$\sum_{j=0}^{j=l}(-1)^j K_{l-j} J_j = \sum_{k=0}^{k=l}\left[\sum_{i=0}^{i=l-k}(-1)^i C_{l-k-i}\, d_i \cdot \sum_{j=0}^{j=k}(-1)^j \delta_{k-j}\gamma_j\right]=0. \tag{4}$$

§ 9. It will also be shown that

$$\begin{aligned} K_l = {}_{\bar{1}}K_{l-1}\cdot J_1 - {}_{\bar{2}}K_{l-2}\cdot {}_{\bar{1}}J_2 + {}_{\bar{3}}K_{l-3}\cdot {}_{\bar{2}}J_3 - \ldots\ldots \\ + (-1)^{j-1}\,{}_{\bar{j}}K_{l-j}\cdot {}_{\overline{j-1}}J_j + \ldots\ldots + (-1)^l\,{}_{\overline{l-2}}J_{l-1}. \end{aligned} \tag{5}$$

From equation (3)

$$K_l = K_{l-1}J_1 - K_{l-2}J_2 + \ldots\ldots + (-1)^{j-1}K_{l-j}J_j + \ldots\ldots + (-1)^{l-1}J_l.$$

By separating the terms of K_l into two classes, according as they do or do not contain a_1,

$$K_l = a_1\,{}_{\bar{1}}K_{l-1} + {}_{\bar{1}}K_l. \tag{6}$$

For the J's the result is $J_l = {}_{\bar{1}}J_l + a_1 J_{l-1}$, and by applying the formula to itself,

$$J_l = {}_{\bar{1}}J_l + a_1\,{}_{\bar{1}}J_{l-1} + a_1^2\,{}_{\bar{1}}J_{l-2} + \ldots\ldots + a_1^h\,{}_{\bar{1}}J_{l-h}. \tag{7}$$

From 5 and 6,

$$\begin{aligned} K_l = a_1\,{}_{\bar{1}}K_{l-1} + {}_{\bar{1}}K_{l-1}\cdot {}_{\bar{1}}J_1 - {}_{\bar{1}}K_{l-2}\cdot {}_{\bar{1}}J_2 + \ldots\ldots \\ + (-1)^{j-1}\,{}_{\bar{1}}K_{l-j}\cdot {}_{\bar{1}}J_j + \ldots\ldots + (-1)^{l-1}\,{}_{\bar{1}}J_{l-1}, \end{aligned} \tag{8}$$

and from 7,

$$\begin{aligned} K_l = {}_{\bar{1}}K_{l-1}\cdot J_1 - {}_{\bar{1}}K_{l-2}\cdot {}_{\bar{1}}J_2 + {}_{\bar{1}}K_{l-3}\cdot {}_{\bar{1}}J_3 - \ldots\ldots \\ + (-1)^{j-1}\,{}_{\bar{1}}K_{l-j}\cdot {}_{\bar{1}}J_j + \ldots\ldots + (-1)^{l-1}\,{}_{\bar{1}}J_{l-1}. \end{aligned} \tag{9}$$

By applying (6) and (7) to the $(j-1)$-th and j-th terms of the right member of (9),

$$\begin{aligned} {}_{\bar{1}}K_{l-j}\cdot {}_{\bar{1}}J_j - {}_{\bar{1}}K_{l-j+1}\cdot {}_{\bar{1}}J_{j-1} = ({}_{\bar{2}}K_{l-j} + a_2\cdot {}_{\bar{2}}K_{l-j-1})\,({}_{\bar{2}}J_j + a_2\cdot {}_{\bar{2}}J_{j-1} + \ldots\ldots + a_2^{j-2}\cdot {}_{\bar{2}}J_2) \\ - ({}_{\bar{2}}K_{l-j+1} + a_2\cdot {}_{\bar{2}}K_{l-j})\,({}_{\bar{2}}J_{j-1} + a_2\cdot {}_{\bar{2}}J_{j-2} + \ldots\ldots + a_2^{j-2}\cdot {}_{\bar{2}}J_2), \end{aligned}$$

and therefore the factor multiplying ${}_{\bar{2}}K_{l-j}$ in (9) is ${}_{\bar{2}}J_j$, and (9) becomes

$$\begin{aligned} K_l = {}_{\bar{1}}K_{l-1}\cdot J_1 - {}_{\bar{2}}K_{l-2}\cdot {}_{\bar{1}}J_2 + {}_{\bar{2}}K_{l-3}\cdot {}_{\bar{2}}J_3 - \ldots\ldots \\ + (-1)^{j-1}\,{}_{\bar{2}}K_{l-j}\cdot {}_{\bar{2}}J_j + \ldots\ldots + (-1)^{j-1}\,{}_{\bar{2}}J_j. \end{aligned}$$

Repeating this reduction a sufficient number of times,

$$\begin{aligned} K_l = {}_{\bar{1}}K_{l-1}\cdot J_1 - {}_{\bar{2}}K_{l-2}\cdot {}_{\bar{1}}J_2 + {}_{\bar{3}}K_{l-3}\cdot {}_{\bar{2}}J_3 - \ldots\ldots \\ + (-1)^{j-1}\,{}_{\bar{j}}K_{l-j}\cdot {}_{\overline{j-1}}J_j + \ldots\ldots + (-1)^{l-1}\,{}_{\overline{l-1}}J_l. \end{aligned} \tag{10}$$

III. *Some Theorems on the Linear Independence of Certain of the Determinants* $\|u_{m\,n}\|^{(m)}$.

§10. THEOREM 1: In $\|u_{m\,m+1}\|^{(m)}$ there are two linearly independent determinants.

Write

$$\Delta \equiv \begin{vmatrix} v_{11}\,v_{12} & \ldots\ldots v_{1\,m+1} \\ u_{11}\,u_{12} & \ldots\ldots u_{1\,m+1} \\ \ldots\ldots & \ldots\ldots \\ u_{m1}\,u_{m2} & \ldots\ldots u_{m\,m+1} \end{vmatrix},$$

and let V_{1k} be the cofactor of v_{1k} in Δ. Then

$$\begin{array}{ll}
u_{11}V_{11}+\quad u_{12}V_{12}+\ldots\ldots+\quad u_{1\,m+1}V_{1\,m+1}=0 & (e_1),\\
u_{21}V_{11}+\quad u_{22}V_{12}+\ldots\ldots+\quad u_{2\,m+1}V_{1\,m+1}=0 & (e_2),\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \\
u_{m-1\,1}V_{11}+v_{m-1\,2}V_{12}+\ldots\ldots+u_{m-1\,m+1}V_{1\,m+1}=0 & (e_{m-1}),\\
u_{m\,1}V_{11}+\quad u_{m\,2}V_{12}+\ldots\ldots+\quad u_{m\,m+1}V_{1\,m+1}=0 & (e_m).
\end{array}$$

(e_1) makes it possible to express one V linearly in terms of the m others by means of determinants of the first order, showing that there are not more than m linearly independent V's. (e_1) and (e_2) make it possible to express two V's linearly in terms of the $m-1$ others by means of determinants of the second order, showing that there are not more than $m-1$ linearly independent V's. Finally, the system (e_1), (e_2), ..., (e_{m-1}) makes it possible to express $m-1$ V's linearly in terms of the two others by means of determinants of the $(m-1)$-th order, showing that there are not more than two linearly independent V's.

It might seem that the system (e_1), (e_2), ..., (e_m) would enable one to express m V's linearly in terms of the other one by means of determinants of the m-th order; but such is not the case, for when the solution is attempted an identity results, as will be shown by an attempt to solve for $V_{1\,k}$ in terms of $V_{1\,m+1}$.

$$V_{1k}=\frac{\begin{vmatrix} u_{11}\ldots u_{1k-1}\;u_{1m+1}\;u_{1k+1}\ldots u_{1m}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u_{m1}\ldots u_{mk-1}\;u_{mm+1}\;u_{mk+1}\ldots u_{mm}\end{vmatrix}}{\begin{vmatrix} u_{11}\ldots\ldots\ldots\ldots\ldots\ldots u_{1m}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u_{m1}\ldots\ldots\ldots\ldots\ldots\ldots u_{mm}\end{vmatrix}}V_{1m+1}=(-1)^{m-k+1}\frac{\begin{vmatrix} u_{11}\ldots u_{1k-1}\;u_{1k+1}\ldots u_{1m+1}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u_{m1}\ldots u_{mk-1}\;u_{mk+1}\ldots u_{mm+1}\end{vmatrix}}{\begin{vmatrix} u_{11}\ldots\ldots\ldots\ldots\ldots u_{1m}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u_{m1}\ldots\ldots\ldots\ldots\ldots u_{mm}\end{vmatrix}}V_{1m+1},$$

or

$$\Delta_{1\,m+1}V_{1\,k}=(-1)^{m-k+1}\Delta_{1\,k}V_{1\,m+1},$$

where $\Delta_{1\,k}$ is the determinant formed by deleting the k-th column of $\|u_{m\,m+1}\|$; that is,

$$\Delta_{1\,k}\equiv\|{}_k u_{m\,m+1}\|^{(m)}.$$

This gives $\Delta_{1\,h}=(-1)^{1+h}V_{1\,h}$, and therefore

$$(-1)^{m+2}V_{1\,m+1}\cdot V_{1\,k}=(-1)^{m-k+1}(-1)^{k+1}V_{1\,k}\cdot V_{1\,m+1},$$

or

$$V_{1\,m+1}\cdot V_{1\,k}=V_{1\,m+1}\cdot V_{1\,k},$$

an identity.

Since the V's are numerically the determinants $\|u_{m\,m+1}\|^{(m)}$, the theorem follows.

The theorem shows that any two of the determinants may be used as the two linearly independent ones. Unless otherwise stated, the two chosen will be

the two having the first $m-1$ columns common; that is,

$$\|_{\overline{m+1}}\, u_{m\ m+1}\|^{(m)} \text{ and } \|_{\overline{m}}\, u_{m\ m+1}\|^{(m)}.$$

§ 11. It will be shown by mathematical induction that the number of linearly independent determinants of the system $\|u_{m\ n}\|^{(m)}$ is $n-m+1$; that is, that all the determinants of the system can be linearly expressed in terms of $n-m+1$ of them.

Before passing from the case just given to the general case, a few special cases will be given to elucidate the process.

§ 12. Next it will be shown that

THEOREM 2: $\|u_{m\ m+2}\|^{(m)}$ can be linearly expressed in terms of

$$\left.\begin{array}{rl} \|_{\overline{m+1,\, m+2}}\, u_{m\ m+2}\|^{(m)} & 1, \\ \|_{\overline{m,\, m+2}}\, u_{m\ m+2}\|^{(m)} & 2, \\ \text{and} \quad \|_{\overline{m,\, m+1}}\, u_{m\ m+2}\|^{(m)} & 3, \end{array}\right\} I_{m,\, m+2}.$$

The determinants

$$\|_{\overline{m+2}}\, u_{m\ m+2}\|^{(m)} \tag{a}$$

and

$$\|_{\overline{m+1}}\, u_{m\ m+2}\|^{(m)} \tag{b}$$

can be so expressed; for, by theorem 1, (a) can be so expressed in terms of 1 and 2 and (b) in terms of 1 and 3.

Since the remaining matrices of $m+1$ columns contain both columns $m+1$ and $m+2$, they contain $m-1$ of the first m columns each. The one that contains the first $m-1$ columns $\|_{\overline{m}}\, u_{m\ m+2}\|^{(m)}$ is linearly expressible, according to theorem 1, in terms of 2 and 3.

For the case of one of the others,

$$\|_{\overline{h}}\, u_{m\ m+2}\|^{(m)}, \qquad h=1, 2, \ldots, m-1,$$

it is linearly expressible, according to theorem 1, in terms of $\|_{\overline{h,\, m+2}}\, u_{m\ m+2}\|^{(m)}$ and $\|_{\overline{h,\, m+1}}\, u_{m\ m+2}\|^{(m)}$, the first of which is included in (a), the second in (b); and (a) and (b) have already been shown to be linearly dependent on $I_{m,\, m+2}$. Thus all the determinants $\|u_{m\ m+2}\|^{(m)}$ are linearly dependent on the system $I_{m,\, m+2}$.

§ 13. Next it will be shown that

THEOREM 3: $\|u_{m\ m+3}\|^{(m)}$ can be linearly expressed in terms of

$$\left.\begin{array}{rl} \|_{\overline{m+1,\, m+2,\, m+3}}\, u_{m\ m+3}\|^{(m)} & 1, \\ \|_{\overline{m,\, m+2,\, m+3}}\, u_{m\ m+3}\|^{(m)} & 2, \\ \|_{\overline{m,\, m+1,\, m+3}}\, u_{m\ m+3}\|^{(m)} & 3, \\ \text{and} \quad \|_{\overline{m,\, m+1,\, m+2}}\, u_{m\ m+3}\|^{(m)} & 4, \end{array}\right\} I_{m,\, m+3}.$$

The determinants

$$\|_{\overline{m+3}}\, u_{m\;m+3}\|^{(m)} \qquad \text{(a)}$$

and

$$\|_{\overline{m+2}}\, u_{m\;m+3}\|^{(m)} \qquad \text{(b)}$$

can be so expressed, for, by theorem 2, (a) can be so expressed in terms of 1, 2 and 3, and (b) in terms of 1, 2 and 4.

Since the remaining matrices of $m+2$ columns contain both columns $m+2$ and $m+3$, they contain also m of the first $m+1$ columns each. The one that contains the first m,

$$\|_{\overline{m+1}}\, u_{m\;m+3}\|^{(m)},$$

is linearly expressible, according to theorem 2, in terms of 1, 3 and 4.

For the case of one of the others, $\|_{\bar{h}}\, u_{m\;m+3}\|^{(m)}$, $h=1, 2, \ldots, m$, it is linearly expressible, according to theorem 2, in terms of

$$\|_{\overline{h,\, m+2,\, m+3}}\, u_{m\;m+3}\|^{(m)},$$

$$\|_{\overline{h,\, m+1,\, m+3}}\, u_{m\;m+3}\|^{(m)},$$

and

$$\|_{\overline{h,\, m+1,\, m+2}}\, u_{m\;m+3}\|^{(m)},$$

the first two of which are included in (a), the third in (b); and (a) and (b) have already been shown to be linearly dependent on the system $I_{m,\,m+3}$. Thus all the determinants

$$\|u_{m\;m+3}\|^{(m)}$$

are linearly dependent on the system $I_{m,\,m+3}$.

§ 14. It will now be shown that

THEOREM 4: The determinants $\|u_{m\;m+r}\|^{(m)}$ are linearly expressible in terms of the $r+1$ determinants

$$\left.\begin{array}{rr}
\|_{\overline{m+1,\ldots,m+r}}\, u_{m\;m+r}\|^{(m)} & 1, \\
\|_{\overline{m,\, m+2,\ldots,\, m+r}}\, u_{m\;m+r}\|^{(m)} & 2, \\
\|_{\overline{m,\, m+1,\, m+3,\ldots,\, m+r}}\, u_{m\;m+r}\|^{(m)} & 3, \\
\cdots\cdots\cdots\cdots\cdots\cdots & \ldots, \\
\|_{\overline{m,\ldots,\, m+h-1,\, m+h+1,\ldots,\, m+r}}\, u_{m\;m+r}\|^{(m)} & h+1, \\
\cdots\cdots\cdots\cdots\cdots\cdots & \ldots, \\
\|_{\overline{m,\ldots,\, m+r-1}}\, u_{m\;m+r}\|^{(m)} & r+1,
\end{array}\right\} I_{m,\,m+r};$$

provided the expressibility is possible when r is replaced by $r-1$.

The determinants

$$\|_{\overline{m+r}}\, u_{m\;m+r}\|^{(m)} \qquad \text{(a)}$$

and

$$\|_{\overline{m+r-1}}\, u_{m\;m+r}\|^{(m)} \qquad \text{(b)}$$

can be so expressed, for, by the assumption, (a) can be so expressed in terms of 1, 2, ..., r, and (b) in terms of 1, 2, ..., $r-1$ and $r+1$.

Since the remaining matrices of $m+r-1$ columns contain both the columns

$m+r-1$ and $m+r$, each of them contains also $m+r-3$ of the first $m+r-2$ columns; that is, they are

$$\|_{\overline{h}}u_{m\ m+r}\|^{(m)}, \quad h=1, 2, \ldots, m+r-2.$$

The matrix $\|_{\overline{m+r-2}}u_{m\ m+r}\|^{(m)}$ is linearly expressible, according to the assumption, in terms of $2, 3, \ldots, r+1$.

For the case of one of the others,

$$\|_{\overline{h}}u_{m\ m+r}\|^{(m)}, \quad h=1, 2, \ldots, m+r-3,$$

it is linearly expressible, according to the assumption, in terms of

$$\|_{\overline{h,\ m+2,\ \ldots,\ m+r}}\,u_{m\ m+r}\|^{(m)},$$

$$\|_{\overline{h,\ m+1,\ m+3,\ \ldots,\ m+r}}\,u_{m\ m+r}\|^{(m)},$$

$$\|_{\overline{h,\ m+1,\ m+2,\ m+4,\ \ldots,\ m+r}}\,u_{m\ m+r}\|^{(m)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\|_{\overline{h,\ m+1,\ldots,\ m+k-1,\ m+k+1,\ \ldots,\ m+r}}\,u_{m\ m+r}\|^{(m)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and

$$\|_{\overline{h,\ m+1,\ldots,\ m+r-1}}\,u_{m\ m+r}\|^{(m)},$$

all but the last of which are included in (a), the last being included in (b). As (a) and (b) have already been shown to be linearly dependent on the system $I_{m,\ m+r}$, it follows that all the determinants $\|u_{m\ m+r}\|^{(m)}$ are linearly dependent on the system $I_{m,\ m+r}$.

§ 15. Theorems 1, 2 and 3 respectively state that the assumption of theorem 4 is true for the cases $r=2$, 3 and 4. Therefore by mathematical induction there follows the more general theorem

THEOREM 5: The number of linearly independent determinants among $\|u_{m\ n}\|^{(m)}$ is $n-m+1$.*

§ 16. It has been pointed out (§ 1) that the condition under which two rational integral equations of degrees μ and ν respectively will have k common roots may be written in the form

$$\|u_{m\ n}\|^{(m)}=0,$$

where $n=\mu+\nu-k+1$ and $m=\mu+\nu-2(k-1)$. It is to be noticed from theorem 5 that the number of linearly independent determinants involved is

$$n-m+1=[\mu+\nu-k+1]-[\mu+\nu-2(k-1)]+1=k.$$

IV. *A Problem on Expressibility.*

§ 17. The expressibility of all the determinants $\|u_{m\ n}\|^{(m)}$ in terms of $n-m+1$ of them having been shown, the problem of actually expressing them will be considered.

* Compare Sommerville, *Proceedings of the Edinburgh Mathematical Society*, Vol. XXIV, November 10, 1905. See also Muir, "On Some hitherto Unproved Theorems in Determinants," *Proceedings of the Royal Society of Edinburgh* for 1890–1891.

PROBLEM 1: To express any determinant $|h_1, h_2, \ldots, h_m|$ of $\|u_{mn}\|^{(m)}$ in terms of the determinants $|1, 2, \ldots, m-1, k|$, $k=m, m+1, \ldots, n$.

The first case considered will be that in which $|h_1, h_2, \ldots, h_m|$ contains none of the first $m-1$ columns of $\|u_{mn}\|$,

$$|h_1, h_2, \ldots, h_m| \equiv \begin{vmatrix} u_{1h_1} & u_{1h_2} & \ldots & u_{1h_m} \\ u_{2h_1} & u_{2h_2} & \ldots & u_{2h_m} \\ \ldots & \ldots & \ldots & \ldots \\ u_{mh_1} & u_{mh_2} & \ldots & u_{mh_m} \end{vmatrix}.$$

Multiply the first row by W_1, the second by W_2, etc., and add to the first row the sum of those following, where W_k is the cofactor of w_k in

$$\begin{vmatrix} u_{11} & \ldots & u_{1\,m-1} & w_1 \\ u_{21} & \ldots & u_{2\,m-1} & w_2 \\ \ldots & \ldots & \ldots & \ldots \\ u_{m1} & \ldots & u_{m\,m-1} & w_m \end{vmatrix}.$$

$$|h_1, h_2, \ldots, h_m| \equiv \frac{1}{W_1 W_2 \ldots W_m} \begin{vmatrix} u_{1h_1}W_1 & u_{1h_2}W_1 & \ldots & u_{1h_m}W_1 \\ u_{2h_1}W_2 & u_{2h_2}W_2 & \ldots & u_{2h_m}W_2 \\ \ldots & \ldots & \ldots & \ldots \\ u_{mh_1}W_m & u_{mh_2}W_m & \ldots & u_{mh_m}W_m \end{vmatrix},$$

or

$$|h_1, h_2, \ldots, h_m| \equiv \frac{1}{W} \begin{vmatrix} |1, 2, \ldots, m-1, h_1| & \ldots & |1, 2, \ldots, m-1, h_m| \\ u_{2h_1} & \ldots & u_{2h_m} \\ \ldots & \ldots & \ldots \\ u_{mh_1} & \ldots & u_{mh_m} \end{vmatrix}.$$

By expanding the determinant in the right member in terms of the elements of its first row,

$$\begin{aligned} |h_1, h_2, \ldots, h_m| &\equiv \frac{1}{W_1} [\,|h_2, \ldots, h_m| \cdot |1, 2, \ldots, m-1, h_1| \\ &\qquad - |h_1, h_3, \ldots, h_m| \cdot |1, 2, \ldots, m-1, h_2| + \ldots] \\ &\equiv \frac{1}{W_1} \sum_{j=1}^{j=m} (-1)^{j-1} |h_{i_1}, \ldots, h_{i_{j-1}}, h_{i_{j+1}}, \ldots, h_{i_m}| \cdot \\ &\qquad \cdot |1, 2, \ldots, m-1, h_{i_j}|, \qquad (11) \end{aligned}$$

where $|h_1, h_2, \ldots, h_{m-1}|$ is a determinant formed from $|h_1, h_2, \ldots, h_{m-1}, h_l|$ by suppressing the first row and the last column.

§ 18. To express $|h_1, h_2, \ldots, h_m|$, in the case under consideration, evidently $n-m+1$ linearly independent determinants, that is, a complete system, are required. To express $|h_1, h_2, \ldots, h_m|$ in terms of linearly independent determinants, in the case where it contains k of the first $m-1$ columns, the argument as given for the other case is valid, but evidently only $n-m+1-k$ of the linearly independent determinants are required, as k of the determinants in the right member of formula (11) have each two common columns and vanish identically.

§ 19. By a slight change in the details of the proof a formula similar to (11) may be obtained in which W_1 is replaced by W_k, $k=1, 2, \ldots, m$. However there follows the

COROLLARY: The expressibility is not possible if

$$\|u_{m\,m-1}\|^{(m-1)} \equiv \left\| \begin{matrix} u_{11} & \ldots & u_{1\,m-1} \\ \ldots & \ldots & \ldots \\ u_{m\,1} & \ldots & u_{m\,m-1} \end{matrix} \right\|^{(m-1)} = 0.$$

V. *The Reduction of the Order of* $\|u_{m\,n}\|^{(m)}=0$.

§ 20. A part of the notation will be repeated at this point for convenient reference.

$$G_{kA} \equiv \|_{\overline{1,2,\ldots,k-1}}\, u_{m\,n}\|^{(m)} \equiv \left\| \begin{matrix} u_{1\,k} & u_{1\,k+1} & \ldots & u_{1\,m} & \ldots & u_{1\,n} \\ u_{2\,k} & u_{2\,k+1} & \ldots & u_{2\,m} & \ldots & u_{2\,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{m\,k} & u_{m\,k+1} & \ldots & u_{m\,m} & \ldots & u_{m\,n} \end{matrix} \right\|^{(m)},$$

$$G_{kB} \equiv \|_{\overline{1,2,\ldots,k-2}}\, u_{m\,m}\|^{(m-k+2)} \equiv \left\| \begin{matrix} u_{1\,k-1} & \ldots & u_{1\,m} \\ u_{2\,k-1} & \ldots & u_{2\,m} \\ \ldots & \ldots & \ldots \\ u_{m\,k-1} & \ldots & u_{m\,m} \end{matrix} \right\|^{(m-k+2)},$$

$$G_{kC} \equiv \|_{\overline{1,2,\ldots,k-2,\,h_1,\ldots,h_{k-2}}}\, u_{m\,m}\|^{(m)} \equiv \left\| \begin{matrix} u_{1\,k-1} & \ldots & u_{1\,m} & u_{1\,h_1} & \ldots & u_{1\,h_{k-2}} \\ u_{2\,k-1} & \ldots & u_{2\,m} & u_{2\,h_1} & \ldots & u_{2\,h_{k-2}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{m\,k-1} & \ldots & u_{m\,m} & u_{m\,h_1} & \ldots & u_{m\,h_{k-2}} \end{matrix} \right\|^{(m)},$$

$$G_{kD} \equiv \|_{\overline{1,2,\ldots,k-2,\,h_1,\ldots,h_{k-3}}}\, u_{m\,m}\|^{(m-1)} \equiv \left\| \begin{matrix} u_{1\,k-1} & \ldots & u_{1\,m} & u_{1\,h_1} & \ldots & u_{1\,h_{k-3}} \\ u_{2\,k-1} & \ldots & u_{2\,m} & u_{2\,h_1} & \ldots & u_{2\,h_{k-3}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ u_{m\,k-1} & \ldots & u_{m\,m} & u_{m\,h_1} & \ldots & u_{m\,h_{k-3}} \end{matrix} \right\|^{(m-1)},$$

$$G_k \equiv G_{kA} + G_{kB} \quad \text{and} \quad G'_k \equiv G_{kA} + G_{kC}.$$

It will be noticed that $G_1 \equiv G'_1 \equiv G_{1A}$, since $G_{1B} \equiv G_{1C} \equiv 0$; that is, no system G_{1B} or G_{1C} is formed.

No system G_{kA} can be formed when $n-(k-1)<m$; that is, when $k>n-m+1$. Likewise no system G_{kB} can be formed when $m-(k-2)<1$; that is, when $k>m+1$. G_{kC} is a single determinant.

§ 21. Since $G_1 \equiv \|u_{m\,n}\|^{(m)}$, $m \leq n$, contains $n-m+1$ linearly independent determinants (theorem 5), the elements of its determinants will be considered to be non-homogeneous functions of degree as previously stated (§ 2), of a set of $n-m+1$ variables; so that the system $\|u_{m\,n}\|^{(m)}$ will be exactly sufficient to determine value systems of the variables.

The determinants of G_1 can be linearly expressed in terms of

$$G_2' \equiv G_{2A} + G_{2C} \equiv \|_1 u_{m\,n}\|^{(m)} + \|u_{m\,m}\|^{(m)}$$

if $G_{3D} \equiv \|_1 u_{m\,m}\|^{(m-1)} \neq 0$. (Problem 1.) Therefore

THEOREM 6: $O_{G_1} \leqq O_{G_2}$ (note $G_2 \equiv G_2'$), where

$$G_2 \equiv G_{2A} + G_{2B} \equiv \|_1 u_{m\,n}\|^{(m)} + \|u_{m\,m}\|^{(m)}.$$

§ 22. But G_1 is not linearly expressible in terms of G_2' if $G_{3D} = 0$ (problem 1, corollory). Hence, if a system of equations can be formed from which values of the variables can be determined that will make G_{3D} and G_2 vanish simultaneously, one will have $G_2 = 0$ without having $G_1 = 0$. In that case, the order of such a system should be deducted from O_{G_2} in determining O_{G_1}. It will be shown that G_3 is such a system.

$G_{3A} \equiv \|_{\overline{1,2}}\, u_{m\,n}\|^{(m)}$ contains $n-m-1$ linearly independent determinants (theorem 5), and, therefore, to make it vanish requires the fixing of $n-m-1$ variables in terms of the remaining ones, and leaves but two to provide for the vanishing of G_{3D} and $G_2 \equiv G_{2A} + G_{2B}$. But if G_{3A} and G_{3D} vanish together, G_{3C} vanishes also, since it can be expanded in terms of G_{3D}, and therefore G_{2A} vanishes, unless $G_{4D} \equiv \|_{\overline{12}\,h_1}\, u_{m\,m}\|^{(m-1)} = 0$, since G_{2A} is linearly expressible in terms of G_3', unless $G_{4D} = 0$ (problem 1); so that the problem is to cause G_{3D} and G_{2B} to vanish by the fixing of two variables. This might seem to be impossible, since G_{3D} contains two and G_{2B} one linearly independent determinant. But the three may not be independent of each other, since the columns of G_{3D} are to be found in G_{2B}; in fact, if the two variables are so fixed as to make $G_{3D} \equiv G_{3B} = 0$, G_{2B} will vanish, since the minors of the elements of its first column vanish. Thus G_3 is a system whose vanishing uniquely determines the values of the $n-m+1$ variables, so as to make $G_2 = 0$ without making $G_1 = 0$. Therefore

THEOREM 7: $O_{G_1} \overline{\gtrless} O_{G_2} - G_{G_3}$.

§ 23. But G_2 is not linearly expressible in terms of G_3' if $G_{4D} = 0$ (problem 1, corollory). Hence, if a system of equations can be formed, from which values of the variables can be determined that will make G_{4D} and G_3 vanish simultaneously, one will have $G_3 = 0$ without having $G_2 = 0$. In this case, not more than the excess of O_{G_3} over the order of such a system should be deducted from O_{G_2} in determining O_{G_1}. It will be shown that G_4 is such a system.

$G_{4A} \equiv \|_{\overline{123}}\, u_{m\,n}\|^{(m)}$ contains $n-m-2$ linearly independent determinants (theorem 5), and, therefore, to make it vanish requires the fixing of $n-m-2$ variables in terms of the remaining ones and leaves but three to provide for the vanishing of G_{4D}, G_{3A}, and G_{3B}. But if G_{4A} and G_{4D} vanish simultaneously, $G_{4C} \equiv \|_{\overline{12}\,h_1\,h_2}\, u_{m\,m}\|^{(m)}$ vanishes also, as it can be expanded in terms of G_{4D} by

Laplace's development, and therefore G_{3A} vanishes also, unless

$$G_{5D} \equiv \| _{\overline{123}\, h_1\, h_2} u_{m\, m} \|^{(m-1)} = 0,$$

since G_{3A} is linearly expressible in terms of $G'_4 \equiv G_{4A} + G_{4C}$, unless $G_{5D} = 0$ (problem 1); so that the problem is, to cause G_{4D} and G_{3B} to vanish simultaneously by the fixing of three variables. This again might seem impossible, since G_{3B} contains two determinants linearly independent of each other and G_{4D} a like number. However, the four need not be linearly independent of each other, as the two matrices have some common columns. The problem can be solved by fixing the remaining three variables so as to make all the determinants of a matrix of a set of common columns (in terms of which all the determinants can be expanded by Laplace's development) vanish, provided they have a sufficient number of common columns to form a matrix containing not more than three linearly independent determinants. They are seen to have $m-2$ common columns, which is just sufficient and therefore uniquely determines the remaining three variables; that is, the $n-m+1$ variables are determined from the system $G_4 = 0$. It follows that

THEOREM 8: $O_{G_1} \leqq O_{G_2} - O_{G_3} + O_{G_4}$.

§ 24. Suppose it has been shown where k is even that

$$O_{G_1} \geqq O_{G_2} - O_{G_3} + \ldots\ldots - O_{G_{k-1}},$$

and where k is odd that

$$O_{G_1} \leqq O_{G_2} - O_{G_3} + \ldots\ldots + O_{G_{k-1}};$$

it will be shown in the former case that

$$O_{G_1} \leqq O_{G_2} - O_{G_3} + \ldots\ldots + O_{G_k},$$

and in the latter that

$$O_{G_1} \geqq O_{G_2} - O_{G_3} + \ldots\ldots - O_{G_k};$$

that is, it will be shown in both cases that the substitution of $O_{G_{k-1}} - O_{G_k}$ for $O_{G_{k-1}}$ reverses the sense of the inequality.

$G_{k-2\,A}$ is not linearly expressible in terms of G'_{k-1} if $G_{kD} = 0$ (problem 1, corollary). Hence, if a system of equations can be formed, from which values of the variables can be determined that will make G_{kD} and G_{k-1} vanish simultaneously, one will have $G_{k-1} = 0$ without having $G_{k-2} = 0$. In that case, not more than the excess of $O_{G_{k-1}}$ over the order of such a system should be deducted from $O_{G_{k-2}}$ in determining O_{G_1}. It will be shown that G_k is such a system.

G_{kA}, where $k < n-m+2$, contains $n-m-k+2$ linearly independent determinants (theorem 5), and, therefore, to make it vanish requires the fixing of $n-m-k+2$ variables in terms of the remaining ones, and leaves but $k-1$ to provide for the vanishing of G_{kD} and G_{k-1}. But if G_{kA} and G_{kD} vanish simul-

taneously, G_{kC} vanishes also, as it can be expanded in terms of G_{kD} by Leplace's development, and therefore G_{k-1A} also vanishes unless $G_{k+1D}=0$, since G_{k-1A} is linearly expressible in terms of G'_k unless $G_{k+1D}=0$ (problem 1); so that the problem is to cause G_{kD} and G_{k-1B} to vanish simultaneously by the fixing of $k-1$ variables. This might seem impossible, since G_{k-1B} contains $k-2$ determinants linearly independent of each other, and G_{kD} two of them. However, these k determinants need not all be linearly independent of each other, as the two matrices may have some common columns. The problem can be solved by fixing the remaining $k-1$ variables so as to make vanish all the determinants of a matrix of a set of common columns (in terms of which all the determinants can be expanded by Laplace's development), provided they have a sufficient number of common columns to form a matrix containing not more than $k-1$ linearly independent determinants. They are seen to have $m-k+2$ common columns, which, if $m-k+2>0$, is just sufficient and therefore uniquely determines the $k-1$ remaining variables; that is, the $n-m+1$ variables are determined from the system $G_k=0$. Therefore

THEOREM 9: The substitution of $O_{G_{k-1}}-O_{G_k}$ for $O_{G_{k-1}}$ reverses the sense of the inequality limiting O_{G_1} when $k<m+2$.

§ 25. But when $k=m+2$, the reduction ceases, for

$$G_{m+2D}\equiv\|_{\overline{1,2,\ldots,m},\,h_1,\,h_2,\,\ldots,\,h_{m-1}}u_{m\,n}\|^{(m-1)}$$

and

$$G_{m+1B}\equiv\|_{\overline{1,2,\ldots,m-1}}u_{m\,m}\|^{(1)}$$

have no common columns, and therefore the two linearly independent determinants of the former and the $m-1$ of the latter are all linearly independent of each other, and the m remaining variables can not be so chosen as to make both sets vanish. Therefore the $n-m+1$ variables can not be so determined as to render $G_{m+1}=0$ without rendering $G_m=0$. Thus no reduction should be made from $O_{G_{m+1}}$ in calculating O_{G_1}, and therefore

THEOREM 10: $O_{G_1}=O_{G_2}-O_{G_3}+\ldots\ldots+(-1)^{m+1}O_{G_{m+1}}$.

§ 26. When $k=n-m+2$, G_{kA} can not be formed. In this case, however, the $n-m+1$ variables can be so chosen as to make G_{kB} vanish, which will in turn cause the vanishing of G_{k-1} without causing the vanishing of G_{k-2}. To express this result by the formula of theorem 10, it is to be observed that when $k=n-m+2$, $G_k\equiv G_{kB}$, and that when $k>n-m+2$, $G_k=0$. It follows for all cases that

THEOREM 11:* $O_{G_1}=O_{G_2}-O_{G_3}+\ldots\ldots+(-1)^{m+1}O_{G_{m+1}}$, which is a reduction formula for the order of $\|u_{m\,n}\|^{(m)}=0$.

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*Compare Brill, "Ueber algebraische Correspondenzen" II, *Clebsch-Neumann Mathematische Annalen*, Vol. XXXVI (1890). (Brill's paper which discusses special groups of points on a curve has been called to the writer's attention since the completion of this paper.)

§ 27. In the search for the actual order of $\|u_{m\,n}\|^{(m)}$, the order of $\|u_{m\,m+1}\|^{(m)}$ will next be investigated. It may for the present be regarded as containing two variables only, for when used in the formula of theorem 11 it will be seen to be combined with a matrix containing $n-m-1$ linearly independent determinants by the use of which the remaining $n-m-1$ variables may be expressed in terms of two of them.

§ 28. A part of the notation will here be repeated:

$$H_1 \equiv \|u_{m\,m+1}\|^{(m)} \equiv \left\| \begin{matrix} u_{11} \ldots\ldots u_{1\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,1} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m)},$$

$$H_{2A} \equiv \|_{\bar{2}}\, u_{m\,m+1}\|^{(m)} \equiv \left\| \begin{matrix} u_{11} \;\; u_{13} \ldots\ldots u_{1\,m+1} \\ \ldots\ldots\ldots\ldots\ldots \\ u_{m\,1}\, u_{m\,3} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m)},$$

$$H_{2B} \equiv \|_{\bar{1}}\, u_{m\,m+1}\|^{(m)} \equiv \left\| \begin{matrix} u_{12} \ldots\ldots u_{1\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,2} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m)},$$

$$H_{3C} \equiv \|_{\overline{1,2}}\, u_{m\,m+1}\|^{(m-1)} \equiv \left\| \begin{matrix} u_{13} \ldots\ldots u_{1\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-1)},$$

$$H_{3A} \equiv \left\| \begin{matrix} u_{13} \ldots\ldots u_{1\,m+1} \\ u_{33} \ldots\ldots u_{3\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{3B} \equiv \left\| \begin{matrix} u_{23} \ldots\ldots u_{2\,m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-1)}, \quad H_{4C} \equiv \left\| \begin{matrix} u_{33} \ldots\ldots u_{3\,m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,3} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-2)},$$

$$H_{4A} \equiv \left\| \begin{matrix} u_{33} \;\; u_{35} \ldots\ldots u_{3\,m+1} \\ u_{43} \;\; u_{45} \ldots\ldots u_{4\,m+1} \\ \ldots\ldots\ldots\ldots\ldots \\ u_{m\,3}\, u_{m\,5} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{4B} \equiv \left\| \begin{matrix} u_{34} \ldots\ldots u_{3\,m+1} \\ u_{44} \ldots\ldots u_{4\,m+1} \\ \ldots\ldots\ldots\ldots \\ u_{m\,4} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-2)}, \quad H_{5C} \equiv \left\| \begin{matrix} u_{35} \ldots\ldots u_{3\,m+1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{m\,5} \ldots\ldots u_{m\,m+1} \end{matrix} \right\|^{(m-3)},$$

$H_{m+1\,A} \equiv u_{m\,m}$ or $u_{m-1\,m+1}$, according as m is odd or even,
$H_{m+1\,B} \equiv u_{m\,m+1}$,
$H_{m+1\,C} \equiv \|u_{m\,m-1}\, u_{m\,m}\|^{(1)}$ or $u_{m-1\,m+1}$, according as m is odd or even,
$H_k \equiv H_{kA} + H_{kB}$.

§ 29. H_1 is expressible in terms of H_2 (theorem 1) unless $H_{3C}=0$. Therefore

$$O_{H_1} \leqq O_{H_2}. \quad \text{(Problem 1.)}$$

But if a system of values can be determined for the two variables such that $H_{3C}=0$, H_1 can not be expressed in terms of H_2. This is possible, as H_{3C} contains but two linearly independent determinants. So if a system of equations can be formed from which values of the variables can be determined so as to make H_{3C} and H_2 vanish simultaneously, one will have $H_2=0$ without having

$H_1=0$. H_{3C} is such a system, for H_2 can be expanded in terms of H_{3C}. Therefore

$$O_{H_1}=O_{H_2}-O_{H_{3C}}.$$

H_{3C} is a matrix like H_1 with m reduced by 1. Hence,

$$O_{H_{3C}}=O_{H_3}-O_{H_{4C}}$$

and

$$\begin{aligned} O_{H_{4C}}&=O_{H_4}-O_{H_{5C}},\\ &\dots\dots\dots\dots\dots,\\ O_{H_{mC}}&=O_{H_m}-O_{H_{m+1C}}. \end{aligned}$$

But $O_{H_{m+1C}}=O_{H_{m+1}}$. Therefore

THEOREM 12: $O_{H_1}=O_{H_2}-O_{H_3}+O_{H_4}-\dots+(-1)^{m+1}O_{H_{m+1}}$.

It is to be observed that H_{kA} and H_{kB} are single determinants and that theorem 12, therefore, gives the order of $\|u_{m\,m+1}\|^{(m)}$ in terms of the orders of determinants.

VI. *The Calculation of the Order of* $\|u_{m\,n}\|^{(m)}=0$.

§ 30. By the use of the symmetric functions already defined (§ 6) this order will be calculated first when m is even and second when m is odd, $u_{r\,s}$ being considered of degree $a_s+\alpha_r$.

When m is even,

$$\begin{aligned} O_{H_1}=&\,({}_{\bar 2}g_1+a_1+\gamma_1)({}_{\bar 2}g_1+a_2+\gamma_1)-({}_{\bar 2}g_1+{}_{\bar 2}\gamma_1+\alpha_1)({}_{\bar 2}g_1+{}_{\bar 2}\gamma_1+\alpha_2)\\ &+({}_{\bar 4}g_1+a_3+{}_{\bar 2}\gamma_1)({}_{\bar 4}g_1+a_4+{}_{\bar 2}\gamma_1)-({}_{\bar 4}g_1+{}_{\bar 4}\gamma_1+\alpha_3)({}_{\bar 4}g_1+{}_{\bar 4}\gamma_1+\alpha_4)\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+({}_{\overline{2k}}g_1+a_{2k-1}+{}_{\overline{2k-2}}\gamma_1)({}_{\overline{2k}}g_1+a_{2k}+{}_{\overline{2k-2}}\gamma_1)-({}_{\overline{2k}}g_1+{}_{\overline{2k}}\gamma_1+\alpha_{2k-1})({}_{\overline{2k}}g_1+{}_{\overline{2k}}\gamma_1+\alpha_{2k})\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &+({}_{\overline{m-2}}g_1+a_{m-3}+{}_{\overline{m-4}}\gamma_1)({}_{\overline{m-2}}g_1+a_{m-2}+{}_{\overline{m-4}}\gamma_1)-({}_{\overline{m-2}}g_1+{}_{\overline{m-2}}\gamma_1+\alpha_{m-3})({}_{\overline{m-2}}g_1+{}_{\overline{m-2}}\gamma_1+\alpha_{m-2})\\ &+({}_{\bar m}g_1+a_{m-1}+{}_{\overline{m-2}}\gamma_1)({}_{\bar m}g_1+a_m+{}_{\overline{m-2}}\gamma_1)-({}_{\bar m}g_1+{}_{\bar m}\gamma_1+\alpha_{\overline{m-1}})({}_{\bar m}g_1+{}_{\bar m}\gamma_1+\alpha_m)\\ =&\,A+B+C, \end{aligned}$$

where

$$\begin{aligned} A&\equiv\sum_{k=1}^{k=m/2}\left[(a_{2k-1}+a_{2k})\,{}_{\overline{2k}}g_1+a_{2k-1}\cdot a_{2k}\right]=\sum_{k=1}^{k=m/2}({}_{\overline{2k-2}}g_2-{}_{\overline{2k}}g_2)\\ &=\overline{g_2-{}_{\bar 2}g_2}+\overline{{}_{\bar 2}g_2-{}_{\bar 4}g_2}+\dots+\overline{{}_{\overline{2m-2}}g_2-{}_{\bar m}g_2}\\ &=g_2, \end{aligned}$$

and

$$\begin{aligned} B\equiv&\,\gamma_1(2\,{}_{\bar 2}g_1+a_1+a_2)-{}_{\bar 2}\gamma_1\cdot 2\,{}_{\bar 2}g_1-{}_{\bar 2}g_1(\alpha_1+\alpha_2)\\ &+{}_{\bar 2}\gamma_1(2\,{}_{\bar 4}g_1+a_3+a_4)-{}_{\bar 4}\gamma_1\cdot 2\,{}_{\bar 4}g_1-{}_{\bar 4}g_1(\alpha_3+\alpha_4)\\ &+\dots\dots\dots\dots\dots\dots\dots\dots\\ &+{}_{\overline{m-4}}\gamma_1(2\,{}_{\overline{m-2}}g_1+a_{m-3}+a_{m-2})-{}_{\overline{m-2}}\gamma_1\cdot 2\,{}_{\overline{m-2}}g_1-{}_{\overline{m-2}}g_1(\alpha_{m-3}+\alpha_{m-2})\\ &+{}_{\overline{m-2}}\gamma_1(2\,{}_{\bar m}g_1+a_{m-1}+a_m)-{}_{\bar m}\gamma_1\cdot 2\,{}_{\bar m}g_1-{}_{\bar m}g_1(\alpha_{m-1}+\alpha_m)\\ =&\,\gamma_1(g_1+{}_{\bar 2}g_1-{}_{\bar 2}g_1)-{}_{\bar 2}\gamma_1\cdot{}_{\bar 2}g_1\\ &+{}_{\bar 2}\gamma_1({}_{\bar 2}g_1+{}_{\bar 4}g_1-{}_{\bar 4}g_1)-{}_{\bar 4}\gamma_1\cdot{}_{\bar 4}g_1\\ &+\dots\dots\dots\dots\\ &+{}_{\overline{m-4}}\gamma_1({}_{\overline{m-4}}g_1+{}_{\overline{m-2}}g_1-{}_{\overline{m-2}}g_1)-{}_{\overline{m-2}}\gamma_1\cdot{}_{\overline{m-2}}g_1\\ &+{}_{\overline{m-2}}\gamma_1({}_{\overline{m-2}}g_1+{}_{\bar m}g_1-{}_{\bar m}g_1)-{}_{\bar m}\gamma_1\cdot{}_{\bar m}g_1\\ =&\,g_1\gamma_1, \end{aligned}$$

and
$$\begin{aligned} C \equiv {} & \gamma_1^2 - {}_2\gamma_1^2 - [(\alpha_1+\alpha_2)\,{}_{\bar{2}}\gamma_1 + \alpha_1\alpha_2] \\ & + {}_{\bar{2}}\gamma_1^2 - {}_{\bar{4}}\gamma_1^2 - [(\alpha_3+\alpha_4)\,{}_{\bar{4}}\gamma_1 + \alpha_3\alpha_4] \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + {}_{\overline{m-4}}\gamma_1^2 - {}_{\overline{m-2}}\gamma_1^2 - [(\alpha_{m-3}+\alpha_{m-2})\,{}_{\overline{m-2}}\gamma_1 + \alpha_{m-3}\cdot\alpha_{m-2}] \\ & + {}_{\overline{m-2}}\gamma_1^2 - \alpha_{m-1}\alpha_m \\ = {} & \gamma_1^2 - \sum_{k=1}^{k=m/2} [(\alpha_{2k-1}+\alpha_{2k})\,{}_{\overline{2k}}\gamma_1 + \alpha_{2k-1}\cdot\alpha_{2k}] \\ = {} & \gamma_1^2 - \sum_{k=1}^{k=m/2} ({}_{\overline{2k-2}}\gamma_2 - {}_{\overline{2k}}\gamma_2) \\ = {} & \gamma_1^2 - \gamma_2 + {}_{\bar{2}}\gamma_2 - {}_{\bar{2}}\gamma_2 + {}_{\bar{4}}\gamma_2 - \cdots\cdots - {}_{\overline{m-2}}\gamma_2 + {}_{\bar{m}}\gamma_2 \\ = {} & \gamma_1^2 - \gamma_2 \\ = {} & \delta_2 . \end{aligned}$$

Therefore
$$O_{H_1} = g_2 + g_1\gamma_1 + \delta_2 = K_2 .$$

When m is odd,
$$\begin{aligned} O_{H_1} \equiv {} & ({}_{\bar{2}}g_1 + a_1 + \gamma_1)({}_2g_1 + a_2 + \gamma_1) - ({}_{\bar{2}}g_1 + {}_{\bar{2}}\gamma_1 + \alpha_1)({}_{\bar{2}}g_1 + {}_{\bar{2}}\gamma_1 + \alpha_2) \\ & + ({}_{\bar{4}}g_1 + a_3 + {}_{\bar{2}}\gamma_1)({}_{\bar{4}}g_1 + a_4 + {}_{\bar{2}}\gamma_1) - ({}_{\bar{4}}g_1 + {}_{\bar{4}}\gamma_1 + \alpha_3)({}_{\bar{4}}g_1 + {}_{\bar{4}}\gamma_1 + \alpha_4) \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + ({}_{\overline{2k}}g_1 + a_{2k-1} + {}_{\overline{2k-2}}\gamma_1)({}_{\overline{2k}}g_1 + a_{2k} + {}_{\overline{2k-2}}\gamma_1) - ({}_{\overline{2k}}g_1 + {}_{2k}\gamma_1 + \alpha_{2k-1})({}_{\overline{2k}}g_1 + {}_{2k}\gamma_1 + \alpha_{2k}) \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + ({}_{\overline{m-1}}g_1 + a_{m-2} + {}_{\overline{m-3}}\gamma_1)({}_{\overline{m-1}}g_1 + a_{m-1} + {}_{\overline{m-3}}\gamma_1) - ({}_{\overline{m-1}}g_1 + {}_{\overline{m-1}}\gamma_1 + \alpha_{m-2})({}_{\overline{m-1}}g_1 + {}_{\overline{m-1}}\gamma_1 + \alpha_{m-1}) \\ & + (a_m + \alpha_m)(a_{m+1} + \alpha_m) \\ = {} & A + B + C \\ = {} & g_2 + g_1\gamma_1 + \delta_2 \\ = {} & K_2 . \end{aligned}$$

Therefore

THEOREM 13: $O_{H_1} = K_2$.

By an interchange of rows and columns it is evident that

THEOREM 14: $O_{\|u_{m+1\,m}\|^{(m)}} = J_2$.

§ 32. The order of $\|u_{m\,m+2}\|^{(m)}$ will next be investigated. It will be obtained by mathematical induction after $O_{\|u_{24}\|^{(2)}}$ and $O_{\|u_{35}\|^{(3)}}$ have been calculated.

By theorems 12, 13 and 14, and formula 5,
$$O_{\|u_{24}\|^{(2)}} = O_{\|{}_{\rm I}u_{24}\|^{(2)}} \cdot O_{\|u_{22}\|^{(2)}} - O_{\|{}_{\rm II}u_{24}\|^{(2)}} \cdot O_{\|{}_{\rm I}u_{22}\|^{(1)}}$$

THEOREM 15: $\qquad = {}_{\rm I}K_2 J_1 - {}_{\rm II}K_1 \cdot {}_{\rm I}J_2 = K_3$,

the reduction ceasing when $k = m+2 = 4$; and similarly

THEOREM 16: $O_{\|u_{42}\|^{(2)}} = J_3$;
$$O_{\|u_{35}\|^{(3)}} = O_{\|{}_{\rm I}u_{35}\|^{(3)}} \cdot O_{\|u_{33}\|^{(3)}} - O_{\|{}_{\rm II}u_{35}\|^{(3)}} \cdot O_{\|{}_{\rm I}u_{33}\|^{(2)}} + O_{\|{}_{\rm III}u_{35}\|^{(3)}} \cdot O_{\|{}_{\rm II}u_{33}\|^{(1)}}$$

THEOREM 17: $={}_{1}K_{2}J_{1}-{}_{12}K_{1}\cdot{}_{\overline{1}}J_{2}+{}_{\overline{12}}J_{3}=K_{3}$;

and similarly

THEOREM 18: $O_{\|u_{53}\|^{(3)}}=J_{3}$.

It is to be noticed that in the case of $\|_{\overline{123}}u_{35}\|^{(3)}$, $k=n-m+2=4$, and therefore its order is to be regarded as 1. Also it is to be noticed that the reduction ceases when $k=m+2=5$ (§ 25).

Let h be a positive integer for which, and for all smaller ones, it has been verified that $O_{\|u_{h\,h-2}\|^{(h-2)}}=J_{3}$; then

$$O_{\|u_{h\,h+2}\|^{(h)}}=O_{\|_{1}u_{h\,h+2}\|^{(h)}}\cdot O_{\|u_{h\,h}\|^{(h)}}-O_{\|_{12}u_{h\,h+2}\|^{(h)}}\cdot O_{\|_{\overline{1}}u_{h\,h}\|^{(h-1)}}$$
$$+O_{\|_{123}u_{h\,h+2}\|^{(h)}}\cdot O_{\|_{\overline{12}}u_{h\,h}\|^{(h-2)}}$$

THEOREM 19: $={}_{1}K_{2}J_{1}-{}_{12}K_{1}\cdot{}_{\overline{1}}J_{2}+{}_{\overline{12}}J_{3}=K_{3}$;

and similarly

THEOREM 20: $O_{\|u_{h+2\,h}\|^{(h)}}=J_{3}$.

From theorems 16, 18, 20, it follows that

THEOREM 21: $O_{\|u_{m\,m+2}\|^{(m)}}=K_{3}$;

and similarly

THEOREM 22: $O_{\|u_{m+2\,m}\|^{(m)}}=J_{3}$.

Suppose j a positive integer for which, and for all smaller ones, it has been verified that

$$O_{\|u_{m\,m+j}\|^{(m)}}=K_{j+1}\quad\text{and}\quad O_{\|u_{m+j\,m}\|^{(m)}}=J_{j+1},$$

and note that $O_{\|_{\overline{1,2,\ldots,m-1}}u_{m\,m}\|^{(1)}}={}_{\overline{m-1}}J_{m}$. Then

$$O_{\|u_{m\,m+j+1}\|^{(m)}}=O_{\|_{1}u_{m\,m+j+1}\|^{(m)}}\cdot O_{\|u_{m\,m}\|^{(m)}}-O_{\|_{12}u_{m\,m+j+1}\|^{(m)}}\cdot O_{\|_{\overline{1}}u_{m\,m}\|^{(m-1)}}+\ldots.$$
$$+(-1)^{m+1}O_{\|_{\overline{1,\ldots,m}}u_{m\,m+j+1}\|^{(m)}}\cdot O_{\|_{\overline{1,\ldots,m-1}}u_{m\,m}\|^{(1)}}$$
$$={}_{1}K_{j}\cdot J_{1}-{}_{2}K_{j-1}\cdot{}_{\overline{1}}J_{1}+{}_{3}K_{j-2}\cdot{}_{\overline{2}}J_{2}-\ldots.$$
$$+(-1)^{m-1}{}_{\overline{m-1}}K_{j-m+3}\cdot{}_{\overline{m-2}}J_{m-1}+(-1)^{m}{}_{m}K_{j-m+2}\cdot{}_{\overline{m-1}}J_{m}$$

THEOREM 23: $=K_{j+2}$ (formula 10);

and by the interchange of rows and columns,

THEOREM 24: $O_{\|u_{m+j+1\,m}\|^{(m)}}=J_{j+2}$.

It follows from theorems 13, 21 and 23, that

THEOREM 25: $O_{\|u_{m\,n}\|^{(m)}}=K_{n-m+1}$, where $n>m$; and $O_{\|u_{m\,n}\|^{(n)}}=J_{m-n+1}$, where $m>n$.

§ 33. Theorem 12 evidently provides a reduction formula for m and n, and the argument of the preceding section may seem unnecessarily extended unless it be observed that the theorem furnishes no reduction for the quantity $n-m$.

It may be seen that the greatest difference between the number of columns and the number of rows occurring in a matrix of the right member is $n-m$.

First it will be observed that this difference is less in G_{kA} than in G_{k-1A}, but greater in G_{kB} than in G_{k-1B}. This difference is therefore greatest either in G_{2A} or in G_{kB}, where k has its greatest value. In G_{2A} it is $n-m+1$. The greatest value of k is $n-m+2$ (§ 26), and

$$G_{n-m+2\,B} \equiv \|_{\overline{1,\,2,\ldots,\,n-m}}\, u_{m\,m}\|^{(2m-n)}$$

and is of the form $\|u_{m'\,n'}\|$, where $m'=m$ and $n'=m-(n-m)=2m-n$, giving $m'-n'=n-m$.

§ 34. In case all the constituents of $\|u_{m\,n}\|^{(m)}$ are of degree l, the a's may be taken each equal to l, and the α's each equal to zero, and in this case

$$K_{n-m+1} = l^{n-m+1}\,{}_nC_{n-m+1} = l^{n-m+1}\cdot\frac{n!}{(n-m+1)!\,(m-1)!}.$$

VII. *A Geometric Interpretation of* K_{n-m+1}.

§ 35. In a space of $n-m+1$ dimensions, two varieties of dimensions, k and $n-m+1-k$, intersect, in general, in a finite number of points. If the $n-m+1$ variables of G_1 be considered as the non-homogeneous coordinates of a point in a space of $n-m+1$ dimensions, G_1 will represent points. These points are among the points of intersection of the two varieties G_{2A} and G_{2B} of dimensions 1 and $n-m$ respectively. But the points G_1 do not include all the points of intersection of the two varieties. From these points of intersection must be deducted all the points of intersection of the two varieties G_{3A} and G_{3B} of dimensions 2 and $n-m-1$, respectively, except those that satisfy a certain other condition, etc. Finally it is found that G_1 represents K_{n-m+1} points.

It is to be observed that if a matrix of l-ary forms represents p points in a space of l dimensions, the same matrix of $(l+k)$-ary forms will represent a variety of order p, having k dimensions in a space of $l+k$ dimensions. Thus if G_1 had been considered as a matrix of non-homogeneous forms containing $n-m+1+k$ variables instead of $n-m+1$ of them, it would represent a variety of k dimensions in a space of $n-m+1+k$ dimensions, and in that case K_{n-m+1} would represent the order of the variety.

One-Parameter Families of Curves.

By Luther Pfahler Eisenhart.

1. It has become an established belief that many of the differential geometric properties of curves and surfaces are more readily studied when the latter are referred to a set of moving axes. In the case of twisted curves the principal directions at a point (*i. e.*, the tangent, principal normal and binormal) afford a good set of axes, and for surfaces it is customary to take the tangent plane at a point for one of the coordinate planes of the axes for this point. In many problems it is convenient to look upon a surface as the locus of a one-parameter family of curves C. In cases of this sort it is often advisable to use for moving axes the principal directions of the curves C. The present paper develops the equations of a surface from this point of view and establishes the fundamental equations of condition to be satisfied by a set of functions determining a surface.

In illustration of the method we consider surfaces: (i) with plane lines of curvature in one system; (ii) with a family of circular generators; (iii) with a family of asymptotic lines of the same constant torsion; (iv) with a family of geodesics of constant torsion.* The general equations lend themselves readily to the consideration of continuous deformations of twisted curves.

If a surface S is generated by a family of curves C, there exist surfaces S' generated by a second family of curves C' such that the tangent, principal normal and binormal at a point of a curve C' are parallel respectively to the binormal, principal normal and tangent at the corresponding point of a curve C. The determination of these surfaces reduces to the integration of a system of ordinary linear differential equations of the first order. The case where S and S' correspond with parallelism of tangent planes is investigated.

General Equations of a Surface.

2. Consider any surface S referred to a family of non-minimal skew curves $v=$const. and any other family of curves $u=$const. We denote by x, y, z the cartesian coordinates of a point M on S, and by α, β, γ; l, m, n; λ, μ, ν the direction-cosines of the tangent, principal normal and binormal at M to the

* These methods have been applied by Dr. R. D. Beetle in an article entitled, "Congruences Associated with a One-Parameter Family of Curves," which will appear in this Journal.

curve $v = \text{const.}$ through M. It is clear that there exist determinate functions p, q, r, t such that

$$\frac{\partial x}{\partial u} = p\,\alpha, \quad \frac{\partial x}{\partial v} = q\,\alpha + r\,l + t\,\lambda, \tag{1}$$

and similar equations in y and z. It is our purpose to determine the conditions which four such functions must satisfy in order that equations of the form (1) define a surface.

In the first place, we observe that the Frenet-Serret formulas* for a curve $v = \text{const.}$ may be written

$$\frac{\partial \alpha}{\partial u} = \frac{p\,l}{\rho}, \quad \frac{\partial l}{\partial u} = -p\left(\frac{\alpha}{\rho} + \frac{\lambda}{\tau}\right), \quad \frac{\partial \lambda}{\partial u} = p\,\frac{l}{\tau}, \tag{2}$$

where ρ and τ denote the radii of first curvature and torsion respectively.

The condition of integrability of equations (1) is reducible by means of (2) to

$$\frac{\partial \alpha}{\partial v} = A_1\,\alpha + A_2\,l + A_3\,\lambda, \tag{3}$$

where

$$\left.\begin{aligned} A_1 &= \frac{1}{p}\frac{\partial q}{\partial u} - \frac{\partial \log p}{\partial v} - \frac{r}{\rho}, \\ A_2 &= \frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau}, \\ A_3 &= \frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau}. \end{aligned}\right\} \tag{4}$$

Since equations analogous to (3) are satisfied by β and γ, and since also

$$\Sigma\,\alpha^2 = 1, \qquad \Sigma\,\alpha\,l = 0, \qquad \Sigma\,\alpha\,\lambda = 0, \tag{5}$$

the function A_1 must be equal to zero.

In like manner the condition of integrability of equation (3) and the first of equations (2) is reducible, by means of (2) and the requirements (5) and

$$\Sigma\,l^2 = 1, \qquad \Sigma\,l\,\lambda = 0, \tag{6}$$

to

$$\frac{\partial l}{\partial v} = -A_2\,\alpha + \left(\frac{\rho}{p}\frac{\partial A_3}{\partial u} - A_2\frac{\rho}{\tau}\right)\lambda, \tag{7}$$

and to the additional condition

$$\frac{\partial A_2}{\partial u} - \frac{\partial}{\partial v}\left(\frac{p}{\rho}\right) + \frac{p}{\tau}A_3 = 0. \tag{8}$$

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* E., p. 17. A reference of this kind is to the author's "Differential Geometry," Ginn & Co., Boston, 1909.

Proceeding in a similar manner with equation (7) and the second of equations (2), we find that λ, μ, ν must satisfy equations of the form

$$\frac{\partial \lambda}{\partial v} = -A_3 \alpha - \left(\frac{\rho}{p}\frac{\partial A_3}{\partial u} - A_2 \frac{\rho}{\tau}\right) l, \tag{9}$$

and that the further condition

$$\frac{\partial}{\partial u}\left(\frac{\rho}{p}\frac{\partial A_3}{\partial u} - A_2 \frac{\rho}{\tau}\right) + \frac{\partial}{\partial v}\left(\frac{p}{\tau}\right) + \frac{p}{\rho} A_3 = 0 \tag{10}$$

must be satisfied. It is readily shown that equation (9) and the third of equations (2) are consistent, when the preceding conditions are satisfied.

The equations of condition sought are accordingly $A_1 = 0$, (8) and (10), which in consequence of (4) are

$$\left.\begin{aligned} \frac{1}{p}\frac{\partial q}{\partial u} - \frac{\partial \log p}{\partial v} - \frac{r}{\rho} &= 0, \\ \frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau}\right) - \frac{\partial}{\partial v}\left(\frac{p}{\rho}\right) + \frac{p}{\tau}\left(\frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau}\right) &= 0, \\ \frac{\partial}{\partial u}\left[\frac{\rho}{p}\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau}\right) - \frac{\rho}{\tau}\left(\frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau}\right)\right] + \frac{\partial}{\partial v}\left(\frac{p}{\tau}\right) + \frac{p}{\rho}\left(\frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau}\right) &= 0. \end{aligned}\right\} \tag{11}$$

Retracing the above steps, we see that if the conditions (11) are satisfied by six functions ρ, τ, p, q, r, t, the determination of the surface requires the solution of equations (2), (3), (7) and (9), which is equivalent to the integration of a Riccati equation,* and the quadratures (1).

It is evident at once from equations (1) that the direction-cosines X, Y, Z of the normal to S are of the form

$$X = \sigma(r\lambda - tl), \quad Y = \sigma(r\mu - tm), \quad Z = \sigma(r\nu - tn), \tag{12}$$

where

$$\sigma = (r^2 + t^2)^{-\frac{1}{2}}. \tag{13}$$

Making use of (2), (3), (7) and (9), we obtain from (12) by differentiation

$$\left.\begin{aligned} \frac{\partial X}{\partial u} &= \sigma\left\{\frac{tp}{\rho}\alpha - \left(pA_3 + \frac{\partial \log \sigma}{\partial u} t\right) l + \left(pA_2 - \frac{pq}{\rho} + r\frac{\partial \log \sigma}{\partial u}\right)\lambda\right\}, \\ \frac{\partial X}{\partial v} &= \sigma\left\{(tA_2 - rA_3)\alpha - \left[\frac{\partial t}{\partial v} + r\left(\frac{p}{\rho}\frac{\partial A_3}{\partial u} - A_2\frac{\rho}{\tau}\right) + \frac{\partial \log \sigma}{\partial v} t\right] l \right. \\ &\quad \left. + \left[\frac{\partial r}{\partial v} - t\rho\left(\frac{1}{p}\frac{\partial A_3}{\partial u} - \frac{A_2}{\tau}\right) + \frac{\partial \log \sigma}{\partial v} r\right]\lambda\right\}. \end{aligned}\right\} \tag{14}$$

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* Cf. Darboux, "Leçons," Vol. I, p. 56.

Particular Curves Parametric.

3. As immediate consequences of equations (1) we have

THEOREM 1. *The necessary and sufficient condition that the parametric curves form an orthogonal system is that $q = 0$.*

THEOREM 2. *The necessary and sufficient condition that the curves $v =$ const. be asymptotic lines is that $t = 0$.*

For in this case the binormals are normal to the surface.

The analytic condition that the parametric curves of a surface form a conjugate system is that*

$$\Sigma \frac{\partial x}{\partial u}\frac{\partial X}{\partial v} = \Sigma \frac{\partial x}{\partial v}\frac{\partial X}{\partial u} = 0.$$

Making use of equations (14) and (4), we have

THEOREM 3. *The necessary and sufficient condition that the parametric curves form a conjugate system is that*

$$t A_2 - r A_3 = \frac{1}{p}\left(t\frac{\partial r}{\partial u} - r\frac{\partial t}{\partial u}\right) + \frac{q t}{\rho} + \frac{1}{\tau}(t^2 + r^2) = 0. \qquad (15)$$

If the parametric lines are lines of curvature, equation (15) must hold and also $q = 0$. In this case we have

$$\frac{\partial \omega}{\partial u} - \frac{p}{\tau} = 0, \quad \text{where } \tan\omega = \frac{t}{r}. \qquad (16)$$

As thus defined, ω is the angle between the principal normal to a curve $v =$ const. and the tangent to the curve $u =$ const. at the same point. Equation (16) expresses the well-known fact that the geodesic torsion is zero.†

When no other condition is put upon the curves $u =$ const., we can in all generality take $p = 1$. This means that the arcs of the curves $v =$ const. between two curves $u =$ const. are equal. With this choice the linear element of S assumes the form

$$d s^2 = d u^2 + 2 q\, d u\, d v + (q^2 + r^2 + t^2)\, d v^2. \qquad (17)$$

We observe furthermore that p continues to be $+1$, if the parameters are changed in accordance with the equations

$$u_1 = u + V, \quad v_1 = v, \qquad (18)$$

where V denotes any function of v. Under this transformation we have, for the corresponding functions q_1, r_1, t_1, the expressions

$$q_1 = q - V', \quad r_1 = r, \quad t_1 = t, \qquad (19)$$

where the prime indicates differentiation.

* E., p. 127.

† Cf. E., p. 138.

As an example, we consider the surfaces whose asymptotic lines in one system have the same constant torsion a. If we put

$$\frac{1}{\tau}=a, \quad t=0, \quad p=1,$$

equations (11) are reducible to

$$\frac{\partial q}{\partial u}-\frac{r}{\rho}=0, \qquad \frac{\partial}{\partial u}\left(\rho\frac{\partial r}{\partial u}+q\right)=0,$$
$$\frac{\partial}{\partial u}\left(\frac{\partial r}{\partial u}+\frac{q}{\rho}\right)-\frac{\partial}{\partial v}\left(\frac{1}{r}\right)-ra^2=0.$$

By a proper choice of the parameter u_1, given by (18), and taking this for the new variable u, we may replace the second of the above equations by

$$\rho\frac{\partial r}{\partial u}+q=0.$$

Combining this with the first, we find that q^2+r^2 must be a function of v alone. As this function is the coefficient of dv^2 in the linear element (17), the parameter v can be chosen so that this quantity is equal to unity. Accordingly we can put

$$q=\cos\omega, \quad r=\sin\omega,$$

thus defining the function ω. Now the above equations of condition reduce to

$$\frac{1}{\rho}=-\frac{\partial\omega}{\partial u}, \quad \frac{\partial^2\omega}{\partial u\,\partial v}=a^2\sin\omega,$$

which are the well-known equations for pseudo spherical surfaces.* It can be shown readily that the curves $u=$ const. are asymptotic, and by means of Enneper's theorem we have that they are of constant torsion $-a$.

4. By definition the curves $v=$const. are geodesics if the principal normals to the curves are normal to the surface. Hence, from equations (1) follows

THEOREM 4. *The necessary and sufficient condition that the curves $v=$ const. be geodesics is that $r=0$.*

Suppose that this condition is satisfied. We take the orthogonal trajectories for the curves $u=$const.; then $q=0$. In consequence of the first of equations (11) we may in all generality take $p=1$, and thus the other equations (11) can be written

$$\left.\begin{aligned}\frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)&=t\frac{\partial}{\partial u}\left(\frac{1}{\tau}\right)+\frac{2}{\tau}\frac{\partial t}{\partial u},\\ \frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)&=\frac{\partial}{\partial u}\left[\rho\left(\frac{t}{\tau^2}-\frac{\partial^2 t}{\partial u^2}\right)\right]-\frac{1}{\rho}\frac{\partial t}{\partial u}.\end{aligned}\right\} \tag{20}$$

* E., p. 190.

These equations are readily reducible to the form to which the Kowaleski* existence theorem applies. Hence we have

THEOREM 5. *In the problem of finding surfaces whose linear element is of the form $ds^2 = du^2 + t^2 dv^2$, each surface is determined by an arbitrary curve which is taken for the curve $v = 0$; then the intrinsic equations of each curve $v = \text{const.}$ are given as power-series in u.*†

When the Curves $v = \text{const.}$ Are Plane.

5. The requirement that the curves $v = \text{const.}$ be plane (that is, $1/\tau = 0$) affects the general development only in that equation (9) does not arise as in the general case, but because of the conditions

$$\Sigma \alpha \lambda = 0, \quad \Sigma l \lambda = 0.$$

Hence, in order to obtain the equations for plane curves it is necessary and sufficient to put $1/\tau = 0$ in the foregoing equations. In this case equations (11) become

$$\left.\begin{aligned} &\frac{1}{p}\frac{\partial q}{\partial u} - \frac{\partial \log p}{\partial v} - \frac{r}{\rho} = 0, \\ &\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho}\right) - \frac{\partial}{\partial v}\left(\frac{p}{\rho}\right) = 0, \\ &\frac{\partial}{\partial u}\left[\frac{\rho}{p}\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial t}{\partial u}\right)\right] + \frac{1}{\rho}\frac{\partial t}{\partial u} = 0. \end{aligned}\right\} \tag{11'}$$

These equations determine t only to within an additive function of v. Consider the cases for t and $t_1 = t + V$. Since p, ρ, τ and r are the same for both, so also are the functions α, β, γ; l, m, n; λ, μ, ν. Hence, from equations (1) we have that the coordinates x, y, z; x_1, y_1, z_1 of the two surfaces are in the relation

$$x_1 - x = \int \lambda V dv, \quad y_1 - y = \int \mu V dv, \quad z_1 - z = \int \nu V dv.$$

We return to the consideration of the general case and write the equation of the plane of a curve $v = \text{const.}$ in the form

$$\Sigma x \lambda = V, \tag{21}$$

where evidently V is a determinate function of v. The characteristic of this plane is defined by (21) and

* Cf. Picard, "Traité d'Analyse," Vol. II (1893), pp. 318–323.

† The first part of this theorem is in accord with the general theory of the determination of applicable surfaces, since the tangent planes to the desired surface along the curve $v=0$ are tacitly given; they are the rectifying planes of the curve. Cf. Darboux, "Leçons," Vol. II, pp. 263–267.

$$\Sigma x\left[\frac{1}{p}\frac{\partial t}{\partial u}\alpha+\frac{\rho}{p}\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial t}{\partial u}\right)l\right]=-V'. \tag{22}$$

From (21) it follows that this characteristic meets the curve in the points for which $\Sigma\lambda\dfrac{\partial x}{\partial v}=0$. Referring to equations (1), we have

THEOREM 6. *The points for which $t=0$, and only these, lie on the characteristic of the plane of the curves $v=const.$*

In other words:

THEOREM 7. *At a point of intersection of a curve $v=const.$ and of the characteristic of its plane, one of the asymptotic lines is tangent to the curve.*

6. We consider in particular the case where the curves $v=$ const. are plane lines of curvature. We take the orthogonal trajectories for the curves $u=$ const. Hence, in accordance with (16) and a theorem of Joachimsthal,* we have

$$q=0,\quad r=\cos V\cdot\sigma,\quad t=\sin V\cdot\sigma, \tag{23}$$

where σ is thus defined and V is a function of v alone. For the present we understand that $\cos V\neq 0$, and we introduce a variable v_1 by means of the equation

$$v_1=\int\cos V\,dv. \tag{24}$$

Equations (11′) may be given the form (dropping the subscript of v_1)

$$\left.\begin{aligned}\frac{\partial\log p}{\partial v}+\frac{\sigma}{\rho}&=0,\\ \frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial\sigma}{\partial u}\right)-\frac{\partial}{\partial v}\left(\frac{p}{\rho}\right)&=0,\\ \frac{\partial}{\partial u}\left[\frac{\rho}{p}\frac{\partial}{\partial u}\left(\frac{1}{p}\frac{\partial\sigma}{\partial u}\right)\right]+\frac{1}{\rho}\frac{\partial\sigma}{\partial u}&=0.\end{aligned}\right\} \tag{25}$$

In accordance with the second of these equations we define a function θ by the equations

$$\frac{1}{p}\frac{\partial\sigma}{\partial u}=\frac{\partial\theta}{\partial v},\quad \frac{p}{\rho}=\frac{\partial\theta}{\partial u}. \tag{26}$$

By means of these equations the last of equations (25) may be reduced to

$$\frac{\partial^2}{\partial u\,\partial v}\log\frac{\partial\theta}{\partial u}+\frac{\partial\theta}{\partial u}\cdot\frac{\partial\theta}{\partial v}=0, \tag{27}$$

of which the first integral is

* E., p. 140.

$$\left(\frac{\partial}{\partial v}\log\frac{\partial\theta}{\partial u}\right)^2+\left(\frac{\partial\theta}{\partial v}\right)^2=V_1^2, \tag{28}$$

where V_1 denotes an arbitrary function of v alone.

This equation is satisfied by $\frac{\partial\theta}{\partial v}=\pm V_1$, but from (26) and the last of (25) it follows that, then, we must have $V_1=0$. Hence, σ is a function of v alone, and likewise r and t. Consequently, the curves $u=$const. are geodesics, and being lines of curvature at the same time, they are necessarily plane curves. Hence, this case is included in that of $\cos V=0$, to be considered later.

Excluding this case, we find that the first integral of (28) is

$$\frac{\partial\theta}{\partial v}=V_1\sin(\theta+V_2), \tag{29}$$

where V_2 is an arbitrary function of v. This equation is reducible to the Riccati form by a suitable change of the dependent variable.

If σ be eliminated from the first equations of (25) and (26), the resulting equation is reducible in consequence of (27) to

$$\frac{\partial}{\partial v}\left(\frac{\partial p}{\partial u}-p\frac{\partial}{\partial u}\log\frac{\partial\theta}{\partial u}\right)=0.$$

The integral of this equation may be written

$$p=(\phi+V_3)\frac{\partial\theta}{\partial u},\quad \text{where } \phi=\int\frac{U}{\frac{\partial\theta}{\partial u}}\partial u, \tag{30}$$

U and V_3 being arbitrary functions of u and v respectively. From (25) and (26) it follows that

$$\rho=\phi+V_3,\quad -\sigma=V_1(\phi+V_3)\cos(\theta+V_2)+\frac{\partial\phi}{\partial v}+V_3', \tag{31}$$

where the prime denotes differentiation.

Equations (2), (3), (7), (9) reduce to

$$\left.\begin{aligned}
&\frac{\partial\alpha}{\partial u}=\frac{\partial\theta}{\partial u}l, && \frac{\partial\alpha}{\partial v}=V_1\sin(\theta+V_2)(l+\lambda\tan V),\\
&\frac{\partial l}{\partial u}=-\frac{\partial\theta}{\partial u}\alpha, && \frac{\partial l}{\partial v}=-V_1[\sin(\theta+V_2)\alpha-\tan V\cos(\theta+V_2)\lambda],\\
&\frac{\partial\lambda}{\partial u}=0, && \frac{\partial\lambda}{\partial v}=-V_1\tan V[\sin(\theta+V_2)\alpha+\cos(\theta+V_2)l],
\end{aligned}\right\} \tag{32}$$

the function V being an arbitrary function of v alone. Moreover, the equations of the surface are

$$\frac{\partial x}{\partial u}=p\alpha,\quad \frac{\partial x}{\partial v}=\sigma(l+\tan V\lambda). \tag{33}$$

From (32) it follows that each solution θ of equation (29) determines the spherical representation of the lines of curvature and that the functions U and V_2 determine a surface of the kind sought with this spherical representation.

In particular, when $U=0$, and only in this case, the curves $v=\text{const.}$ are circles, as follows from (31).

Again, in order that the planes of the curves $v=\text{const.}$ shall envelop a cylinder whose generators are parallel to the z-axis, it is necessary and sufficient that $\nu=0$. Hence, it follows from the last of equations (32), when λ is replaced by ν, that we may take

$$\gamma=\cos(\theta+V_2), \quad n=-\sin(\theta+V_2).$$

When these values are substituted in the other equations (32), it is found that V_2 is constant, which may in all generality be taken equal to zero. In this case equation (29) can be integrated by a quadrature.

7. We consider now the case $\cos V=0$, which has been excluded. Since the curves $v=\text{const.}$ are plane geodesics, S is a surface of Monge.* From the first two of equations (11′) it follows that p and ρ are functions of u alone. This parameter may be chosen so that $p=1$. Since ρ is a function of u alone, the curves $v=\text{const.}$ are congruent to one another.

The third of equations (11′) admits the first integral

$$\rho^2\left(\frac{\partial^2 t}{\partial u^2}\right)^2+\left(\frac{\partial t}{\partial u}\right)^2=V^2. \tag{34}$$

This equation is satisfied by $\dfrac{\partial t}{\partial u}=\pm V$, but this is a solution of (11′) only in case $V=0$; that is, when t is a function of v alone. It follows from (17) that S is a developable surface under these conditions.

Excluding this case, the integral of (34) is

$$\frac{\partial t}{\partial u}=V\sin(\theta+V_1),$$

where V_1 is an arbitrary function of v and θ is the function of u defined by

$$\theta=\int\frac{du}{\rho}.$$

Hence, t is given by a quadrature. As surfaces of Monge have been thoroughly discussed by many writers, we will not give any further details.

When the Curves $v=\text{const.}$ Are Circles.

8. We choose the curves $u=\text{const.}$ so that $p=1$; and we impose the condition $\rho=1/V$, where V is a function of v alone. Hence, the plane curves $v=\text{const.}$ are circles. In this case equations (11′) become

* "Application de l'Analyse à la Géométrie," Paris (1849), § 17.

$$\frac{\partial q}{\partial u} - r V = 0, \quad \frac{\partial}{\partial u}\left(\frac{\partial r}{\partial u} + q V\right) - V' = 0, \quad \frac{\partial^3 t}{\partial u^3} + V^2 \frac{\partial t}{\partial u} = 0. \tag{35}$$

Eliminating r from the first two equations, we obtain an equation whose first integral is

$$\frac{\partial^2 q}{\partial u^2} + q V^2 = V (u V' + V_1),$$

where V_1 is an arbitrary function of v. The integral of this equation is

$$q V_2 \sin (V u + V_3) + \frac{1}{V} (V' u + V_1), \tag{36}$$

where V_2 and V_3 are arbitrary functions of v alone.

From the last of equations (35) we have by integration

$$t = V_4 \sin (u V + V_5) + V_6, \tag{37}$$

V_4, V_5 and V_6 being arbitrary functions of v. Since uV measures the angle between radii of a circle $v = \text{const.}$, it follows from Theorem 6 and equation (37) that, when $V_6 = 0$, the characteristics of the planes of the circles pass through the respective centers.

From the first of (35) we have

$$r = - V_2 \cos (V u + V_3) + \frac{V'}{V_2}. \tag{38}$$

Hence, the further determination of the surface requires the finding of α, β, γ; $\ldots$; λ, μ, ν, which is reducible to the solution of a Riccati equation.

Since there has been no special choice of v, we may in all generality take $V = v$. Again, in accordance with § 3, we can by a suitable choice of the curves $u = \text{const.}$ reduce V_3 or V_5 to zero. Accordingly we have

THEOREM 8. *The determination of surfaces possessing a family of circles reduces to the integration of a Riccati equation and quadratures; the general solution involves five arbitrary functions of the parameter of the circles.*

The latter part of this theorem is evident also from the point of view of the determination of such a family of circles by the coordinates of the center, the radius and the inclination of the axis of the circle.

Finite and Infinitesimal Deformation of Curves.

9. It is evident that if we take $p = 1$ in the general equations, the arcs of all the curves $v = \text{const.}$ between any two curves $u = \text{const.}$ are equal, and consequently we may look upon the surface as generated by the continuous deformation of one of the curves $v = \text{const.}$

If we write the fundamental equations of § 2 in the form

$$\left.\begin{aligned}&\frac{\partial q}{\partial u}-\frac{r}{\rho}=0,\quad A_2=\frac{\partial r}{\partial u}+\frac{q}{\rho}+\frac{t}{\tau},\quad A_3=\frac{\partial t}{\partial u}-\frac{r}{\tau},\\ &\frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)=\frac{\partial A_2}{\partial u}+\frac{A_3}{\tau},\quad \frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)=\frac{\partial}{\partial u}\left(\frac{\rho}{\tau}A_2-\rho\frac{\partial A_3}{\partial u}\right)-\frac{A_3}{\rho},\end{aligned}\right\}\quad(39)$$

the last two equations give the variation of the intrinsic functions ρ and τ.

General deformations of a twisted curve have little significance, but the interesting cases are those in which all of the curves possess similar properties. For example, in § 3 we treated the case in which a curve of constant torsion is deformed into curves of the same torsion, the direction of deformation of any point of the curve being in its osculating plane at the point; in other words, the osculating planes are tangent to the surface-locus of the curves, and consequently the deformed curves are asymptotic lines.

It is our purpose now to consider the deformations of a curve of constant torsion into curves of constant torsion, the direction of deformation of a point being that of the binormal to the curve at the point. Consequently, the deforms are geodesics on the surface locus of the curves. Surfaces of this kind have been studied by Fibbi* and Bianchi.† We consider the general case for which the torsion varies from curve to curve. Since the curves $v=\text{const.}$ are geodesics, we may in all generality take

$$q=r=0,\quad p=1,\quad \frac{\partial}{\partial u}\left(\frac{1}{\tau}\right)=0.$$

The first of equations (39) is satisfied identically; the second may be replaced by

$$\frac{1}{2\rho}=\frac{\partial\theta}{\partial u},\quad \frac{t}{\tau}=\frac{\partial\theta}{\partial v},$$

θ being a function thus defined; and the last equation may be written

$$\frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)=\frac{\partial}{\partial u}\left[\frac{1}{2\dfrac{\partial\theta}{\partial u}}\left(\frac{1}{\tau}\frac{\partial\theta}{\partial v}-\tau\frac{\partial^3\theta}{\partial u^2\,\partial v}\right)\right]-2\tau\frac{\partial\theta}{\partial u}\frac{\partial^2\theta}{\partial u\,\partial v}.$$

Hence, the problem of the determination of these surfaces reduces to the integration of this equation of the fourth order.

Sannia ‡ has studied at length the infinitesimal deformation of twisted curves. His fundamental equations follow at once from (39). In fact, if q and t are any functions of u, and r is the function of u given by the first of

* *Annali della Scuola Normale Superiore di Pisa* (1888).

† "Sulla theoria delle trasformazioni delle curve di Bertrand," *Memorie della Società italiana delle Scienze*, Ser. 3, Vol. XVIII.

‡ *Rendiconti del Circolo Matematico di Palermo*, Vol. XXI (1906), pp. 229–256.

(39), we have from the last two of (39) that the intrinsic equations of the new curve are

$$\frac{1}{\rho'}=\frac{1}{\rho}+\left(\frac{\partial A_2}{\partial u}+\frac{A_3}{\tau}\right)\varepsilon,\quad \frac{1}{\tau'}=\frac{1}{\tau}+\left[\frac{\partial}{\partial u}\left(\frac{\rho}{\tau}A_2-\rho\frac{\partial A_3}{\partial u}\right)-\frac{A_3}{\rho}\right]\varepsilon,$$

where ε denotes an infinitesimal constant.

Surfaces Conjugate to S.

10. If C is a curve whose functions satisfy the Frenet-Serret formulas (2), the functions defined by

$$\alpha'=\lambda,\quad \beta'=\mu,\quad \gamma'=\nu;\qquad l'=-l,\quad m'=-m,\quad n'=-n;$$
$$\lambda'=\alpha,\quad \mu'=\beta,\quad \nu'=\gamma;\qquad \rho'=-\tau,\quad \tau'=-\rho,\quad p'=p,\quad u'=u$$

satisfy similar equations. Hence, these functions define a curve C',* whose tangent, principal normal and binormal are parallel respectively to the binormal, principal normal and tangent of C. We shall show that there exist surfaces S' such that each of its curves $v=$ const. bears this relation to the corresponding curve on S. Such a surface S' is *conjugate* to S.

If A'_2 and A'_3 denote functions for S' analogous to A_2 and A_3 for S, the equations analogous to (3), (7) and (9) may be put in the form

$$\frac{\partial\lambda}{\partial v}=-A'_2 l+A'_3\alpha,\quad \frac{\partial l}{\partial v}=A'_2\lambda+\left(\tau\frac{\partial A'_3}{\partial u}+A'_2\frac{\tau}{\rho}\right)\alpha,$$
$$\frac{\partial\alpha}{\partial v}=-A'_3\lambda-\left(\tau\frac{\partial A'_3}{\partial u}+A'_2\frac{\tau}{\rho}\right)l.$$

In order that these equations be consistent with (3), (7) and (9), we must have

$$A'_2=\rho\left(\frac{\partial A_3}{\partial u}-\frac{A_2}{\tau}\right),\quad A'_3=-A_3,\quad \tau\left(\frac{\partial A'_3}{\partial u}+\frac{A'_2}{\rho}\right)=-A_2.$$

The last of these equations is a consequence of the first two. Furthermore, when these expressions for A'_2 and A'_3 are substituted in equations for S' analogous to (8) and (10), we obtain the latter equations in inverse order. Hence, it follows from (4) and (11) that the determination of a surface S' reduces to the solution of the linear system

$$\frac{\partial q'}{\partial u}+\frac{r'}{\tau}=0,\quad \frac{\partial r'}{\partial u}-\frac{q'}{\tau}-\frac{t'}{\rho}=\rho\frac{\partial A_3}{\partial u}-A_2\frac{\rho}{\tau},\quad \frac{\partial t'}{\partial u}+\frac{r'}{\rho}=-A_3. \quad (40)$$

If we compare these equations with (2), we note that if a set of functions q', r', t' satisfy (40), so also do

* Cf. Bianchi, "Lezioni," Vol. I, p. 53.

$$\left.\begin{aligned} q_1' &= q' + V_1\lambda + V_2\mu + V_3\nu, \\ r_1' &= r' - V_1 l - V_2 m - V_3 n, \\ t_1' &= t' + V_1\alpha + V_2\beta + V_3\gamma, \end{aligned}\right\} \tag{41}$$

where V_1, V_2, V_3 are arbitrary functions of v.

From equations (12) it follows that the necessary and sufficient condition that the tangent planes to S and S' at corresponding points be parallel is that

$$\sigma(r\lambda - tl) = \sigma'(r'\alpha + t'l), \quad \sigma(r\mu - tm) = \sigma'(r'\beta + t'm),$$
$$\sigma(r\nu - tn) = \sigma'(r'\gamma + t'n),$$

from which it follows that

$$r = r' = 0.$$

We may take $q = 0$; then

$$A_2 = \frac{t}{\tau}, \quad A_3 = \frac{\partial t}{\partial u},$$

and from (40) we have

$$q' = V, \quad t' = V_1 - t, \quad -\frac{V}{\tau} + \frac{t - V_1}{\rho} = \rho\left(\frac{\partial^2 t}{\partial u^2} - \frac{t}{\tau^2}\right), \tag{42}$$

where V and V_1 are arbitrary functions of v. The solution of the problem reduces to the determination of three functions ρ, τ, t satisfying equations (11′) and the last of (42). It is readily seen that such solutions exist.

PRINCETON UNIVERSITY, *May* 20, 1914.

On a Porism Connected with the Theory of Maxwell's Equations and a Method of Obtaining the Lines of Electric Force Due to a Moving Point Charge

By H. Bateman.

§ 1. I have shown elsewhere* that when Maxwell's equations are written in Silberstein's concise form

$$\text{rot } M = -\frac{i}{c}\frac{\partial M}{\partial t}, \qquad \text{div } M = 0, \tag{1}$$

where M denotes the complex vector $H + iE$ and c is a constant, a solution may be obtained by writing for the three components of M

$$M_x = f(\alpha, \beta)\frac{\partial(\alpha,\beta)}{\partial(y, z)}, \quad M_y = f(\alpha, \beta)\frac{\partial(\alpha,\beta)}{\partial(z, x)}, \quad M_z = f(\alpha, \beta)\frac{\partial(\alpha,\beta)}{\partial(x, y)}, \tag{2}$$

where f is an arbitrary function of two quantities α, β which satisfy the partial differential equations †

$$\left.\begin{aligned} c\partial(\alpha, \beta | y, z) &= i\partial(\alpha, \beta | x, t), \\ c\partial(\alpha, \beta | z, x) &= i\partial(\alpha, \beta | y, t), \\ c\partial(\alpha, \beta | x, y) &= i\partial(\alpha, \beta | z, t). \end{aligned}\right\} \tag{3}$$

A general solution of these equations has already been obtained by taking x, y, α, β as new independent variables and transforming the equations. We shall now indicate a method by which the transformation may be performed very quickly.

It is easily seen that equations (3) can be written in the form

$$\left.\begin{aligned} c\partial(\alpha, \beta, x, t | y, z, x, t) &= i\partial(\alpha, \beta, y, z | x, t, y, z), \\ c\partial(\alpha, \beta, y, t | z, x, y, t) &= i\partial(\alpha, \beta, z, x | y, t, z, x), \\ c\partial(\alpha, \beta, z, t | x, y, z, t) &= i\partial(\alpha, \beta, x, y | z, t, x, y). \end{aligned}\right\} \tag{4}$$

Now multiply each Jacobian in equation (4) by $\partial(x, y, z, t | \alpha, \beta, x, y)$ and make use of the multiplication theorem for Jacobians. We then obtain a set of equations similar to (4) but with α, β, x, y as independent variables instead of x, y, z, t. The new equations reduce to the form

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* "The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell's Equations," pp. 12, 122, 125, *Cambr. Univ. Press*; also *Bulletin of the American Mathematical Society*, Vol. XXI (1915), pp. 299–309. This paper will be cited later as "B."

† A different notation is now adopted for the Jacobians already used.

$$\frac{\partial t}{\partial y}=-\frac{i}{c}\frac{\partial z}{\partial x}, \quad \frac{\partial t}{\partial x}=\frac{i}{c}\frac{\partial z}{\partial y}, \quad \frac{\partial(z, t)}{\partial(x, y)}=\frac{i}{c}. \tag{5}$$

These equations may be solved as before, and the result is

$$z-ct=\phi+\theta(x+iy), \quad z+ct=\psi-\frac{1}{\theta}(x-iy), \tag{6}$$

where θ, ϕ, ψ are arbitrary functions of α and β.

Now let complex values be assigned to α and β and let x, y, z, t, be real. We proceed to find under what circumstances equations (6) can be satisfied. Write $\phi=\phi_1+i\phi_2$, $\psi=\psi_1+i\psi_2$, $\theta=\theta_1+i\theta_2$, where θ_1, θ_2, ϕ_1, ϕ_2, ψ_1, ψ_2 are all real; then, on equating the real and imaginary terms in equations (6), we find that

$$\left.\begin{aligned} z-ct&=\phi_1+\theta_1 x-\theta_2 y, & \phi_2+\theta_2 x+\theta_1 y&=0,\\ z+ct&=\psi_1-\frac{\theta_1 x-\theta_2 y}{\theta_1^2+\theta_2^2}, & \psi_2+\frac{\theta_2 x+\theta_1 y}{\theta_1^2+\theta_2^2}&=0. \end{aligned}\right\} \tag{7}$$

Hence, it appears that equations (6) can only be satisfied by real values of x, y, z, t, if the complex numbers α, β are chosen so that

$$\psi_2(\theta_1^2+\theta_2^2)=\phi_2. \tag{8}$$

When this condition is satisfied there are ∞^1 sets of values of x, y, z, t which satisfy equations (7). *Hence, the problem of finding real values of x, y, z, t corresponding to two complex numbers α, β is poristic.*

On solving equations (7) for x, y, z, we find that

$$\left.\begin{aligned} (1+v)z&=ct(1-v)+\phi_1+v\psi_1,\\ (1+v)y&=2\theta_2 ct+\theta_2(\phi_1-\psi_1)-\phi_2\theta_1(1+v^{-1}),\\ (1+v)x&=-2\theta_1 ct-\theta_1(\phi_1-\psi_1)-\phi_2\theta_2(1+v^{-1}), \end{aligned}\right\} \tag{9}$$

where $v=\theta_1^2+\theta_2^2$. *Hence, the different real values of x, y, z, t corresponding to a possible set of complex values of θ, ϕ, ψ are associated with a point which travels along a straight line with velocity c.*

We may regard the equations

$$X=\phi(\alpha, \beta), \quad Y=\psi(\alpha, \beta), \quad Z=\theta(\alpha, \beta) \tag{10}$$

as the equations of a surface S in the (X, Y, Z) space. Each point on this surface whose coordinates satisfy the condition (8) will correspond to a straight line which is described by a moving point with velocity c. Hence, with each surface S there is associated a complex of ∞^3 real straight lines described by points moving with velocity c. On account of the porism which has just been mentioned, one of the statements on p. 127 of my book is incorrect. It is easily seen that equations (280) on p. 124, viz.,

$$\left.\begin{aligned} (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2&=c^2(t-\tau)^2,\\ l(x-\xi)^2+m(y-\eta)^2+n(z-\zeta)^2&=c^2p(t-\tau),\\ l^2+m^2+n^2&=c^2p^2, \end{aligned}\right\} \tag{11}$$

can be thrown into the form (6). For, if we write $ict=s$, $ic\tau=\sigma$, and interpret x, y, z, s; ξ, η, ζ, σ as rectangular coordinates of points in a space of four dimensions, the first of equations (11) represents a hypersphere of zero radius, and the second, one of its tangent hyperplanes. The two equations consequently represent two planes, each of which passes through a generator of the sphere at infinity. Moreover, it is easy to see that the equations of such a plane can be thrown into the form (6). It is not correct, then, to say that if $\xi, \eta, \zeta, \tau, l, m, n, p$ are functions of two complex variables α, β, there are generally two real space-time points x, y, z, t corresponding to given values of α and β, for the special case to which I have referred on p. 127 is the only one which can arise.

The method which I have suggested for a specification of a real space-time point x, y, z, t by means of two complex numbers α, β thus has the disadvantage that there may be either no real point x, y, z, t or an infinite number of such points corresponding to a given pair of values of α and β.

§ 2. I have shown that a line of electric force in the electromagnetic field due to a moving point charge may be obtained by considering the positions at time t of a series of particles which are projected in certain directions from the different positions of the moving charge and travel along straight lines with the velocity of light. The direction cosines (l_0, m_0, n_0) of the direction of projection at time τ were found to satisfy three differential equations of type*

$$\mu \frac{dl_0}{d\tau} = \lambda\xi'' + (\xi' - cl_0)(l_0\xi'' + m_0\eta'' + n_0\zeta''), \tag{12}$$

where $\mu = c^2 - \xi'^2 - \eta'^2 - \zeta'^2$, $\lambda = c - l_0\xi' - m_0\eta' - n_0\zeta'$ and (ξ, η, ζ) denote the coordinates of the point charge at time τ. These equations are easily seen to be consistent with the relation $l_0^2 + m_0^2 + n_0^2 = 1$, for on multiplying the three equations by l_0, m_0, n_0, respectively, and adding, we find that $l_0\dfrac{dl_0}{d\tau} + m_0\dfrac{dm_0}{d\tau} + n_0\dfrac{dn_0}{d\tau} = 0$.

Let us now write

$$ks = l_0 + im_0, \quad k\sigma = l_0 - im_0, \quad k = 1 + n_0. \tag{13}$$

Then it is easy to see that s satisfies the *Riccatian equation*

$$2\mu\frac{ds}{d\tau} = (\xi'' + i\eta'')(c - \zeta') + \zeta''(\xi' + i\eta') - 2s[c\zeta'' + i(\xi'\eta'' - \xi''\eta')] + s^2[\zeta''(\xi' - i\eta') - (\zeta' + c)(\xi'' - i\eta'')],$$

and that σ satisfies a similar equation in which $-i, \sigma$ are written in place of i and s respectively.

If these equations can be solved in any special case, l_0, m_0, n_0 can be expressed as functions of τ with the aid of equations (13), and then the coordinates of an arbitrary point on a line of electric force can be expressed as functions of the parameter τ.

February 8, 1915.

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* B., p. 308. The primes are used to denote differentiation with regard to τ.

The Abstract Definitions of Groups of Degree 8.

By Josephine E. Burns.

Introduction.

Abstract group theory may be said to begin in the work of Cayley. In 1854 he first defined a group by means of relations existing among its operators. In two papers published in the *Philosophical Magazine** he defined a few elementary groups by means of such abstract relations. In a sequel to these papers which appeared in 1859, he defined abstractly the five groups of order 8 and the system of dihedral groups. On the other hand, Cayley did not attempt to formulate a general abstract definition of a group until several years later (1878), not, in fact, until after such a formulation had been made by others. From the number of pioneers in abstract group theory the name of Sir William Hamilton should not be omitted; for to him are due the first definitions of the important category of groups known as the groups of the regular polyhedrons. These were included in an article published in the *Proceedings of the Royal Irish Academy*† in 1856, just two years after Cayley's first articles. However, in spite of these beginnings on English soil, the first actual attempts at an abstract definition of a group were found in the work of a German mathematician some fifteen years later. Kronecker,‡ in 1870, gave a really abstract definition of a group, although he considered only the case in which the operators were commutative. Dyck, in 1882, published quite an extensive article in the *Mathematische Annalen,* in which he made marked advance over anything which had previously appeared. He explicitly defined the simple group of order 168, and a group of order $2mn$ defined by the relations

$$s_1^2 = s_2^2 = s_3^2 = s_4^2 = s_1 s_2 s_3 s_4 = (s_1 s_2)^m = (s_3 s_2)^n = 1.$$

The contributions of Weber (1882) and of Frobenius (1887) should also be noted in this connection.

* "Papers," 2 (1854), p. 123.

† *Proceedings of the Royal Irish Academy,* 6 (1853–1857), p. 415.

‡ Kronecker, "Werke," 1, p. 274.

Netto* was the first to examine the possible orders of groups which may be generated by operations satisfying certain defining relations. He sought the number of possible products of s_1 and s_2 when they are connected by the equation $s_1 s_2 = s_2 s_1^m$. He also enunciated a theorem in regard to the order of the group generated by three operators satisfying quite elementary conditions. In recent years, much has been done by Professor G. A. Miller in regard to the groups generated by a small number of operators satisfying simple equations of relation. In particular, his article entitled "The Abstract Definitions of All the Substitution Groups Whose Degree Does not Exceed Seven"† forms the starting-point for the present paper.

The first part of this paper is devoted to the proof of a few general theorems relative to the groups generated by two operators satisfying certain defining relations. The second part consists of the abstract definitions of the substitution groups of degree 8, including applications of the theorems of the first part.

I. GENERAL THEOREMS.

In the theorems which follow, the equation $s^n = 1$ implies that s is of exactly order n, and when relations are given connecting two operators it is supposed that no other relations exist except those stated and such others as may be derivable from them.

1. *The Group Generated by Two Operators Satisfying the Conditions* $s_1^4 = s_2^4,\ (s_1 s_2)^2 = 1,\ s_1^{-1} s_2 s_1 = s_2^{\beta} s_1^{\alpha}$.

The groups generated by two operators s_1 and s_2 satisfying the relations $s_1^4 = s_2^4$, $(s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_2^{\beta} s_1^{\alpha}$ depend upon the congruences of α and β, mod 4. Four cases may be distinguished according as $\beta \equiv 0, 1, 2$ or 3, mod 4. In each case the values of α reduced by the same modulus will give subcases.

CASE 1. $\beta \equiv 0$, mod 4. The third equation in this case reduces to $s_2 = s_1^{\alpha+4k}$. The group is then cyclic and the order depends upon α. The largest possible group arises when $\alpha \equiv 3$, mod 4, in which case G is of order 8. All other values of α give subgroups of this cyclic group.

CASE 2. $\beta \equiv 1$, mod 4. The third equation now becomes $s_1^{-1} s_2 s_1 = s_2 s_1^{\alpha} \cdot s_2^{\beta-1}$ $= s_2 s_1^{\alpha} s_2^{4k'}$, where k' is integral. The order of G again depends upon α. The second equation gives the relations

$$s_1^{-2} s_2^{-1} = s_2^{4k+1}, \quad s_1^{-2} = s_2^{4k+2}, \quad s_2^8 = 1.$$

* Netto, "Substitutionentheorie" (1882), p. 37.

† AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), p. 363.

Hence, the orders of s_1 and s_2 divide 8. If this is exactly the order, two possibilities arise. If k is odd, G is of order 16, and may be generated by the two substitutions $s_1 = adcfehgb$, $s_2 = abcdefgh$. In the table of abstract definitions of groups of degree 8, which appears as Part II of the present paper, this is G_{16}, No. 8. If k is even, G is abelian of order 16 and type $(3, 1)$. It may be generated by the two substitutions $s_1 = abcdefgh \cdot ij$, $s_2 = ahgfedcb$. If $s_1^4 = s_2^4 = 1$, G is abelian of order 8 and type $(2, 1)$. From smaller possible orders of s_1 and s_2 arise the four-group, the octic group and the group of order 2.

CASE 3. $\beta \equiv 2$, mod 4. The third equation now becomes $s_1^{-1} s_2 s_1 = s_2^2 s_1^{\alpha} s_2^{4k'}$. If $\alpha \equiv 0$, mod 4, the order of s_1 reduces to 2, and s_2 must be identity. If $\alpha \equiv 1$, mod 4, the relations follow:

$$s_1^{-1} s_2 s_1 = s_2^2 s_1 \cdot s_1^{4k}, \quad s_1^{-1} = s_2^{4k+1}, \quad (s_1 s_2)^2 = (s_2^{-4k})^2 = s_2^{-8k} = 1; \quad s_1^4 = s_2^{-4}, \quad s_2^8 = 1, \quad s_1^8 = 1.$$

The group is then the cyclic group of order 8. If $\alpha \equiv 2$, mod 4, $s_1^4 = s_2^2 = 1$ and $s_1^2 = s_2$; hence, G is the cyclic group of order 4. If s_2 reduces to the identical operation, the conditions are satisfied and G is of order 2. If $\alpha \equiv 3$, mod 4, the conditions are contradictory unless $s_1^2 = s_2^2 = 1$ and $s_1 = s_2$.

CASE 4. $\beta \equiv 3$, mod 4. Equation three now becomes $s_1^{-1} s_2 s_1 = s_2^3 s_1^{\alpha} \cdot s_2^{4k'}$. If $\alpha \equiv 0$, mod 4, the following equations result:

$$s_1^{-1} s_2 s_1 = s_2^{4k'+3}, \quad s_1^{-2} s_2^{-1} = s_2^{4k'+3}, \quad s_1^{-2} = s_2^{4k}, \quad s_1^4 = s_2^{-4k} = s_2^4, \quad s_2^{4k+4} = 1.$$

Now, $s_1 s_2$ and $s_2 s_1$ have a common square. Hence, the product of one into the inverse of the other is transformed into its inverse by each operation.

$$(s_1 s_2^2 s_1)^2 = s_1 s_2^2 s_1^2 s_2^2 s_1 = s_1 s_2^4 s_1 s_1^2 = s_1^8.$$

Hence, s_1^8 both is invariant and is transformed into its inverse by $s_1 s_2$ and $s_2 s_1$. The order of s_1 and s_2 divides 16. We then have as the only possibilities $s_1^2 = s_2^4 = (s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_2^3$, which define the octic group, or $s_1^2 = s_2^2 = (s_1 s_2)^2 = 1$, which define the four-group.

If $\alpha \equiv 1$, mod 4, the conditions are contradictory unless $s_1 = 1$ and $s_2^2 = 1$. If $\alpha \equiv 2$, mod 4, the following conditions arise:

$$(s_1 s_2^2 s_1)^2 = s_1 s_2^2 s_1^2 s_2^2 s_1 = s_2^2 s_1^3 s_2^2 s_1 \cdot s_1^{4k} = s_2^2 s_1^3 \cdot s_1 s_2^2 = s_2^8, \text{ since } s_1^{-1} s_2 s_1 = s_2^3 s_1^2 \cdot s_1^{4k}.$$

The order of s_1 and s_2 then divides 16. But no groups exist for the maximum values of the orders of s_1 and s_2. If $s_1^8 = s_2^8 = 1$, the conditions result:

$$s_2^{-1} s_1^2 s_2 = s_2^3 s_1^2 s_2 s_1^4 = s_1^{-2} s_2^{-1} s_2 \cdot s_1^{4k-4} = s_1^2 s_1^{-4k}.$$

If k is even, s_1^2 and s_2^2 are invariant in G and generate the central. $[s_2, s_1^2]$ is abelian of order 16, type $(3, 1)$. This subgroup is invariant under s_1. G is of

order 32. Its existence is proved, since it may be generated by two operators $s_1 = adc'f'ehg'b' \cdot a'd'cfe'h'gb$ and $s_2 = abcdefgh \cdot a'b'c'd'e'f'g'h'$. If k is odd, s_1^2 and s_2^2 are commutative, each is of order 4 and they generate an invariant subgroup of order 8, which is abelian, of type (2, 1). s_2 transforms this H_8 into itself, giving a subgroup of order 16, H_{16}, which is in turn invariant under s_1. G is then of order 32. Its existence is established by the substitutions $s_1 = abchefgd$, $s_2 = abcdefgh$. In the table of definitions of groups of degree 8, this group is No. 9, of order 32.

If $s_1^4 = s_2^4 = 1$, then s_1^2 and s_2^2 are invariant and generate the four-group. Extending this by s_2 gives an abelian subgroup H_8, type (2, 1), which is invariant under s_1. G is therefore of order 16. It may be generated by $s_1 = abef \cdot cdgh$, $s_2 = abcd \cdot efgh$. In the table this group is No. 5 of order 16. Degenerate cases are given if $s_1^4 = s_2^4 = 1$, the octic group, or if $s_1^2 = s_2^2 = 1$, the four-group.

If $\alpha \equiv 3$, mod 4, the conditions are contradictory unless s_2 is of order 2 and s_1 reduces to the identical operation.

The preceding results may be collected in the following theorem:

If two operators s_1 and s_2 satisfy the three conditions $s_1^4 = s_2^4$, $(s_1s_2)^2 = 1$, $s_1^{-1}s_2s_1 = s_2^{\beta}s_1^{\alpha}$, they generate one of eleven groups, two of which are of order 32, *one of order* 16 *and the others subgroups of these three. Six are non-abelian; and five are abelian, three cyclic and two non-cyclic abelian.*

2. *The Groups Generated by Two Operators Satisfying the Conditions*

$$s_1^4 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^{-1}s_2s_1 = s_1^{\alpha}s_2^{\beta}.$$

When $s_1^4 = s_2^4$, $(s_1s_2)^2 = 1$, $s_1^{-1}s_2s_1 = s_1^{\alpha}s_2^{\beta}$, four cases are possible, according as $\alpha \equiv 0, 1, 2$ or 3, mod 4. Within each case are considered the groups which arise when $\beta \equiv 0, 1, 2$ or 3, mod 4.

CASE 1. $\alpha \equiv 1$, mod 4. If $\beta \equiv 0$, mod 4, the third equation reduces to equation $s_1^{-1}s_2s_1 = s_1^{4k+1}$. Hence, G must be cyclic and its order can evidently not exceed 4. If $\beta \equiv 1$, mod 4, the conditions are contradictory except for the trivial case $s_1 = s_2^2 = 1$. If $\beta \equiv 2$, mod 4, equation three becomes $s_1^{-1}s_2s_1 = s_1s_2^2s_2^{4k}$ or $s_2s_1 = s_1^2s_2^2 \cdot s_2^{4k}$. Then

$$s_1^{-1}s_2^{-1} = s_1^2s_2^2 \cdot s_2^{4k}, \quad s_1^{-3}s_2^{-3} = s_1^{4k}, \quad s_1s_2 = s_1^{4k+3}, \quad s_2 = s_1^{4k+7}.$$

The group is then evidently cyclic and the order of s_1 is limited to 8. G is therefore the cyclic group of order 8 or a subgroup. If $\beta \equiv 3$, mod 4, the conditions can be satisfied if and only if $s_1 = s_2^2 = 1$.

CASE 2. $\alpha \equiv 2$, mod 4. If $\beta \equiv 0$, mod 4, the conditions are satisfied only if $s_1^2 = s_2 = 1$. If $\beta \equiv 1$, mod 4, the conditions readily reduce to $s_1^2 = s_2^2 = (s_1s_2)^2 = 1$

and the four-group results. If $\beta \equiv 2$, mod 4, the conditions again degenerate to the trivial case $s_1^2 = s_2 = 1$. If $\beta \equiv 3$, mod 4, the three defining relations cease to be independent. The third equation may be deduced from the second. The orders of s_1 and s_2 can not exceed 8. The equations then become

$$s_1^8 = s_2^8 = 1, \quad s_1^4 = s_2^4, \quad (s_1 s_2)^2 = 1, \quad \text{and} \quad s_1^4 = s_2^4 = (s_1 s_2)^2 = 1.$$

It is well-known that these conditions are not in themselves sufficient to define groups of finite order. In the trivial case $s_1^4 = s_2^2 = 1$, the octic group is defined and, if $s_1^2 = s_2^2 = 1$, the four-group.

CASE 3. $\alpha \equiv 3$, mod 4. If $\beta \equiv 0$, mod 4, equation three reduces to $s_1^{-1} s_2 s_1 = s_1^3 s_2^{4k}$ or $s_1 = s_2^{4k+3}$. Hence, G is cyclic and its order can evidently not exceed 8. All such groups are then subgroups of the cyclic group of order 8. If $\beta \equiv 1$, mod 4, the group is again cyclic and of order 4 or 2. The cases $\beta \equiv 2$ and $\beta \equiv 3$, mod 4, also give rise to these two cyclic groups and only these.

CASE 4. $\alpha \equiv 0$, mod 4. Condition three now becomes $s_1^{-1} s_2 s_1 = s_1^{4k'}$, $s_2^{\beta} = s_2^{4k+\beta}$. If $\beta \equiv 0$, mod 4, the only conditions possible are evidently $s_1^2 = s_2 = 1$. If $\beta \equiv 1$, mod 4, the following equations result:

$$s_1^{-1} s_2 s_1 = s_2^{4k+5} = s_1^{-2} s_2^{-1}, \quad s_1^{-2} = s_2^{4k+6}, \quad \therefore\ s_1^6 = s_2^6 = 1.$$

If k is odd, the group is abelian of order 8 and type (2, 1). If k is even, $s_1^{-2} = s_2^6$, $s_1^2 = s_2^2$ and $s_1^{-1} s_2 s_1 = s_2^5$. The result is the group of order 16 containing a cyclic subgroup of order 8, each operator of which is transformed into its fifth power. In the table this group is No. 9 of order 16. If $\beta \equiv 2$, mod 4, the equations again degenerate so that only a group of order 2 results. If $\beta \equiv 3$, mod 4, s_1 and s_2 generate either the octic group or the four-group.

These results may be collected in the theorem:

If two operators s_1 and s_2 satisfy the three conditions $s_1^4 = s_2^4$, $(s_1 s_2)^2 = 1$, $s_1^{-1} s_2 s_1 = s_1^{\alpha} s_2^{\beta}$, they generate one of seven groups, two of which are the octic and G_{16}, containing an operator of order 8 which is transformed into its fifth power. All others are subgroups of G_{16} and are abelian. The case in which $\alpha \equiv 2$ and $\beta \equiv 3$, mod 4, is excepted, for when these conditions are fulfilled, s_1 and s_2 may generate a group of infinite order.

3. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^4 = s_2^4, \quad (s_1 s_2)^2 = 1, \quad s_2^2 s_1^2 = s_1^2 s_2^2.$

In order to find the largest possible orders for s_1 and s_2 when they satisfy the above conditions, the following theorem is applied:

If two operators have a common square, the product of one and the inverse of the other is transformed into its inverse by each operator.

$s_1 s_2$ and $s_2 s_1$ being of order 2, their product is transformed into its inverse by each operator. $(s_1 s_2^2 s_1)^2 = s_1 s_2^2 s_1^2 s_2^2 s_1 = s_1^8$. Hence, the order of s_1 and s_2 divides 16. The possible cases then are

$$1)\ s_1^{16} = s_2^{16} = 1, \quad 2)\ s_1^8 = s_2^8 = 1, \quad 3)\ s_1^4 = s_2^4 = 1, \quad 4)\ s_1^2 = s_2^2 = 1, \quad 5)\ s_1 s_2 = s_2 s_1.$$

In the last case the groups are abelian.

CASE 1. $s_1^{16} = s_2^{16} = 1$. s_1^4 is an invariant operator of order 4. s_1^4, $s_1 s_2^2 s_1$, $s_1^2 s_2^2$ generate an abelian group of order 16 and type $(2,1,1)$ which is invariant under s_1 and s_2.

$$(s_1^2 s_2^2)^{-1} s_1 s_2^2 s_1 (s_1^2 s_2^2) = s_1^8 s_2^2 s_1^8 s_2^2 s_1^8 s_2^2 = s_1^8 s_1^2 s_2 s_1^{-2} s_2 s_1^2 = s_1^4 s_1^2 s_2 s_1^2 s_2 s_1^2 = s_1^4 s_1 s_2^{-2} s_1 = s_1 s_2^2 s_1,$$
$$s_1^{-1} \cdot s_1 s_2 s_1 \cdot s_1 = s_2^2 s_1^2, \quad s_1^{-1} s_1^2 s_2^2 s_1 = s_1 s_2^2 s_1, \quad s_2^{-1} \cdot s_2^2 s_1^2 . s_2 = s_2 s_1^2 s_2 = s_1 s_2^2 s_1 s_1^8,$$
$$s_2^{-1} \cdot s_1 s_2^2 s_1 \cdot s_2 = s_2^{-1} s_1 s_2 s_1^{-1} = s_2^2 s_1^2 \cdot s_1^8.$$

However, the abelian group of order 16, type $(2, 1, 1)$, does not admit such an isomorphism as $s_1 s_2$ requires. The group does not exist.

CASE 2. $s_1^8 = s_2^8 = 1$, $s_1 s_2^2 s_1$, $s_1^2 s_2^2$, $s_1 s_2$ and s_1^4 are all of order 2 and are all commutative, and hence generate an abelian group of order 16, type $(1,1,1,1)$. This is invariant under s_1 and s_2.

$$s_1^{-1} s_1 s_2^2 s_1 s_1 = s_2^2 s_1^2, \quad s_1^{-1} s_1^2 s_2^2 s_1 = s_1 s_2^2 s_1, \quad s_1^{-1} s_1 s_2 s_1 = s_2 s_1 = s_1 s_2 \cdot s_1 s_2^2 s_1,$$
$$s_2^{-1} s_1 s_2^2 s_1 s_2 = s_2^{-1} s_1 s_2 s_1^{-1} = s_2^{-2} s_1^{-2} = s_2^2 s_1^2, \quad s_2^{-1} s_1^2 s_2^2 s_2 = s_2 s_1^2 s_2 = s_2 s_1 \cdot s_1 s_2,$$
$$s_2^{-1} s_1 s_2 s_2 = s_2^{-1} s_1 s_2^2 = s_2^{-1} s_1^{-1} s_1^2 s_2^2 = s_1 s_2 . s_1^2 s_2^2.$$

If this group of order 16 be extended by s_1, a group of order 64 results. The subgroup contains $s_1 s_2$; hence G_{64} involves s_2. The existence of such a group is established by the substitutions $s_1 = aebfcgdh$, $s_2 = agbfcedh$. In the table this is No. 2 of order 64.

CASE 3. $s_1^4 = s_2^4 = 1$. $s_1^2 s_2^2$, $s_1 s_2$ and $s_1 s_2^2 s_1$ are all of order 2 and all commutative. They generate an abelian group of order 8, type $(1, 1, 1)$, which is invariant under s_1.

$$s_1^{-1} s_1^2 s_2^2 s_1 = s_1 s_2^2 s_1, \quad s_1^{-1} s_1^2 s_2^2 s_1 = s_1 s_2^2 s_1, \quad s_1^{-1} s_1 s_2 s_1 = s_2 s_1 = s_1 s_2 \cdot s_1 s_2^2 s_1.$$

The order of G is therefore 32. Its existence is proved by the substitutions $s_1 = abcd \cdot efgh$, $s_2 = ag \cdot bfhd$. This group is No. 2 of order 32 in the table.

CASE 4. $s_1^2 = s_2^2 = 1$, $(s_1 s_2)^2 = 1$. The four-group results.

CASE 5. If $s_1 s_2 = s_2 s_1$, one of three groups results. $s_1^8 = s_2^8 = 1$ gives the abelian group of order 16, type $(3, 1)$; $s_1^4 = s_2^4 = 1$, the abelian group of order 8, type $(2, 1)$; and $s_1^2 = s_2^2 = 1$, the four-group.

The following theorem has thus been proved:

If two operators satisfy the three conditions $s_1^4=s_2^4$, $(s_1s_2)^2=1$, $s_1^2s_2^2=s_2^2s_1^2$, *they will generate one of two non-abelian or one of four abelian groups. The non-abelian groups are of orders* 64 *and* 32 *respectively, and the abelian groups are of orders* 16, 8, 4 *and* 2.

4. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^6=s_2^6$, $(s_1s_2)^2=1$, *when* s_1^3 *and* s_2^3 *Are Invariant.*

When s_1 and s_2 satisfy the above conditions, the theorem quoted in the preceding section may be applied to find the maximum order. s_1s_2 and s_2s_1 are of order 2; hence their product is transformed into its inverse by each operator.

$$(s_1s_2^2s_1)^2=s_1s_2^2s_1^2s_2^2s_1=s_1s_2^{-1}s_2^3s_1^3s_1^{-1}s_2^2s_1=s_1s_2^{-1}s_1^{-1}s_2^2s_1\cdot s_1^3s_2^3=s_1^2s_2^3s_1\cdot s_1^3s_2^3=s_1^6s_2^6=s_1^{12}.$$

The order of s_1 and s_2 then divides 24. If the common order of s_1 and s_2 is 2, G is the four-group. If this order is 3 or 4, G is the tetrahedral group or the abelian group of order 8 and type (2, 1) respectively. If $s_1^6=s_2^6=1$, the group generated is the first group of order 48 in the following list of substitution groups. $s_1^2s_2^2$, $s_2^2s_1^2$ are both of order 2 and generate the four-group which is invariant under s_1, so that $[s_1^2, s_1^2s_2^2]$ is the tetrahedral group.

$$(s_1^2s_2^2)^2=s_1^2s_2^2s_1^2s_2^2=s_1^2s_2^{-1}s_1^{-1}s_2^2s_1^3s_2^3=s_1^3s_2^3s_1^3s_2^3=1;$$
$$s_1^3s_1^2s_2^2s_1=s_1s_2^2s_1=s_1^2s_2^2\cdot s_2^2s_1^2,\quad s_1^4\cdot s_1^2s_2^2s_1^2=s_2^2s_1^2,\quad (s_1^2\cdot s_1^2s_2^2)^3=1.$$

This tetrahedral group is extended by s_1, giving a group of order 24 which is invariant under s_2; for $s_2^{-1}s_1s_2=s_2^4s_1^6$, $s_2^{-1}s_1^2s_2^2s_2=s_2^2s_1^2$. Hence, G is of order 48. It may be generated by the two operators $s_1=abd\cdot gh$, $s_2=abc\cdot ef$.

If $s_1^6=s_2^3=1$, it may be easily verified that the group is the non-twelve group of order 24, and may be generated by $s_1=ahdbgc\cdot ef$, $s_2=cehdfg$. If the order of both operators is 8, they are of course commutative and G is the abelian group of order 32 and type (3, 2). When $s_1^{12}=s_2^{12}=1$, G is of order 96. An invariant abelian subgroup of order 8 and type (2, 1) is generated by s_1^3 and s_2^3, and G is the direct product of this group and the tetrahedral group. It may be generated by the two substitutions $s_1=abc\cdot efgh\cdot ijkl$, $s_2=abd\cdot eigk\cdot fjhl$. The maximum group arises when $s_1^{24}=s_2^{24}=1$. It is easily shown that this group is of order 192 and is the direct product of the tetrahedral group and the abelian group of order 16 and type (3, 1). The following theorem results:

If two operators satisfy the conditions $s_1^6=s_2^6$, $(s_1s_2)^2=1$, $s_2^{-1}s_1^3s_2s_1^{-3}=s_1^{-1}s_2^3s_1s_2^{-3}=1$, *then they generate one of eight possible groups of orders* 192, 96, 48, 32, 24, 12, 8, 4, *respectively, three of which are abelian.*

5. *The Groups Generated by Two Operators Satisfying the Conditions*
$$s_1^8=s_2^4,\quad s_1^2s_2=s_2s_1^2,\quad (s_1s_2)^2=1.$$

In order to find the upper limits for the orders of s_1 and s_2 when they satisfy the above relations, we proceed as before. Since s_1s_2 is of order 2, $s_1s_2^2s_1$ must be transformed into its inverse by s_1s_2 and s_2s_1.

$$(s_1s_2^2s_1)^4=s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1^2s_2^2s_1=s_1^8s_2^8=s_1^{24}.$$

Since s_1^{24} is invariant under the group generated, the order of s_1 must divide 48. Five trivial cases arise when the orders of s_1 and s_2 are less than 8 and 4 respectively. If $s_1^2=s_2^2=1$, the group is the four-group, and if $s_1^4=s_2^2=1$, it is the octic group. $s_1^4=s_2^4=1$ define the group which is No. 5 of the list of substitution groups of order 16. If both operators are of order 3 or if one is of order 6 and the second of order 3, they generate the cyclic group of order 3 or the cyclic group of order 6, respectively.

If $s_1^6=s_2^4=1$, s_1 and s_2 generate an abelian subgroup of order 16, type (2, 2), which is invariant under s_1, for $s_1^{-1}s_2s_1=(s_2s_1^2)^{-1}$. G is then of order 32 and is the sixth of that order in the list of substitution groups. It may be generated by the two substitutions $s_2=abcd\cdot ef\cdot gh$, $s_1=aebfcgdh$.

When the common order of s_1 and s_2 is 12, it is readily seen that s_1^3 and s_2^3 satisfy the conditions for the group of order 16 mentioned in the first paragraph of this section and that G is the direct product of this group of order 16 and a cyclic group of order 3.

$s_1^{24}=s_2^{12}=1$ likewise gives rise to a group which is a direct product. s_1^3 and s_2^3 generate the above group of order 32, and G is the direct product of this and the cyclic group of order 3. It is of order 96. The only condition remaining to be considered is $s_1^{48}=s_2^{24}=1$. That these conditions are contradictory follows, if we consider the transform $(s_1s_2)^{-1}s_2s_1s_1s_2=s_1s_2^2s_1^2s_2=s_1^3s_2^3$. $s_1^3s_2^3$ should be of order 2, while this assumption leads to a contradiction.

$$(s_1^3s_2^3)^2=(s_2^{-3}s_1^{-3})^2=(s_2^{-4}s_1^{-8}s_2s_1^5)^2=(s_1^{-12}\cdot s_2s_1)^2=s_1^{24}\neq 1.$$

If two operators s_1 *and* s_2 *satisfy the conditions* $s_1^8=s_2^4$, $s_1s_2s_1^{-2}s_2^{-1}=(s_1s_2)^2=1$, *they generate one of three abelian groups or five non-abelian groups. The non-abelian groups are of orders* 96, 48, 32, 16 *and* 8, *and the abelian groups are the cyclic groups of orders* 6 *and* 3 *and the four-group.*

6. *The Groups Generated by Two Operators Satisfying the Two Conditions*
$$s_1^4=s_2^4,\quad (s_1s_2)^3=(s_1^2s_2)^2=1.$$

In order to find the upper limit for the orders of s_1 and s_2, consider the product $s_1^2s_2\cdot s_2s_1^2$. $s_1^2s_1$ is of order 2; hence, the product $s_1^2s_2\cdot s_2s_1^2$ is transformed into its inverse by each operation. $(s_2s_1^2\cdot s_1^2s_2)^2=(s_2^2s_1^4)^2=s_2^{12}$. s_2^{12} is invariant

and hence the order of s_1 and s_2 must divide 24. $s_1^2=s_2^2=1$ define the symmetric group of order 6; $s_1^3=s_2^3=1$ define the cyclic group of order 3. That $s_1^4=s_2^4=1$ define the octahedral group follows if the operators s_1 and s_1s_2 are considered. $s_1^4=(s_1s_2)^3=(s_1^2s_2^2)^2=1$. Thus they fulfil the well-known relations of the octahedron group. The group of order 18 which is represented as a substitution group by (abc) all (def) is defined abstractly by the given relations when $s_1^6=s_2^6=1$.

When $s_1^8=s_2^8=1$, s_1^2 and s_2^2 generate a quaternion subgroup which is invariant under s_1s_2.

$$(s_1^2)^4=(s_2^2)^4=s_1^4s_2^4=1, \quad s_1^6s_2^2s_1^2=s_2^6, \quad (s_1s_2)^{-1}s_1^2s_1s_2=s_2^7s_1^2s_2=s_1^2s_2s_1^2\cdot s_1^2s_2=s_1^2s_2^6,$$
$$(s_1s_2)^{-1}s_2^2s_1s_2=s_2^7s_1^7s_2^2s_1s_2=s_1^2s_2s_1s_2^2s_1s_2=s_1s_2^7s_1^7s_2s_1s_2=s_1s_2^7s_1^6s_2^7s_1^7=s_1^6s_2^8s_1^7=s_1^2.$$

$[s_1^2, s_2^2, s_1s_2]$ is evidently the non-twelve group of order 24. s_2^2 is in this subgroup, and the subgroup is invariant under s_2. Hence, G is of order 48.

$$s_2^{-1}\cdot s_1s_2\cdot s_2=s_2^7s_1s_2^2=s_2s_1^3=s_2s_1\cdot s_1^2, \quad s_2^{-1}s_1^2s_2=s_1^2s_2s_1^2\cdot s_1^2s_2=s_1^2s_2^6.$$

This group may be generated by $s_1=acegdfbh$, $s_2=aehfdbgc$. It is the fourth group of order 48 in the list which follows.

That the two orders $s_1^{12}=s_2^{12}=1$ and $s_1^{24}=s_2^{24}=1$ are contradictory is seen as follows:

$$s_1^{-2}s_2s_1^2=s_2^{-1} \text{ and } s_1^{-2}s_2^4s_1^2=s_2^{-4}=s_2^4, \ \therefore\ s_2^8=1.$$

If two operators satisfy the conditions $s_1^4=s_2^4$, $(s_1s_2)^3=(s_1^2s_2)^2$, *they generate one of four non-abelian groups or one abelian. The orders of the non-abelian groups are* 48, 24, 8 *and* 6; *and of the abelian*, 3.

7. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^6=s_2^6$, $(s_1s_2)^2=1$, $s_2^{-1}s_1^2s_2=s_1^2$.

In order to find the maximum order of s_1 and s_2, the same process is used as in the preceding problems. Since s_1s_2 and s_2s_1 are of order 2, their product must be transformed into its inverse by each operator.

$$(s_1s_2\cdot s_2s_1)^3=s_1s_2^2s_1^2s_2^2s_1^2s_2^2s_1=s_1s_2^6s_1^6=s_1^{12}.$$

The order of s_1 and s_2 must then divide 24, since s_1^{12} is invariant and is transformed into its inverse.

The conditions are satisfied if $s_1^2=s_2^2=1$, in which case G is the four-group, if $s_1^2=s_2^3=1$, where G is the dihedral group of order 6, when G is cyclic of order 3 and when $s_1^4=s_2^4=1$, which gives the abelian group of order 8 and type $(2,1)$. The dihedral group of order 12 arises when $s_1^2=s_2^6=1$. When $s_1^6=s_2^3=1$, $[s_1^2, s_2]$ is the non-cyclic group of order 9 and is invariant under s_1. Hence, G is of order 18. It may be generated by $s_1=acdfgi\cdot bhe$, $s_2=abc\cdot def\cdot ghi$.

When $s_1^6=s_2^6=1$, $[s_1^2,\ s_1s_2^2s_1]$ is an invariant non-cyclic subgroup of order 9. $s_1s_2\cdot s_1s_2^2s_1\cdot s_1s_2=s_2s_1^2s_2=(s_1s_2^2s_1)^{-1}$. Hence, $[s_1^2,\ s_1s_2^2s_1,\ s_1s_2]$ is of order 18. It is invariant under s_1, giving 36 as the order of G. This is the second group of order 36 in the list of substitution groups, and may be generated by the substitutions $s_1=adcbfe$, $s_2=abc\cdot gh$.

If $s_1^8=s_2^8=1$, $[s_1^2,\ s_1s_2]$ gives the abelian group of order 8 and type $(2, 1)$ which is invariant under s_2, $s_2^{-1}s_1s_2^2=s_2^{-1}s_1^3=(s_1^5s_2)^{-1}$. G is the eighth group of order 16 in the list of substitution groups. When $s_1^{12}=s_2^{12}=1$, $[s_2,\ s_1^2]$ is an invariant abelian subgroup of order 36 with the invariants 12 and 3. G is of order 72. It is generated by

$$s_1=agce\cdot bhdf\cdot imlpoj\cdot kqn,\quad s_2=abcd\cdot efgh\cdot ijk\cdot lmn\cdot opq.$$

When $s_1^4=s_2^{24}=1$, G is the group of order 24 which contains a cyclic subgroup of order 12, each operator of which is transformed into its fifth power by an operator of order 2. $s_1^{-1}s_2s_1=s_1^{-2}s_2^{-1}=s_2^5$. When $s_1^8=s_2^{24}=1$, G is of order 48 and is the direct product of the symmetric group of order 6 and the cyclic group of order 8.

If $s_1^{24}=s_2^{24}=1$, $[s_2,\ s_1^2]$ is abelian of order 72, with invariants 24 and 3. It is invariant under s_1. Hence, G is of order 144. As generating substitutions may be taken $s_1=ahcbedgf\cdot imlpoj\cdot kqn$, $s_2=abcdefgh\cdot ijk\cdot lmn\cdot opq$.

If two operators satisfy the conditions $s_1^6=s_2^6$, $(s_1s_2)^2=1$, $s_2^{-1}s_1^2s_2=s_1^2$, *they generate one of nine non-abelian groups of orders* 144, 72, 48, 36, 24, 18, 16, 12 *and* 6 *respectively, or one of four abelian groups of orders* 8, 6, 4 *or* 3.

8. *The Groups Generated by Two Operators Satisfying the Conditions* $s_1^6=s_2^6,\quad (s_1s_2)^3=1,\quad s_1^2s_2^2=s_2^2s_1^2,\quad (s_1^3s_2^3)^2=1.$

It is possible to apply again the theorom stated, that if two operators have a common square, the product of the one into the inverse of the other is transformed into its inverse by each operator. s_1^3 and s_2^3 have a common square.

$$(s_1^3s_2^{-3})^2=s_1^3s_2^{-3}s_1^3s_2^{-3}=s_1^3s_2^3s_1^3s_2^3s_2^{-12}=s_2^{-12}.$$

Since s_2^{-12} both is invariant and is transformed into its inverse, the orders of s_1 and s_2 divide 24.

If now s_2s_1 be transformed by $s_1^3s_2^3$, the resulting operator must be of order 3.

$$s_1^3s_2^3\cdot s_2s_1\cdot s_1^3s_2^3=s_1^3s_2^4s_1^4s_2^3=s_1s_2\cdot s_1^{12}.$$

From this it is readily seen that the order of s_1 must divide 36. Twelve is then the largest possible order for s_1 and s_2.

$s_1^3=s_2^3=1$ gives the non-cyclic group of order 9. $s_1^2=s_2^3=1$ generates the tetrahedral group. When $s_1^2=s_2^6=1$, G may be generated by $s_1=ab$, $s_2=aecbfd$,

and is of order 24. When $s_1^3=s_2^6=1$, s_2^3 and $s_1^{-1}s_2^3s_1$ generate the four-group, since

$$s_1^{-1}s_2^3s_1s_2^3\cdot s_1^{-1}s_2^3s_1s_2^3=s_1^2s_2s_1s_2s_1^2s_2s_1s_2^3=s_1s_2^{-1}s_2^{-1}s_1^2s_2^2=s_1^3=1.$$

s_1 with this generates the tetrahedral group which is invariant under s_2.

$$s_2^{-1}s_1^{-1}s_2^3s_1s_2=s_1s_2s_1s_2^4s_1s_2=s_1s_2s_1^2s_2^5=s_1s_2^3s_1^2s_2^3,\quad s_2^{-1}s_1s_2=s_2^3s_1s_2^3.$$

G is of order 36 and may be generated by the two substitutions

$$s_1=abg\cdot cdi\cdot efk\cdot hlj,\quad s_2=abcdef\cdot ghijkl.$$

When $s_1^4=s_2^{12}=1$, s_1 and s_2 generate the dicyclic group of order 24.

$$s_1^{-1}s_2s_1=s_1^3s_2s_1=s_2s_1s_2^{11}s_1=s_2s_1s_2^2s_1\cdot s_2^9=s_2s_1^2s_2s_1^2\cdot s_2^9=s_2^2s_2^9=s_2^{11}.$$

When $s_1^6=s_2^6=1$, s_1^3, s_2^3, $s_1^{-1}s_2^3s_1$ and $s_2^{-1}s_1^3s_2$ generate an abelian subgroup of order 16 and type $(1,1,1,1)$.

$$(s_1^3\cdot s_1^5s_2^3s_1)^2=s_1^2s_2^3s_1s_2^2s_2^3s_1=s_1^2s_2^3s_1^3s_2^3s_1=s_1^2s_2^6s_1^4=1,$$
$$(s_2^3\cdot s_1^5s_2^3s_1)^2=s_2^3s_1^5s_2^3s_1s_2^3s_1^5s_2^3s_1=s_2^3s_1^2s_2^3s_1s_2^3s_1^5s_2^3s_1^4=s_2^5s_1^2s_2s_1s_2s_1^2s_2^5s_1=s_2^5s_1s_2^5s_1s_2^5s_1=1,$$
$$(s_1^3\cdot s_2^5s_1^2s_2)^2=s_2^3s_1^3s_2^2s_1^3s_2s_1^3s_2^5s_1^3s_2=s_2^3s_1s_2^2s_1^5s_2s_1s_2^2s_1^5s_2^4=s_2^3\cdot(s_1^5s_2)^3s_2^3=1,$$
$$(s_2^3\cdot s_2^5s_1^3s_2)^2=s_2^2s_1^3s_2^3s_1^3s_2=s_2^5s_1^6s_2=s_2^6=1.$$

This abelian subgroup is invariant under s_1 and

$$s_2\cdot s_1^{-1}\cdot s_1^5s_2^3s_1\cdot s_1^2=s_1^5s_2^3s_1s_2^3,\ \ s_2^{-1}s_1^5s_2^3s_1\cdot s_2=s_2^5s_1^3s_2^3s_1s_2=s_1^3s_2s_1^5s_2s_1s_2^3s_1^4s_2=s_1^5s_2s_1^4s_2^2s_1^3=s_1^5s_2^3s_1;$$
$$s_1^{-1}\cdot s_2^5s_1^3s_2\cdot s_2=s_2^4\cdot s_1^3s_2^2=s_2^4\cdot s_2^4s_1^3s_2s_1^3=s_2^5s_1^3s_2s_1^3.$$

The group is then generated by this subgroup and s_1, s_2, and is of order 144. It is generated by the two substitutions $s_1=abc\cdot ef\cdot gh$, $s_2=ab\cdot cd\cdot efg$. In the list of substitution groups which follows, this is the first group of order 144.

When the common order of s_1 and s_2 is 12, it is possible again to consider the subgroup generated by the four operators s_1^3, s_2^3, $s_1^{-1}s_2^3s_1$, $s_2^{-1}s_1^3s_2$. These now generate the group of order 32.

$$s_1^{-1}s_2^9s_1\cdot s_1^3s_1^{-1}s_2^3s_1=s_1^{-1}s_2^9s_1^3s_2^3s_1=s_1^3,\quad s_1^{-1}s_2^9s_1\cdot s_2^3s_1^{-1}s_2^3s_1=s_1^{-1}s_2^3s_1s_2^3s_1^{-1}s_2s_1=s_2^9,$$
$$s_2^{-1}s_1^9s_2s_1^3s_2^{-1}s_1^3s_2=s_2^5s_1^3s_2s_1^3s_2^5s_1^3s_2=s_1^9,\quad s_2^{-1}s_1^9s_2s_2^3s_2^{-1}s_1^3s_2=s_2^3,$$
$$s_1^{-1}s_2^9s_1s_2^{-1}s_1^3s_2s_1^{-1}s_2^3s_1=s_2^{-1}s_1^3s_2.$$

This G_{32} is invariant under s_1 and s_2, and G is of order 288. In this G_{288}, s_1^6 is an invariant operator of order 2. With respect to this invariant operator of order 2, the quotient group is the G_{144} considered above. This G_{144} contains an invariant abelian subgroup of order 16 and type $(1,1,1,1)$. Corresponding to this in G_{288} must be an invariant subgroup of order 32, all of whose operators must be of order 2 or 4. The invariant subgroup in the quotient group is transformed according to a non-cyclic group of order 9, in the group of isomorphisms of the quotient group. Hence, the corresponding sets in G_{288} must be generated in the same way. We find on trial that the group of order 32 which admits

this isomorphism is the group of order 32 which involves nineteen operators of order 2 and twelve of order 4, of which the latter have a common square. That such a group of order 288 exists, is established by the two substitutions

$$s_1 = actnp\varepsilon bdsmo\theta \cdot eayixvf\beta zjwu \cdot glhk \cdot q\gamma r\delta,$$
$$s_2 = ac\alpha pnvbd\beta omn \cdot elfk \cdot gtri\gamma\varepsilon hsqj\delta\theta \cdot wzxy.$$

The above results may be stated in the following theorem:

If two operators s_1 *and* s_2 *satisfy the conditions* $s_1^6=s_2^6$, $(s_1s_2)^3=(s_1^3s_2^3)^2=1$, $s_1^2s_2^2=s_2^2s_1^2$, *they must generate one of seven groups of order* 9, 12, 24, 36, 144 *or* 288.

II. TABLE OF ABSTRACT DEFINITIONS OF THE SUBSTITUTION GROUPS OF DEGREE 8.

The substitution groups used in this table are taken from the list of Professor G. A. Miller (AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 326). The notation there used has been practically retained. There are, in all, 200 distinct substitution groups of degree 8. As abstract groups, however, these are not all distinct. The same abstract group may be represented in more than one way on eight letters. Some of the groups of degree 8 may be abstractly the same as substitution groups of lower degree. The abstract definitions of the substitution groups whose degree does not exceed 7 have already been given.* The definitions which are given in the present table exclude the groups which may be represented on a smaller number of letters, and include only the distinct abstract groups of degree 8. There are 82 such groups to be defined.

	Order.	No.	Designation.	Abstract Definition.
1	8	1	$(abcdefgh)$	$s_1^8=1.$
2		2	$(abcdefgh)_8$	$s_1^4=(s_1s_2^{-1})^4=s_1^2s_2^2=1.$
3	15	1	$(abcde)(fgh)$	$s_1^{15}=1.$
4	16	1	$(ab)(cd)(ef)(gh)$	$(s_is_j)^2=1\ (i, j=1, 2, 3, 4).$
5		2	$(abcd)(ef)(gh)$	$s_1^4=s_2^2=s_3^2=s_i^{-1}s_j^{-1}s_is_j=1\ (i, j=1, 2, 3).$
6		3	$(abcd)(efgh)$	$s_1^4=s_2^4=s_1^3s_2^3s_1s_2=1.$
7		4	$[(abcd)(efgh)_8]\ \mathrm{dim}_1$	$s_1^4=s_2^4=s_1^3s_2s_1s_2=1.$
8		5	$[(abcd)(efgh)_8]\ \mathrm{dim}_1$	$s_1^4=s_2^4=(s_1s_2)^2=(s_1^2s_2^2)^2=1.$
9		6	$[(abcd)(efgh)_8]\ \mathrm{dim}_2$	$s_1^4=s_2^2=s_3^2=(s_1s_2)^2=(s_2s_3)^2=s_1^3s_2s_1s_2=1.$

* G. A. Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), p. 383.

	Order.	No.	Designation.	Abstract Definition.
10		7	$[(abcd)(efgh)]\,\mathrm{pos}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^4=s_2^2=s_1^3s_2s_1s_2=s_3^2$ $=s_3s_1^3s_3s_1=s_3s_2s_3s_1^2s_2=1.$
11		8	$[(abcd)(efgh)]\,\mathrm{pos}\,(aefbgchd)$	$s_1^8=s_2^2=s_2s_1s_2s_1^3=1.$
12		9	$(abcd\cdot efgh)_{8,2}\,(aebfcgdh)$	$s_1^8=s_2^2=s_2s_1s_2s_1^5=1.$
13		10	$(abcd)_8(efgh)_8(afbgchde)$	$s_1^8=s_2^2=(s_1s_2)^2=1.$
14	18	1	$(abc)(def)(gh)$	$s_1^6=s_2^3=s_2^2s_1^5s_2s_1=1.$
15	24	1	$(abc\cdot def)(afdc\cdot egbh)$	$s_1^3=s_2^6=s_1s_2^4s_1s_2=1.$
16	30	1	$(abcde)(fgh)\,\mathrm{all}$	$s_1^{15}=s_2^2=s_2s_1s_2s_1^4=1.$
17		2	$(abcde)_{10}(fgh)$	$s_1^{15}=s_2^2=s_2s_1s_2s_1^{11}=1.$
18		3	$[(abcde)_{10}(fgh)\,\mathrm{all}]\,\mathrm{dim}$	$s_1^2=s_2^2=(s_1s_2)^{15}=1.$
19	32	1	$(abcd)_8(ef)(gh)$	$s_1^4=s_2^2=s_3^2=s_4^2=(s_1s_2)^2=s_3s_1^3s_3s_1$ $=(s_3s_2)^2=s_4s_1^3s_4s_1=(s_4s_3)^2$ $=(s_4s_2)^2=1.$
20		2	$(abcd)_8(efgh)$	$s_1^4=s_2^4=(s_1s_2)^2=(s_1^2s_2^2)^2=1.$
21		3	$[(abcd)_8(efgh)_8]\,\mathrm{dim}$	$s_1^4=s_2^4=s_3^2=s_1^3s_2^3s_1s_2=(s_2s_3)^2$ $=(s_1s_3)^2=1.$
22		4	$[(abcd)_8(efgh)_8]\,\mathrm{dim}$	$s_1^4=s_2^4=s_3^2=(s_1s_2)^2=(s_1s_3)^2$ $=s_3s_2s_3s_2^3=s_1^3s_2^2s_1s_2^2=1.$
23		5	$[(abcd)_8(efgh)_8]\,\mathrm{dim}$	$s_1^4=s_2^4=s_3^2=(s_1s_2)^2=s_1^3s_2^2s_1s_2^2$ $=(s_1s_3)^2=s_3s_2s_3s_1^2s_2^3=1.$
24		6	$(abcd)(efgh)(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_2^4=(s_1s_2)^2=s_1^6s_2^3s_1^2s_2=1.$
25		7	$[(abcd)_8(efgh)_8]_{2,2}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^4=s_2^4=s_3^2=s_4^2=s_1^3s_2s_1s_2=s_3s_1s_3s_1^3$ $=s_2^3s_1s_2s_1=(s_4s_1)^2=(s_4s_2)^2$ $=(s_4s_3)^2=(s_3s_2)^2=1.$
26		8	$[(abcd)_8(efgh)_8]_{2,2}(aebf\cdot cgdh)$	$s_1^4=s_2^2=(s_1s_2)^4$ $=s_1^2s_2s_1s_2s_1^2s_2s_1^3s_2=1.$
27		9	$[(abcd)_8(efgh)_8]\,(aebfcgdh)$	$s_1^8=(s_1s_2)^2=s_1^4s_2^4=s_2^2s_1^{-1}s_2^2s_1=1.$
28		10	$[(abcd)_8(efgh)_8]_{2,2}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_2^2=s_3^2=s_2s_1s_2s_1^7=s_3s_1s_3s_1^5$ $=(s_2s_3)^2=1.$
29	36	1	$[(abc)\,\mathrm{all}\,(def)\,\mathrm{all}]\,\mathrm{pos}\,gh$	$s_1^6=s_2^6=s_2^5s_1^5s_2s_1=s_3^2=(s_1s_3)^2$ $=(s_2s_3)^2=s_1^3s_2^3=1.$
30		2	$(abcdef)_{18}(gh)$	$s_1^6=s_2^6=(s_1s_2)^2=s_1^4s_2^5s_1^2s_2=1.$
31	48	1	$(abcd)\,\mathrm{pos}\,(ef)(gh)$	$s_1^6=s_2^6=(s_1s_2)^2=s_2^3s_1^5s_2^3s_1$ $=s_1^3s_2^5s_1^3s_2=1.$
32		2	$(abcd)\,\mathrm{pos}\,(efgh)$	$s_1^{12}=s_2^{12}=s_1^6s_2^9=(s_1s_2)^2=1.$
33		3	$[(abcd)\,\mathrm{pos}\,(efgh)\,\mathrm{pos}]_{4,4}$	$s_1^3=s_2^3=s_3^3=(s_1s_2)^2=(s_1s_3)^2$ $=(s_2^2s_3)^2=(s_3s_2s_1^2)^2=1.$
34		4	$(abc\cdot def)(aefgdbch)$	$s_1^8=s_2^8=(s_1s_2)^3=s_1^4s_2^4$ $=s_2^5s_1^6s_2s_1^2=1.$

	Order.	No.	Designation.	Abstract Definition.
35	56	1	$(abcdefgh)_8(bcedghf)$	$s_1^7=s_2^7=(s_1s_2)^2=s_1s_2^3s_1^4s_2^2=1.$
36	60	1	$(abcde)_{10}(fgh)$ all	$s_1^6=s_2^2=(s_1s_2)^{10}=(s_1s_2^2)^2=1.$
37		2	$(abcde)_{20}(fgh)$	$s_1^{15}=s_2^4=s_2^3s_1s_2s_1^2=1.$
38		3	$[(abcde)_{20}(fgh)$ all$]$ pos	$s_1^{15}=s_2^4=s_2^3s_1s_2s_1^7=1.$
39	64	1	$(abcd)_8(efgh)_8$	$s_1^2=s_2^2=s_3^2=s_4^2=(s_1s_2)^4=(s_1s_3)^2$ $=(s_1s_4)^2=(s_3s_4)^2=(s_2s_3)^2$ $=(s_2s_4)^2=1.$
40		2	$[(abcd)_8(efgh)_8]$ dim $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^8=s_1^4s_2^4=(s_1s_2)^2=s_1^7s_2^2s_1s_2^2$ $=s_3^2=s_3s_1s_3s_1^5=(s_3s_2)^2=1.$
41		3	$[(abcd)_8(efgh)_8]$ dim $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^4=s_2^2=s_3^2=(s_1s_2)^4=(s_1s_3)^2$ $=(s_2s_3)^2=s_1^2s_2s_1s_2s_1^2s_2^2s_1^3s_2=1.$
42		4	$(abcd)_8(efgh)_8$ dim $(aecg\cdot bf\cdot dh)$	$s_1^4=s_2^4=(s_1s_2)^2=(s_1s_2^2)^4$ $=(s_1^3s_2)^4=1.$
43		5	$[(abcd)_8(efgh)_8$ pos$]$ dim $(aebfcgdh)$	$s_1^8=s_2^4=(s_1s_2)^2=(s_1^2s_2^2)^2=1.$
44	72	1	(abc) all (def) all (gh)	$s_1^6=s_2^6=s_3^2=(s_1s_2)^3=s_1^5s_2^2s_1s_2^2$ $=s_2^5s_1^2s_2s_1^2=s_3s_1s_3s_1^5=s_3s_2s_3s_2^5=1.$
45		2	$(abcdef)_{36}(gh)$	$s_1^4=s_2^4=s_3^2=(s_1s_2)^2=(s_1^2s_2^2)^3$ $=(s_1s_2^3)^3=s_3s_1s_3s_1^3=s_3s_2s_3s_2^2=1.$
46	96	1	$(abcd)$ all $(ef)(gh)$	$s_1^2=s_2^6=s_3^2=(s_1s_2^2)^4=(s_1s_2^5)^2$ $=(s_1s_3)^2=s_3s_2s_3s_2^5=1.$
47		2	$(abcd)$ all $(efgh)$	$s_1^{12}=s_2^{12}=s_3^2=s_1^3s_2^9=(s_1s_2)^2$ $=s_3s_1s_3s_1^7=s_3s_2s_3s_2^7=1$
48		3	$(abcd)$ pos $(efgh)_8$	$s_1^{12}=s_2^{12}=s_3^2=s_1^3s_2^9=(s_1s_2)^2$ $=s_3s_1s_3s_1^5=s_3s_2s_3s_2^5=1.$
49		4	$[(abcd)$ all $(efgh)_8$ com$]$ dim	$s_1^6=s_2^4=(s_1s_2^{-1})^4=(s_1s_2)^2$ $=(s_1^3s_2^2)^2=1.$
50		5	$[(abcd)$ all $(efgh)_8$ cyc$]$ dim	$s_1^{12}=s_2^4=(s_1^6s_2^2)^2=(s_1^5s_2)^2$ $=(s_1s_2^2)^3s_1^9=1.$
51		6	$[(abcd)$ all $(efgh)$ all$]_{4,4}$	$s_1^4=s_2^4=(s_1s_2)^3=(s_1^2s_2^2)^2$ $=(s_1^2s_2)^4=1.$
52		7	$[(abcd)$ pos $(efgh)$ pos$]_{4,4}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^6=s_2^6=(s_1s_2)^2=(s_1^3s_2^3)^2$ $=(s_1s_2^3)^3=1.$
53		8	$[(ab)(cd)(ef)(gh)]$ pos $(ach\cdot bdg)(ade\cdot bcf)$	$s_1^6=s_2^2s_1^3=s_2^2s_3^3=s_1^2s_3s_1^2s_3^4$ $=s_1s_2s_1^2s_3s_2s_3^5=1.$
54	120	1	$(abcde)_{20}(fgh)$ all	$s_1^{15}=s_2^4=s_3^2=s_2^3s_1s_2s_1^7=s_3s_2s_3s_2^3$ $=s_3s_1s_3s_1^4=1.$

	Order.	No.	Designation.	Abstract Definition.
55	128	1	$(abcd)_8(efgh)_8(ae \cdot bf \cdot cg \cdot dh)$	$s_1^8 = s_1^4 s_2^4 = (s_1 s_2)^2 = (s_1^3 s_2)^2 = s_3^2 = s_4^2 = s_1^7 s_2^2 s_1 s_2^2 = s_3 s_1 s_3 s_1^5 = (s_3 s_2)^2 = s_4 s_1 s_4 s_2^7 = (s_4 s_3)^2 s_1^3 s_2^7 = 1.$
56	144	1	$(abcd)\mathrm{pos}(efgh)$	$s_1^6 = s_2^6 = (s_1 s_2)^3 = (s_1^3 s_2^3)^2 = s_1^2 s_2^2 s_1^4 s_2^4 = 1.$
57		2	$(abcdef)_{72}(gh)$	$s_1^2 = s_2^4 = s_3^2 = (s_1 s_2)^6 = (s_1 s_2^3 s_1 s_2)^2 = (s_1 s_3)^2 = s_3 s_2 s_3 s_2^3 = 1.$
58	168	1	$(abcdefgh)_8(bcedghf)_{21}$	$s_1^6 = s_2^6 = (s_1 s_2)^7 = (s_1^3 s_2^3)^2 = (s_1 s_2^3)^3 = s_2^2 s_1^3 s_2 s_1^5 s_2^3 s_1^4 = s_1^5 s_2^3 s_1 s_2^4 s_1^3 s_2^5 = 1.$
59	180	1	$(abcde)\mathrm{pos}(fgh)$	$s_1^{15} = (s_1^6 s_2^6)^2 = (s_2^5 s_1^{10}) = (s_2^6 s_1^6 s_2^3)^3 = 1.$
60	192	1	$(abcd)\mathrm{all}(efgh)^8$	$s_1^6 = s_2^4 = (s_1 s_2)^2 = (s_1 s_2^{-1})^4 = (s_1^3 s_2^2)^2 = s_3^2 = (s_3 s_1)^2 = s_3 s_2 s_3 s_1^3 s_2 s_1^3 = 1.$
61		2	$[(abcd)\mathrm{all}(efgh)\mathrm{all}]_{4,4}(ae \cdot bf \cdot cg \cdot dh)$	$s_1^6 = s_2^4 = (s_2 s_1^3)^2 = (s_1^2 s_2^2)^3 = (s_1 s_2)^4 = 1.$
62		3	$[(ab)(cd)(ef)(gh)](ace \cdot bdf)(ceg \cdot dfh)$	$s_1^6 = s_2^6 = (s_1 s_2)^4 = s_1^3 s_2 s_1^3 s_2^5 = 1.$
63		4	$+[(ab)(cd)(ef)(gh)]\mathrm{pos}(ace \cdot bdf)(eg \cdot fh)$	$s_1^4 = s_2^6 = (s_1 s_2)^4 = s_1^3 s_2^3 s_1 s_2 = (s_1^2 s_2^2)^3 s_2^3 = s_1^3 s_2^6 s_1^2 s_2 s_1 s_2 s_1^2 s_2^2 = 1.$
64		5	$\pm[(ab)(cd)(ef)(gh)](ace \cdot bdf)(agbh)$	$s_1^8 = s_2^8 = (s_1^2 s_2^6)^2 = (s_1 s_2)^3 = (s_1 s_2^{-1})^3 = (s_1 s_2^2)^4 = (s_1^2 s_2)^4 = 1.$
65	288	1	$(abcd)\mathrm{all}(efgh)\mathrm{pos}$	$s_1^{12} = s_2^{12} = (s_1 s_2)^6 = (s_2 s_1)^2 s_2^9 s_1^9 = (s_1^4 s_2 s_1 s_2)^2 = (s_1^2 s_2^{-1})^2 = s_2^3 s_1^4 s_2^9 s_1^3 = s_1^3 s_2^4 s_1^9 s_2^3 = 1.$
66		2	$[(abcd)\mathrm{all}(efgh)\mathrm{all}]\mathrm{pos}$	$s_1^6 = s_2^6 = s_3^2 = (s_1 s_2)^3 = s_1^2 s_2^2 s_1^4 s_2^4 = (s_1^3 s_2^3)^2 = (s_1 s_3)^2 = (s_2 s_3)^2 = 1.$
67		3	$(abcd)\mathrm{pos}(efgh)\mathrm{pos}(ae \cdot bf \cdot cg \cdot dh)$	$s_1^6 = s_2^2 = (s_1 s_2)^6 = (s_1^2 s_2)^2 (s_2 s_1^2)^{-2} = (s_1^3 s_2)^4 = 1.$
68	336	1	$(abcdefgh)_{336}$	$s_1^7 = s_2^3 = (s_1^5 s_2)^6 = (s_1 s_2)^2 = (s_2 s_1^4 s_2 s_1^3)^2 = (s_2 s_2^4 s_2)^4 = 1.$
69	360	1	$(abcde)\mathrm{all}(fgh)$	$s_1^{12} = s_2^5 = (s_1^6 s_2)^3 = (s_1^9 s_2^3)^2 = s_1^5 s_2 s_1^7 s_2^4 = 1.$
70		2	$(abcde)\mathrm{pos}(fgh)\mathrm{all}$	$s_1^{15} = s_2^5 s_1^{10} = (s_1^6 s_2^6)^2 = (s_2^6 s_1^6 s_2^3)^3 = s_3^2 = (s_3 s_2)^2 = (s_3 s_1)^2 = 1.$
71		3	$[(abcde)\mathrm{all}(fgh)\mathrm{all}]\mathrm{pos}$	$s_1^{15} = s_2^5 s_1^{10} = s_3^2 = (s_2^6 s_1^6)^2 = (s_2^6 s_1^6 s_2^3)^3 = s_3 s_1 s_3 s_2^4 = s_3 s_2 s_3 s_1^4 = 1.$
72	384	1	$(ab)(cd)(ef)(gh)(ace \cdot bdf)(cg \cdot dh)$	$s_1^4 = s_2^6 = (s_1 s_2)^5 = (s_2 s_1 s_2^3 s_1)^3 = (s_1 s_2^{-1})^4 = (s_1^2 s_2 s_1 s_2)^2 = 1.$

	Order.	No.	Designation.	Abstract Definition.
73	576	1	$(abcd)\text{all}(efgh)\text{all}$	$s_1^{12}=s_2^{12}=s_1^4s_2^4s_1^8s_2^8=s_1^3s_2^3s_1^9s_2^9$ $=(s_1^3s_2^4)^2=(s_2^3s_1^4)^2=1.$
74		2	$[(abcd)\text{all}(efgh)\text{all}]\text{pos}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^6=s_2^2=s_3^2=(s_1s_2)^6=(s_1^3s_2)^4$ $=(s_1^2s_2)^2(s_2s_1^2)^{-2}=(s_3s_1)^2$ $=(s_3s_2)^2=1.$
75		3	$[(abcd)\text{all}(efgh)\text{all}]\text{pos}$ $(aebf\cdot cg\cdot dh)$	$s_1^6=s_2^4=(s_1s_2^2)^3=(s_1^3s_2^3)^4$ $=(s_2^3s_1s_2s_1)^3=1.$
76	720	1	$(abcde)\text{all}(fgh)\text{all}$	$s_1^{12}=s_2^{10}=(s_1^6s_2^2)^3=(s_1^9s_2^6)^2$ $=(s_1s_2)^2=s_1^{11}s_2^5s_1^5s_2^5$ $=s_2^9s_1^4s_2s_1^4=s_1^4s_2^2s_1^8s_2^{10}=1.$
77		2	$(abcdef)\text{pos}(gh)$	$s_1^8=s_2^4=s_3^2=(s_1s_2^2)^4=(s_1s_2^3s_1^2s_2)^2$ $=s_3s_1s_3s_1^2=s_3s_2s_3s_2^3=1.$
78	1152	1	$(abcd)\text{all}(efgh)\text{all}$ $(ae\cdot bf\cdot cg\cdot dh)$	$s_1^{12}=s_2^2=(s_1^4s_2)^2(s_1^8s_2)^2$ $=(s_1^3s_2)^2(s_1^9s_2)^2=(s_1^3s_2s_1^4s_2)^2=1.$
79	1344	1	$(abcdefgh)_{1344}$	$s_1^7=s_2^7=s_3^3=(s_1s_2)^3=(s_1^2s_2^2)^2$ $=(s_1^3s_2^3)^2=s_1s_2^3s_1^4s_2^2=(s_1s_3)^2$ $=(s_3s_1^5)^4=(s_3s_2^{-1})^2=(s_3s_2^2)^4=1.$
80	1440	1	$(abcdef)\text{all}(gh)$	$s_1^6=s_2^4=(s_1^2s_2^2)^4=(s_1^2s_2^3s_1^4s_2)^2$ $=s_1^5s_2^3s_1(s_1^2s_2)^2s_2(s_1^2s_2)^3s_2$ $=s_3^2=s_3s_1s_3s_1^5=s_3s_2s_3s_2^3=1.$
81	20160	1	$(abcdefgh)\text{pos}$	* $s_1^3=s_i^2=1\ (i=1,2,\ldots,7).$ $(s_is_{i+1})^3=1\ (i=1,2,\ldots,6).$ $(s_is_j)^2=1\ (i=1,2,\ldots,5)$ $(j>i+1).$
82	40320	1	$(abcdefgh)\text{all}$	* $s_i^2=1\ (i=1,2,\ldots,8).$ $(s_is_{i+1})^3=1\ (i=1,2,\ldots,7).$ $(s_is_j)^2=1\ (i=1,2,\ldots,6)$ $(j>i+1).$

III. EXPLANATORY NOTES.

Order 32, No. 8. The abstract definition is $s_1^4=s_2^2=(s_1s_2)^4=s_1^2s_2s_1s_2s_1^2s_2s_1^3s_2=1$. s_1^2, $(s_1s_2)^2$ and $(s_2s_1)^2$ generate an abelian subgroup of order 8, type $(1,1,1)$, which is invariant under s_1. This H_{16}, thus obtained, is invariant under s_2. Hence, G is of order 32. Its generating substitutions are $s_1 = aehf\cdot cgdb$, $s_2 = ac\cdot eg$.

Order 48, No. 2. This group comes under one of the generalizations of

* E. H. Moore, *Proc. of the London Math. Soc.*, Vol. XXVIII (1897), pp. 357–366.

the tetrahedral group, *i. e.*, the groups generated by two operators with a common cube, whose product is of order 2.*

Order 48, No. 3. $s_1^3 = s_2^3 = s_3^3 = (s_1 s_2)^2 = (s_1 s_3)^2 = (s_2^2 s_3)^2 = (s_3 s_2 s_1^2)^2 = 1$. $s_1 s_2$, $s_2 s_1$, $s_1 s_3$ and $s_3^2 s_1 s_2^2$ generate an abelian group of order 16 and type $(1, 1, 1, 1)$ which is invariant under the group. s_1 may be added, giving G_{48}. The group contains 32 operators of order 3 and 15 of order 2. Its generating substitutions may be chosen $s_1 = abc \cdot efg$, $s_2 = abd \cdot ehg$, $s_3 = acb \cdot efh$.

Order 56, No. 1. $s_1^7 = s_2^7 = (s_1 s_2)^2 = s_1 s_2^3 s_1^4 s_2^2 = 1$. $s_2 s_1$, $s_2^2 s_1^2$, $s_2^3 s_1^3$ generate an abelian subgroup of order 8 and type $(1, 1, 1)$ which is invariant under s_1, so that G is of order 56. $s_1 = abcdefg$, $s_2 = abdhcge$.

Order 60, No. 1. $s_1^2 = s_2^6 = (s_1 s_2)^{10} = (s_1 s_2^2)^2 = 1$. $[s_2, (s_1 s_2)^2]$ is of order 30, in which $(s_1 s_2)^2$ generates an invariant subgroup of order 5.

$$s_2^{-1} \cdot (s_1 s_2)^2 s_2 = (s_2 s_1)^2 = (s_1 s_2)^8, \quad s_2^{-1} (s_1 s_2)^2 s_2 = (s_2 s_1)^2 = (s_1 s_2)^8.$$

G is generated by $s_1 = ad \cdot bc \cdot efg$, $s_2 = ab \cdot ce \cdot fhg$.

Order 64, No. 2. This group of order 64, which comes under the theorem of paragraph 2, contains as invariant subgroups both G_{32}, No. 9, and G_{32}, No. 6. The third group of this same order is defined by extending the definition of G_{32}, No. 8. It likewise involves G_{32}, No. 7.

Order 64, No. 4. $s_1^4 = s_2^4 = (s_1 s_2)^2 = (s_1 s_2^2)^4 = (s_2 s_1^3)^4 = 1$. s_1 and s_2^2 satisfy the conditions of G_{32}, No. 8. $s_1^4 = (s_2^2)^2 = (s_1 s_2^2)^4 = s_1^2 s_2^2 s_1 s_2^2 s_1^2 s_2^2 s_1^3 s_2^2 = s_1^2 s_2 s_1 s_2 s_1^3 = 1$. This H_{32} is invariant for $s_2^{-1} s_1 s_2 = s_2^2 s_1$. The group is generated by $s_1 = aebf \cdot cgdh$, $s_2 = agce \cdot bh \cdot df$.

Order 64, No. 5. $s_1^8 = s_2^4 = (s_1^2 s_2^2)^2 = (s_1 s_2)^2 = 1$. $[s_2, (s_1^2 s_2^2)]$ is a subgroup of order 32. It is simply isomorphic with G_{32}, No. 8, above, for

$$s_2^4 = (s_2^2 s_1^2)^2 = (s_2^3 s_1^2)^4 = s_2^2 \cdot s_2^2 s_1^2 \cdot s_2 \cdot s_2^2 s_1^2 \cdot s_2^2 \cdot s_2^2 s_1^2 s_2^3 \cdot s_2^2 s_1^2 = 1.$$

This is invariant under s_1 for $s_1^{-1} \cdot s_2^2 s_1^2 \cdot s_1 = s_1^7 s_2^2 s_1^3 = s_2 s_1^2 s_2 \cdot s_1^4$. The square of s_1 is in G_{32}. Hence, G is of order 64. Its generating substitutions may be chosen $s_1 = afdechbg$, $s_2 = afch \cdot be \cdot dg$.

Order 96, No. 4. $s_1^6 = s_2^4 = (s_1 s_2^{-1})^4 = (s_1 s_2)^2 = (s_2^2 s_1^3)^2 = 1$. s_1^3, s_2^2, $(s_1 s_2^{-1})^2$, $(s_2^{-1} s_1)^2$ are all operators of order 2 and are all commutative. The abelian group of order 16 and type $(1, 1, 1, 1)$ generated by them is invariant under s_1^2, making a subgroup of order 48 which is invariant under s_2. $s_1 = acb \cdot ef . gh$, $s_2 = abcd \cdot efgh$ may be chosen as generating operators.

Order 96, No. 5. $s_1^{12} = s_2^4 = (s_1^6 s_2^2)^2 = (s_2 s_1^5)^2 = (s_1 s_2^2)^3 s_1^9 = 1$. s_1^5 and $s_1 s_2^2$ satisfy the conditions $t_1^{12} = t_2^{12} = (t_1 t_2)^2 = 1$, $t_1^3 = t_2^3$, and hence generate a group of order

* G. A. Miller, "Generalization of the Groups of Genus Zero," *Trans. of the Am. Math. Soc.*, Vol. VIII (1907), p. 8.

48 which is invariant under s_2. $s_1 = abc \cdot efgh$, $s_2 = abcd \cdot fh$ are two substitutions satisfying these conditions.

Order 96, No. 6. $s_1^4 = s_2^4 = (s_1 s_2)^3 = (s_2 s_1^2)^4 = (s_1^2 s_2^2)^2 = 1$. The transforms of $s_1^2 s_2^2$, by s_1, s_2 and $s_1 s_2$, give, with $s_1^2 s_2^2$, four operators of order 2 which are all commutative. This abelian subgroup of order 16 is invariant under the group. It may be extended by $s_1 s_2$, making a subgroup of order 48 which is again invariant and may be extended by s_1. $s_1 = abcd \cdot eg$, $s_2 = ab \cdot ehfg$.

Order 96, No. 7. $s_1^6 = s_2^6 = (s_1 s_2)^2 = (s_1^3 s_2^3)^2 = (s_1 s_2^3)^3 = 1$. s_1 and $s_2^3 s_1^2$ generate a subgroup of order 24 for $s_1^6 = (s_2^3 s_1^2)^6 = (s_2^3 s_1^2 s_1)^2 = 1$ and $(s_2^3 s_1^2)^3 = s_1^3$. But the subgroup of order 24 is transformed into itself by no operator outside itself. G_{96}, however, contains G_{48}, No. 3, as an invariant subgroup. $s_1 = afcebg \cdot dh$, $s_2 = ahdgbf \cdot ce$.

Order 96, No. 8. $s_1^6 = s_2^2 s_1^3 = s_2^2 s_3^3 = s_1^2 s_3 s_1^2 s_3^4 = s_1 s_2 s_1^2 s_3 s_2 s_3^5 = 1$. s_2 and $s_1^{-1} s_2 s_1$ generate the quaternion group which is invariant under $s_3 s_2 s_1^2$. The subgroup of order 32 $[s_3, s_1^{-1} s_2 s_1, s_3 s_2 s_1^2]$ is invariant under s_1. $s_1 = aedbfc \cdot gh$, $s_2 = acbd \cdot ehfg$, $s_3 = adhbcg \cdot ef$.

Order 168, No. 1. $s_1^6 = s_2^6 = (s_1 s_2)^7 = (s_1^3 s_2^3)^2 = s_2^2 s_1^3 s_2 s_1^5 s_2^3 s_1^4 = (s_1 s_2^3)^3 = s_1^5 s_2^3 s_1 s_2^4 s_1^3 s_2^5 = 1$. s_1^3, s_2^3, $s_2^{-1} s_1^3 s_2$ generate an abelian group of order 8 and type $(1, 1, 1)$. If this is extended by $s_1 s_2$, the group generated is of order 56 and simply isomorphic with the group of order 56 defined above. Extend this H_{56} by s_1 and the order of the group is 168. It may be generated by $s_1 = ab \cdot cfgdeh$, $s_2 = agbfde \cdot ch$.

Order 180, No. 1. $s_1^{15} = s_2^{15} = s_1^5 s_2^{10} = (s_1^6 s_2^6)^2 = (s_2^6 s_1^6 \cdot s_2^3)^3 = 1$. $s_2^6 s_1^6$ and s_2^3 generate the icosahedral group which is invariant in G_{180}. s_2 and this icosahedral subgroup give the entire group. $s_1 = acebd \cdot fgh$, $s_2 = acedb \cdot fgh$.

Order 192, No. 1. $s_1^6 = s_2^4 = (s_1 s_2^{-1}) = (s_1 s_2^{-1})^4 = (s_1^3 s_2^2)^2 = s_3^2 = s_3 s_2 s_3 s_1^3 s_2 s_1^3 = (s_1 s_3)^2 = 1$. This is simply G_{96}, No. 4, extended by an operator of order 2 which is commutative with s_1 and transforms s_2 just as s_1 does.

Order 192, No. 2. $s_1^6 = s_2^4 = (s_2 s_1^2)^4 = (s_2 s_1^3)^2 = (s_2^2 s_1^2)^3 = (s_1 s_2)^4 = 1$. s_2 and $s_2 s_1^2$ generate G_{96}, No. 6. This is invariant under s_1, thus with s_1 generating the whole group. $s_1 = aebhdg \cdot ef$, $s_2 = afch \cdot bgde$.

Order 192, No. 3. $s_1^6 = s_2^6 = (s_1 s_2)^4 = (s_1^3 s_2 s_1^3 s_2^5) = 1$. $s_2 s_1$, $s_2 s_1 s_2^3$ and $s_1^{-1} s_2 s_1 s_2^3 s_1$ generate a group of order 64 which is simply isomorphic with G_{64}, No. 3. With s_2^3, this subgroup generates the entire group. $s_1 = ab \cdot cfgdeh$, $s_2 = ace \cdot bfd \cdot gh$.

Order 192, No. 4. $s_1^4 = s_2^6 = (s_1 s_2)^4 = s_1^3 s_2^3 s_1 s_2^3 = (s_1^2 s_2^2)^3 s_2^3 = s_1^3 s_2^5 s_1^2 s_2 s_1^3 s_2^5 s_1^2 s_2 = 1$. s_1^2, s_2^3 and $s_2^5 s_1^2 s_2$ generate an abelian group of order 8, type $(1, 1, 1)$, which is invariant under s_1 and s_2. The quotient group as regards this invariant subgroup is of order 24, the octahedral group. $s_1 = agce \cdot bhdf$, $s_2 = aegbfh \cdot cd$.

Order 192, No. 5. $s_1^8=(s_1^6s_2^6)^2=(s_1s_2^2)^4=(s_1^2s_2)^4=(s_1s_2)^3=(s_1s_2^{-1})^3=s_1^4s_2^4=1$. s_1^2, s_2^2, $(s_1s_2^2)^2$, $(s_1^2s_2)^2$ generate a group of order 32 which is invariant under the whole group. The quotient group as regards this invariant subgroup is the dihedral group of order 6. $s_1=agfdbhec$, $s_2=agdfbhce$.

Order 288, No. 1. This group is the direct product of the symmetric group of order 24 and the alternating group of order 12. It is generated by $s_1=acbd\cdot ehg$, $s_2=abcd\cdot efg$. $s_1^{12}=s_2^{12}=(s_1s_2)^6=(s_1^4s_2s_1s_2)^2=s_2^3s_1^4s_2^9s_1^8=s_1^3s_2^4s_1^9s_2^8$ $=(s_2s_1)^2s_2^9s_1^9=(s_1^2s_2^{-1})^2=1$. s_1^3 and $(s_1s_2)^2$ generate the symmetric group of order 24 which is invariant. With respect to this, the quotient group is the tetrahedral group.

Order 288, No. 2. This is simply an extension of G_{144}, No. 1, by an operator of order 2 which transforms each generator into its inverse.

Order 288, No. 3. $s_1^6=s_2^2=(s_1s_2)^6=(s_1^3s_2)^4=(s_1^2s_2)^2(s_2s_1^2)^2=1$. s_1 and $s_2s_1s_2$ generate a group of order 144, which is simply isomorphic with the first group of order 144 above. When this is extended by s_2, the entire group is given. $s_1=abc\cdot ef\cdot gh$, $s_2=ae\cdot bf\cdot cg\cdot dh$.

Order 336, No. 1. $s_1^7=s_2^8=(s_1^5s_2)^6=(s_1s_2)^2=(s_2s_1^4s_2s_1^3)^2=(s_2s_1^4s_2)^4=1$. s_1 and $(s_1^5s_2)^2$ satisfy the conditions for the simple group of order 168, and it is invariant under s_2. Since s_2^2 is in this subgroup, the order of the group is 336. $s_1=bfhgdec$, $s_2=achfgedb$.

Order 360, No. 1. $s_1^{12}=s_2^5=(s_1^6s_2)^3=(s_1^9s_2^3)^2=s_1^4s_2s_1^8s_2^4=1$. s_2 and s_1^6 generate the icosahedral group, for they satisfy the conditions $t_1^2=t_2^5=(t_1t_2)^3=1$. This is invariant under s_1 and contains the sixth power of s_1. Hence, G is of order 360. $s_1=abcd\cdot fgh$, $s_2=adbec$.

Order 360, No. 2. This group is simply G_{180} defined above extended by an operator of order 2 which transforms $s_1^6=s_2^6$ into its inverse, and s_1^3 and s_2^3 into themselves. The next group of order 360 likewise involves G_{180}. The extending operator is of order 2 and transforms s_1 and s_2 into their eleventh powers.

Order 384, No. 1. This group is composed of all the substitutions on eight letters that leave invariant $ac\cdot bd\cdot eg\cdot fh$. It involves three Sylow subgroups of order 128 and sixteen of order 3. It also contains G_{192}, No. 4, and G_{192}, No. 5, invariantly. $s_1^4=s_2^6=(s_1s_2)^8=(s_2s_1s_2^3s_1)^3=(s_1s_2^{-1})^4=(s_1^2s_2s_1s_2)^2=1$. s_2s_1 and $s_2^3s_1$ generate G_{192}, No. 5. $s_1=abe\cdot cdg\cdot fh$, $s_2=abef\cdot cdgh$.

Order 576, No. 1. This is simply the product of two octahedral groups. No. 2 of the same order is G_{288}, No. 8, extended by an operator of order 2 which is commutative with s_2 and transforms s_1 into its inverse.

Order 576, No. 3. $s_1^6 = s_2^4 = (s_1 s_2^2)^2 = (s_1^3 s_2^3)^4 = (s_2^3 s_1 s_2 s_1)^3 = 1$. This group involves G_{144}, No. 1, invariantly and is generated by $G_{144}\ [s_1,\ s_2^3 s_1 s_2]$ and s_2. $s_1 = abc \cdot ef \cdot gh,\ s_2 = aebf \cdot cg \cdot dh$.

Order 720, No. 1. $s_1^{12} = s_2^{10} = (s_1^6 s_2^2)^3 = (s_1^9 s_2^6)^2 = (s_1 s_2)^2 = s_1^{11} s_2^5 = s_2^5 s_1^7$. $s_1^4 s_2^2 = s_2^2 s_1^4$. s_1^8 and s_2^2 generate the symmetric group of order 120 which is invariant under s_1 and s_2. If G_{120} is extended by s_1, the resulting group of order 360 is likewise invariant under s_2. $s_1 = abcd \cdot fgh,\ s_2 = aedcb \cdot fh$.

Order 1152, No. 1. $s_1^{12} = s_2^{12} = s_1^4 s_2 s_1^4 s_2 s_1^8 s_2 s_1^8 s_2 = (s_1^8 s_2)^2 (s_1^9 s_2)^2 = (s_1^3 s_2 s_1^4 s_2)^2 = 1$. s_1 and $s_2 s_1 s_2$ satisfy the defining conditions given above for G_{576}, No. 1. This G_{576} is invariant under s_2. $s_1 = abcd \cdot efg,\ s_2 = ae \cdot bf \cdot cg \cdot dh$.

Order 1344, No. 1. This group is the holomorph of the abelian group of order 8 and type $(1, 1, 1)$. It is generated by $s_1 = bfhgdec,\ s_2 = ahcdbfg$, $s_3 = abf \cdot cdh$.

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*Projective Differential Geometry of One-parameter Families of Space Curves, and Conjugate Nets on a Curved Surface.**

BY GABRIEL M. GREEN.

Introduction.

In studying a geometric configuration, it is often convenient to consider the configuration as part of another, which is characterized by some peculiar geometric property. A one-parameter family of curves on a surface, for instance, is best studied—or so, at least, it seems to the writer—by considering it in connection with the family conjugate thereto. This is, in fact, the point of view of the present paper, in which a one-parameter family of curves is investigated as a component family of a conjugate net.

There seems, at first sight, to be a certain loss of generality in adopting this procedure, since the determination of the family of curves conjugate to a given family requires the integration of a partial differential equation of the first order, an integration which it is, in general, impossible to perform explicitly. It is shown, however, that the restriction is only an apparent one; the possibility of removing it results from the unique analytic apparatus employed. The one-parameter family of curves is defined by a fundamental system of solutions of a completely integrable system of partial differential equations. To Wilczynski is due the credit of recognizing the importance of completely integrable systems in projective differential geometry. By the use of such a system conjugate nets appear in a new light. Darboux's classic treatment depends upon a single partial differential equation; we have associated therewith a second equation, forming with the first a completely integrable system, and are thus enabled to found a purely projective theory of conjugate nets.

The present paper is little more than an introduction to the subject; we have reserved for another occasion the treatment of various interesting topics, in particular the study of certain congruences associated with the configuration.

§ 1. *The Problem, and the System of Differential Equations.*

Let the homogeneous coordinates $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ of a point in space be given by the equations

* Read before the American Mathematical Society, October 25, 1913.

$$y^{(k)} = f^{(k)}(u, v), \qquad (k = 1, 2, 3, 4). \tag{1}$$

These equations define a surface, referred to the parametric curves $u = \text{const.}$, $v = \text{const.}$ It is our purpose to study the single family of curves $v = \text{const.}$, which we shall denote by C_u. The method employed will be that of Wilczynski, in which a projective theory consists in the discussion of the invariants and covariants, under suitable transformations, of a completely integrable system of differential equations.

The curves $u = \text{const.}$, or in our notation the curves C_v, may be asymptotic curves on the surface defined by equations (1). If such be the case, we may always make a proper transformation of the form

$$\bar{u} = U(u, v), \quad \bar{v} = V(v),$$

which alters the parametric curves C_v, but leaves unchanged the curves C_u; in this way we may always select for C_v a set of curves which are not asymptotic, except in the trivial case where all curves are asymptotic, *i. e.*, the surface is a plane.* We shall suppose, then, that the curves C_v are not asymptotic curves. It follows that the functions (1) can not be solutions of a differential equation of the form †

$$\alpha y_{vv} + \beta y_u + \gamma y_v + \delta y = 0. \tag{2}$$

In other words, the four functions (1) can not each satisfy identically the same relation of form (2); hence, *the determinant*

$$D = \begin{vmatrix} y_{vv}^{(1)} & y_u^{(1)} & y_v^{(1)} & y^{(1)} \\ y_{vv}^{(2)} & y_u^{(2)} & y_v^{(2)} & y^{(2)} \\ y_{vv}^{(3)} & y_u^{(3)} & y_v^{(3)} & y^{(3)} \\ y_{vv}^{(4)} & y_u^{(4)} & y_v^{(4)} & y^{(4)} \end{vmatrix} \tag{3}$$

is not identically zero.

Consider the two partial differential equations

$$\left.\begin{aligned} y_{uu} &= a\, y_{vv} + b\, y_u + c\, y_v + d\, y, \\ y_{uv} &= a'\, y_{vv} + b'\, y_u + c'\, y_v + d'\, y. \end{aligned}\right\} \tag{4}$$

The coefficients in these equations may be so determined that the functions (1) will be solutions of system (4). In fact, by substituting successively in the first of equations (4) the quantities $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$, we obtain four equations which, since the determinant D is not zero, may be solved for a, b, c, d. Similarly, we may calculate a', b', c', d'.

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* Systems of curves in the plane have already been studied from the standpoint of projective differential geometry by Wilczynski, "One-parameter Families and Nets of Plane Curves," *Transactions of the American Mathematical Society*, Vol. XII (1911); also by the author in a paper entitled "One-parameter Families of Curves in the Plane," *ibid.*, Vol. XV (1914).

† Darboux, "Théorie Générale des Surfaces," Vol. I, p. 144.

The system (4) thus set up will be completely integrable, unless, as we shall see, $a'^2 - a = 0$. To make the discussion perfectly general, let us suppose given any system of differential equations, of form (4), in which the coefficients are perfectly arbitrary. In order that the system be completely integrable, with just four fundamental solutions, it is necessary that all derivatives of y be expressible linearly, in a unique way, in terms of the same four quantities. The derivatives y_{uu}, y_{uv} are so expressed in terms of y_{vv}, y_u, y_v, y; let us see if it is the same with the higher derivatives of y. Differentiating each of equations (4) with respect to u and v, we obtain the four equations:

$$\left.\begin{aligned} y_{uuu} - a\,y_{uvv} &= b\,y_{uu} + c\,y_{uv} + a_u\,y_{vv} + (b_u + d)\,y_u + c_u\,y_v + d_u\,y,\\ y_{uuv} - a\,y_{vvv} &= b\,y_{uv} + (c + a_v)\,y_{vv} + b_v\,y_u + (c_v + d)\,y_v + d_v\,y,\\ y_{uuv} - a'\,y_{uvv} &= b'\,y_{uu} + c'\,y_{uv} + a'_u\,y_{vv} + (b'_u + d')\,y_u + c'_u\,y_v + d'_u\,y,\\ y_{uvv} - a'\,y_{vvv} &= b'\,y_{uv} + (c' + a'_v)\,y_{vv} + b'_v\,y_u + (c'_v + d')\,y_v + d'_v\,y. \end{aligned}\right\} \quad (5)$$

We may replace y_{uu}, y_{uv} in these equations by their equivalents from (4); the system may then be solved for y_{uuu}, y_{uuv}, y_{uvv}, y_{vvv} linearly in terms of y_{vv}, y_u, y_v, y, provided the determinant of the coefficients on the left,

$$\begin{vmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & -a \\ 0 & 1 & -a' & 0 \\ 0 & 0 & 1 & -a' \end{vmatrix} = a'^2 - a,$$

is not zero. We shall suppose that $a'^2 - a \neq 0$, leaving for a time the geometric interpretation of the excepted cases.

To find the fourth derivatives, we differentiate equations (5) with respect to u, and also with respect to v, thus obtaining eight equations. But differentiation of the first and third of (5) with respect to v gives the same equations as differentiation of the second and fourth with respect to u. It follows that there are just six equations to determine the five fourth derivatives of y. One of the fourth derivatives may therefore be calculated in two ways; but the two expressions must be identically equal, in order that the system (4) be completely integrable. Equating these two expressions, which we suppose expressed (linearly) in terms of y_{vv}, y_u, y_v, y, we obtain a relation of form (2), viz.,

$$\alpha\,y_{vv} + \beta\,y_u + \gamma\,y_v + \delta\,y,$$

which is to be satisfied identically. But we have assumed the curves C_v to be other than asymptotic, so we must have $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$. We thus obtain four equations connecting the coefficients of (4) and their first and second derivatives. If these *conditions of complete integrability* be satisfied, the fourth derivatives of y will be expressible uniquely in terms of y_{vv}, y_u, y_v, y. The higher derivatives will then also be uniquely expressible; in fact, further differentiation of (5) will always give just enough new equations to determine

the derivatives of next higher order. We shall not write out until later the integrability conditions $\alpha=\beta=\gamma=\delta=0$.

System (4) is therefore completely integrable, with four fundamental solutions, provided $a'^2-a\neq 0$ and the integrability conditions are satisfied. If equations (1), viz.,

$$y^{(k)}=f^{(k)}(u,v), \qquad (k=1,2,3,4),$$

be a fundamental system of solutions of (4), then any fundamental system of solutions will be given by

$$\bar{y}^{(k)}=c_{k1}\,y^{(1)}+c_{k2}\,y^{(2)}+c_{k3}\,y^{(3)}+c_{k4}\,y^{(4)}, \qquad (k=1,2,3,4),$$

where the c_{kl}'s are constants such that their determinant $|c_{kl}|\neq 0$. In geometric language, *the configuration represented by any fundamental system of solutions of* (4) *is a projective transformation of the configuration represented by any other fundamental system of solutions.*

In the present paper we shall deal exclusively with a completely integrable system of form (4), for which of course $a'^2-a\neq 0$. If, however, for a system of form (4), $a'^2-a=0$, the third derivatives can not be calculated uniquely from (5). Two cases may arise; either (a) the equations (5) are incompatible, in which case there are fewer than three fundamental solutions of system (4), or (b) one equation of system (5) is a consequence of the other three. In this case one of the third derivatives is perfectly arbitrary, so that there are an infinite number of fundamental solutions of (4). Evidently, neither of these cases can yield a projective theory in three-dimensional space, because for such a theory we need four fundamental solutions. For case (a), it may be shown that a system of four solutions represents geometrically a plane or line.* The curves C_u, which we wish to study, can not lie on a line, but they may lie in a plane. If, now, we associate with system (4) a third differential equation, of form (2), the resulting system will be completely integrable if certain new integrability conditions be satisfied. The completely integrable system will have, not four, but three fundamental solutions. Such a system has indeed been studied by Wilczynski in his memoir, "One-parameter Families and Nets of Plane Curves."† There is no necessity therefore for considering the problem here.

In case (b), one of the third derivatives is arbitrary. This is the *involutory* case, and a system of four solutions represents either a curve or a developable surface.‡ The first of these is of no interest to us; but the second is

* E. J. Wilczynski, first memoir on curved surfaces, *Transactions of the American Mathematical Society*, Vol. VIII (1907).

† *Transactions of the American Mathematical Society*, Vol. XII (1911).

‡ E. J. Wilczynski, first memoir on curved surfaces, *Transactions of the American Mathematical Society*, Vol. VIII (1907).

important. System (4) by itself is in this case not completely integrable; but by associating with it a differential equation of the third order, independent of equation (5), all the third-order derivatives are calculable in terms of y_{vv}, y_u, y_v, y. If, further, integrability conditions be satisfied for the new system of differential equations, that system will be completely integrable, with four fundamental solutions. Such a system would be suitable for studying systems of curves on a developable surface; in fact, a projective theory of developable surfaces has recently been constructed on the basis of such a system of differential equations.* *We shall suppose in this paper that the system of curves C_u does not lie on a developable surface.*

We shall make a further restriction on the curves C_u, the reason for which will become apparent in the sequel. We suppose that the curves C_u are not asymptotic curves on the surface (1), so that† $a \neq 0$. To study a system of asymptotic curves would require a procedure different from that which we propose to follow; Wilczynski has already studied surfaces referred to their asymptotic curves,‡ so that we may well leave aside such systems. *The systems of ∞^1 curves C_u which we do not study in the present paper are those which either lie on a developable (including the case of a plane) or are asymptotic curves on the surface which they determine. In the system of differential equations* (4) *we therefore suppose that*

$$a'^2 - a \neq 0, \tag{6}$$

$$a \neq 0. \tag{7}$$

§ 2. *Intermediate Form of the System of Differential Equations.*

The most general transformations of the variables which leave fixed the system of curves C_u are

$$y = \lambda \bar{y}, \tag{8}$$

where λ is any function of u and v, and

$$\bar{u} = U(u, v), \quad \bar{v} = V(v). \tag{9}$$

These transformations disturb any property not connected intrinsically with the system of curves C_u. The system of differential equations (4) has its general form unchanged when subjected to the transformations (8) and (9), but the coefficients of the system are altered. Since the system is characteristic, not only of the single geometric configuration represented by a funda-

* W. W. Denton, "Projective Differential Geometry of Developable Surfaces," *Transactions of the American Mathematical Society*, Vol. XIV (1913).

† Darboux, "Théorie Générale des Surfaces," t. I, p. 144.

‡ First three of the five memoirs on curved surfaces, *Transactions of the American Mathematical Society*, Vols. VIII–X (1907–09).

mental system of solutions, but also of all its projective transformations, it follows that any property, whose expression in terms of the differential equations remains unchanged under the transformations (8) and (9), is characteristic of the curves C_u and their projective transformations; *i.e.*, it is a projective property. Accordingly, *we call invariants those functions of the coefficients of system* (4) *and their derivatives which remain unchanged, except for a factor, when transformed by* (8) *and* (9); *if the variable y and its derivatives also appear explicitly in such an invariant function, we call the function a covariant.* If similar functions remain unchanged under (8), but not necessarily under (9), we shall call them *seminvariants* and *semi-covariants.*

It will simplify matters to introduce an intermediate form for the system of differential equations. To do this, we make the transformation

$$\bar{u} = U(u, v), \quad \bar{v} = v, \tag{10}$$

which is a subgroup of (9). Denoting $\partial y/\partial \bar{u}$, $\partial y/\partial \bar{v}$, etc., by $\bar{y}_u$, $\bar{y}_v$, etc., we find

$$\left.\begin{aligned} y_u &= \bar{y}_u U_u, \quad y_v = \bar{y}_u U_v + \bar{y}_v, \\ y_{uu} &= \bar{y}_{uu} U_u^2 + \bar{y}_u U_{uu}, \\ y_{uv} &= \bar{y}_{uu} U_u U_v + \bar{y}_{uv} U_u + \bar{y}_u U_{uv}, \\ y_{vv} &= \bar{y}_{uu} U_v^2 + 2\bar{y}_{uv} U_v + \bar{y}_{vv} + \bar{y}_u U_{vv}. \end{aligned}\right\} \tag{11}$$

Substitute these in (4), obtaining a system of two partial differential equations. These may be solved for $\bar{y}_{uu}$ and $\bar{y}_{uv}$, and the resulting system will be precisely of the form (4). Denoting its coefficients by $\bar{a}$, $\bar{b}$, ..., $\bar{a}'$, $\bar{b}'$, ..., we easily find for them the following expressions, in which we have written

$$\Delta = U_u (U_u^2 - 2a' U_u U_v + a U_v^2), \tag{12}$$

and performed no divisions:

$$\left.\begin{aligned} \Delta \bar{a} &= a U_u, \\ \Delta \bar{b} &= a U_u U_{vv} - 2 a U_v U_{uv} - (U_u - 2a' U_v) U_{uu} \\ &\quad + [b (U_u - 2a' U_v) + 2ab' U_v] U_u \\ &\quad + [c (U_u - 2a' U_v) + 2ac' U_v] U_v, \\ \Delta \bar{c} &= c (U_u - 2a' U_v) + 2ac' U_v, \\ \Delta \bar{d} &= d (U_u - 2a' U_v) + 2ad' U_v, \\ \Delta \bar{a}' &= U_u (a' U_u - a U_v), \\ \Delta \bar{b}' &= (U_u U_v - a' U_v^2) U_{uu} - (U_u^2 - a U_v^2) U_{uv} + U_v (a' U_u - a U_v) U_{vv} \\ &\quad + [b' (U_u^2 - a U_v^2) - b (U_u U_v - a' U_v^2)] U_u \\ &\quad + [c' (U_u^2 - a U_v^2) - c (U_u U_v - a' U_v^2)] U_v, \\ \Delta \bar{c}' &= c' (U_u^2 - a U_v^2) - c (U_u U_v - a' U_v^2), \\ \Delta \bar{d}' &= d' (U_u^2 - a U_v^2) - d (U_u U_v - a' U_v^2). \end{aligned}\right\} \tag{13}$$

We may make $\bar{a}'$ vanish by choosing the function $U(u, v)$ as a solution of the differential equation

$$a' U_u - a U_v = 0. \tag{14}$$

It is easy to show that with this choice of U the expression Δ can not be zero. In fact, it takes the form, since $a \neq 0$,

$$\Delta = U_u^2 (a - a'^2)/a. \tag{15}$$

But U_u can not be zero for a proper transformation, and moreover $a - a'^2 \neq 0$, by (7). We may therefore reduce the system (4) to one for which $\bar{a}' = 0$, by solving equation (14).

The geometric interpretation of this transformation is very simple. If in the second of equations (4) we put $a' = 0$, we see from its form that the surface (1) will be referred to the family of curves C_u and to the family of curves conjugate to C_u.* We may therefore state the result:

By the integration of the partial differential equation

$$a' U_u - a U_v = 0, \tag{14}$$

we may always determine the system of curves conjugate to a system of curves C_u which lie on a non-developable surface and are not asymptotic. The system of differential equations (4) *will thereby be thrown into the form*

$$\left.\begin{aligned} y_{uu} &= a y_{vv} + b y_u + c y_v + d y, \\ y_{uv} &= b' y_u + c' y_v + d' y. \end{aligned}\right\} \tag{16}$$

We call this the *intermediate form* of the system of differential equations.

It is easily seen that *the most general transformation of the independent variables which leaves fixed the system of curves C_u and its conjugate system C_v is of the form*

$$\bar{u} = \phi(u), \quad \bar{v} = \psi(v), \tag{17}$$

where ϕ and ψ are arbitrary. This is of course the most general transformation of the independent variables which leaves the system (16) in the intermediate form (characterized by the condition $a' = 0$), without interchanging the parameters u and v.

We shall accordingly study the system (16) under the transformations (8) and (17). Our theory then becomes equivalent to that of a surface referred to a conjugate net as parameter curves. Yet we may legitimately call it the theory of the single system of curves C_u, because the conjugate system C_v is uniquely determined by the system C_u. Moreover, we shall find it possible to express the invariants and covariants of (16), under the transformations (8) and (17), in terms of the coefficients of the original system of differential equations (4), *without requiring the actual integration of the partial differential equation* (14).

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* Darboux, "Théorie Générale des Surfaces," t. I, p. 122.

We may use equation (14) to express the coefficients of the intermediate form in terms of the original quantities a, b, c, etc. Differentiation of (14) with respect to u and with respect to v gives the two relations

$$\left.\begin{aligned} a' U_{uu} - a U_{uv} + a'_u U_u - a_u U_v = 0, \\ a' U_{uv} - a U_{vv} + a'_v U_u - a_v U_v = 0. \end{aligned}\right\} \tag{18}$$

Using these and equation (14), we may remove from (13) all derivatives of U except U_u and U_{uu}. The result is:

$$\left.\begin{aligned}
&\bar{a} = a^2 / U_u^2 (a - a'^2), \\
&\bar{b} = [a (a a'_v - a' a_v) - a' (a a'_u - a' a_u) + b (a^2 - 2 a a'^2) + 2 a^2 a' b' \\
&\qquad + c (a a' - 2 a'^3) + 2 a a'^2 c'] / U_u a (a - a'^2) - U_{uu} / U_u^2, \\
&\bar{c} = a [c (a - 2 a'^2) + 2 a a' c'] / U_u^2 (a - a'^2), \\
&\bar{d} = a [d (a - 2 a'^2) + 2 a a' c'] / U_u^2 (a - a'^2), \\
&\bar{b}' = [a (a b' - a' b) + a' (a c' - a' c) + a' a_u - a a'_u] / a^2, \\
&\bar{c}' = (a c' - a' c) / a U_u, \quad \bar{d}' = (a d' - a' d) / a U_u.
\end{aligned}\right\} \tag{19}$$

It will be noticed that $\bar{b}$ is the only coefficient in which U_{uu} occurs.

§ 3. *The Integrability Conditions for the System in its Intermediate Form.*

We proceed to calculate the integrability conditions for the system in its intermediate form. Specializing equations (5) by putting $a' = 0$, solving for $y_{uuu}, y_{uuv}, y_{uvv}, y_{vvv}$, and substituting for y_{uu}, y_{uv} their values from (16), we obtain

$$\left.\begin{aligned}
y_{uuu} = \alpha^{(1)} y_{vv} + \beta^{(1)} y_u + \gamma^{(1)} y_v + \delta^{(1)} y, \\
y_{uuv} = \alpha^{(2)} y_{vv} + \beta^{(2)} y_u + \gamma^{(2)} y_v + \delta^{(2)} y, \\
y_{uvv} = \alpha^{(3)} y_{vv} + \beta^{(3)} y_u + \gamma^{(3)} y_v + \delta^{(3)} y, \\
y_{vvv} = \alpha^{(4)} y_{vv} + \beta^{(4)} y_u + \gamma^{(4)} y_v + \delta^{(4)} y,
\end{aligned}\right\} \tag{20}$$

where

$$\left.\begin{aligned}
&\alpha^{(1)} = a (b + c') + a_u, \quad \beta^{(1)} = b_u + a b'_v + d + b^2 + b' (c + a b'), \\
&\qquad \gamma^{(1)} = c_u + a c'_v + a d' + b c + c' (c + a b'), \\
&\qquad \delta^{(1)} = d_u + a d'_v + b d + d' (c + a b'), \\
&\alpha^{(2)} = a b', \quad \beta^{(2)} = b' (b + c') + b'_u + d', \quad \gamma^{(2)} = b' c + c'^2 + c'_u, \\
&\qquad \delta^{(2)} = b' d + c' d' + d'_u, \\
&\alpha^{(3)} = c', \quad \beta^{(3)} = b'^2 + b'_v, \quad \gamma^{(3)} = b' c' + c'_v + d', \quad \delta^{(3)} = b' d' + d'_v, \\
&\alpha^{(4)} = \frac{1}{a} (a b' - c - a_v), \quad \beta^{(4)} = \frac{1}{a} (b' c' + b'_u - b_v + d'), \\
&\qquad \gamma^{(4)} = \frac{1}{a} [b' c + c' (c' - b) + c'_u - c_v - d], \\
&\qquad \delta^{(4)} = \frac{1}{a} [b' d + d' (c' - b) + d'_u - d'_v].
\end{aligned}\right\} \tag{21}$$

In § 1, we saw that there could be just one integrability condition of the form

$$\alpha y_{vv} + \beta y_u + \gamma y_v + \delta y = 0, \qquad (2\ bis)$$

which gives four equations, $\alpha = \beta = \gamma = \delta = 0$. We may obtain these equations as follows. We must have

$$\frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \quad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \quad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}. \qquad (22)$$

The second of these is satisfied identically by equations (20). Each of the other two gives a condition of the form (2), but the two sets of equations $\alpha = \beta = \gamma = \delta = 0$ found from them can not be independent, as may be verified by actual calculation. We need therefore use only one of equations (22), say the last. We obtain

$$\frac{\partial y_{uvv}}{\partial v} = (\alpha^{(3)}\alpha^{(4)} + \gamma^{(3)} + \alpha^{(3)}_v)\, y_{vv} + (\alpha^{(3)}\beta^{(4)} + b'\beta^{(3)} + \beta^{(3)}_v)\, y_u$$
$$+ (\alpha^{(3)}\gamma^{(4)} + c'\beta^{(3)} + \gamma^{(3)}_v + \delta^{(3)})\, y_v + (\alpha^{(3)}\delta^{(4)} + d'\beta^{(3)} + \delta^{(3)}_v)\, y,$$
$$\frac{\partial y_{vvv}}{\partial u} = (\alpha^{(4)}\alpha^{(3)} + a\beta^{(4)} + \alpha^{(4)}_u)\, y_{vv} + (\alpha^{(4)}\beta^{(3)} + b\beta^{(4)} + b'\gamma^{(4)} + \beta^{(4)}_u + \delta^{(4)})\, y_u$$
$$+ (\alpha^{(4)}\gamma^{(3)} + c\beta^{(4)} + c'\gamma^{(4)} + \gamma^{(4)}_u)\, y_v + (\alpha^{(4)}\delta^{(3)} + d\beta^{(4)} + d'\gamma^{(4)} + \delta^{(4)}_u)\, y.$$

The two expressions on the right being identically equal, we find as the conditions of complete integrability:

$$\left.\begin{aligned}
\gamma^{(3)} + \alpha^{(3)}_v &= a\beta^{(4)} + \alpha^{(4)}_u,\\
(\alpha^{(3)} - b)\,\beta^{(4)} + b'\beta^{(3)} + \beta^{(3)}_v &= \alpha^{(4)}\beta^{(3)} + b'\gamma^{(4)} + \beta^{(4)}_u + \delta^{(4)},\\
c'\beta^{(3)} + \gamma^{(3)}_v + \delta^{(3)} &= \alpha^{(4)}\gamma^{(3)} + c\beta^{(4)} + \gamma^{(4)}_u,\\
\alpha^{(3)}\delta^{(4)} + d'\beta^{(3)} + \delta^{(3)}_v &= \alpha^{(4)}\delta^{(3)} + d\beta^{(4)} + d'\gamma^{(4)} + \delta^{(4)}_u.
\end{aligned}\right\} \qquad (23)$$

Use may be made of (21) to express these in terms of a, b, c, etc.; in fact, in the third of equations (23) we have already used the relation $\alpha^{(3)} = c'$. The first of equations (23) is found without difficulty to reduce to the equation

$$b_v + 2c'_v = \alpha^{(4)}_u + b'_u,$$

i. e., to

$$\frac{\partial}{\partial v}(b + 2c') = \frac{\partial}{\partial u}\left(\frac{2ab' - c - a_v}{a}\right).$$

We may therefore find by a quadrature a function p, given by

$$p_u = b + 2c', \quad p_v = \frac{2ab' - c - a_v}{a}. \qquad (24)$$

§ 4. *Canonical Form of the System. The Seminvariants and Semicovariants.*

Let us carry out the transformation (8),

$$y = \lambda\bar{y},$$

on the system in its intermediate form. We have

$$\left.\begin{aligned} y_u &= \lambda\bar{y}_u+\lambda_u\bar{y}, \quad y_v = \lambda\bar{y}_v+\lambda_v\bar{y}, \\ y_{uu} &= \lambda\bar{y}_{uu}+2\lambda_u\bar{y}_u+\lambda_{uu}\bar{y}, \\ y_{uv} &= \lambda\bar{y}_{uv}+\lambda_v\bar{y}_u+\lambda_u\bar{y}_v+\lambda_{uv}\bar{y}, \\ y_{vv} &= \lambda\bar{y}_{vv}+2\lambda_v\bar{y}_v+\lambda_{vv}\bar{y}. \end{aligned}\right\} \tag{25}$$

Substituting in (16), we obtain a system of the same form in $\bar{y}$. Denoting its coefficients by $\bar{a}$, $\bar{b}$, etc., we find

$$\left.\begin{aligned} &\bar{a}=a, \quad \bar{b}=b-\frac{2\lambda_u}{\lambda}, \quad \bar{c}=c+\frac{2a\lambda_v}{\lambda}, \\ &\bar{d}=d+b\frac{\lambda_u}{\lambda}+c\frac{\lambda_v}{\lambda}-\frac{\lambda_{uu}}{\lambda}+a\frac{\lambda_{vv}}{\lambda}, \\ &\bar{a}'=0, \quad \bar{b}'=b'-\frac{\lambda_v}{\lambda}, \quad \bar{c}'=c'-\frac{\lambda_u}{\lambda}, \\ &\bar{d}'=d'+b'\frac{\lambda_u}{\lambda}+c'\frac{\lambda_v}{\lambda}-\frac{\lambda_{uv}}{\lambda}. \end{aligned}\right\} \tag{26}$$

The quantities p_u and p_v given by (24) are for the new system

$$\left.\begin{aligned} \bar{p}_u &= \bar{b}+2\bar{c}' = b+2c'-4\frac{\lambda_u}{\lambda}, \\ \bar{p}_v &= \frac{2\bar{a}\bar{b}'-\bar{c}-\bar{a}_v}{\bar{a}} = \frac{2ab'-c-a_v}{a}-4\frac{\lambda_v}{\lambda}. \end{aligned}\right\} \tag{27}$$

We may therefore make them vanish by choosing λ to satisfy the equations

$$p_u-4\frac{\lambda_u}{\lambda}=0, \quad p_v-4\frac{\lambda_v}{\lambda}=0,$$

i. e.,

$$\lambda = e^{p/4}. \tag{28}$$

Putting this value of λ into equations (26), we obtain a new set of coefficients, which we denote by capital letters:

$$\left.\begin{aligned} &A=a, \quad B=b-\frac{1}{2}p_u, \quad C=c+\frac{a}{2}p_v, \\ &D=d+\frac{1}{4}bp_u+\frac{1}{4}cp_v-\frac{1}{4}p_{uu}+\frac{1}{4}ap_{vv}-\frac{1}{16}p_u^2+\frac{1}{16}ap_v^2, \\ &B'=b'-\frac{1}{4}p_v, \quad C'=c'-\frac{1}{4}p_u, \\ &D'=d'+\frac{1}{4}b'p_u+\frac{1}{4}c'p_v-\frac{1}{4}p_{uv}-\frac{1}{16}p_up_v. \end{aligned}\right\} \tag{29}$$

The system of differential equations, of which these are the coefficients, will be called the *canonical form* of system (4). The canonical form is unique for any given system of the form (4), being completely characterized by the conditions

$$B+2C'=0, \quad 2AB'-C-A_v=0. \tag{30}$$

It is easily verified that, except for a factor, each of the coefficients (29) of

the canonical form remains unchanged under the transformation (8). We call a *seminvariant* any function of the coefficients of (16) and their derivatives which remains thus unchanged under the transformation (8). If such a function contain also y and its derivatives, we call it a *semi-covariant*. The seven quantities (29) are, then, seminvariants, as are also their derivatives for any order. But of the seven, at most five are independent in virtue of equations (30). We may reject two from our system, say B and C; we call the remaining five, viz., A, D, B', C', D', the *fundamental seminvariants*. The reason for the name is that *any seminvariant is a function of the five fundamental seminvariants and their derivatives*. In fact, a seminvariant must be unchanged by any transformation of the form (8), in particular by the transformation which puts the system of differential equations into the canonical form, and hence transforms the seminvariant into a function of the coefficients of the canonical form.

We now calculate the semi-covariants. We need seek only those containing y, y_u, y_v, y_{vv}, since all the derivatives of y are expressible in terms of these quantities. It is readily verified that

$$y, \quad \rho = y_u - c'y, \quad \sigma = y_v - b'y, \quad \tau = y_{vv} - 2b'y_v + (b'^2 - b'_v)y = \sigma_v - b'\sigma \tag{31}$$

satisfy the equations

$$\lambda\bar{y} = y, \quad \lambda\bar{\rho} = \rho, \quad \lambda\bar{\sigma} = \sigma, \quad \lambda\bar{\tau} = \tau, \tag{32}$$

and are therefore *relative* semi-covariants.

§ 5. *The Invariants.*

We found in § 2 that the most general transformation of the independent variables which leaves system (16) in the intermediate form, is of the form

$$\bar{u} = \phi(u), \quad \bar{v} = \psi(v), \tag{17}$$

where ϕ and ψ are any functions of their arguments. Let us apply this transformation to the system (16). We have

$$y_u = \bar{y}_u \phi_u, \quad y_v = \bar{y}_v \psi_v, \quad y_{uu} = \bar{y}_{uu}\phi_u^2 + \bar{y}_u \phi_{uu},$$
$$y_{uv} = \bar{y}_{uv}\phi_u \psi_v, \quad y_{vv} = \bar{y}_{vv}\psi_v^2 + \bar{y}_v \psi_{vv},$$

where $\bar{y}_u \equiv \partial y / \partial \bar{u}$, etc. Substituting in (16), we obtain for the coefficients of the new system

$$\left.\begin{aligned} \bar{a} = \frac{\psi_v^2}{\phi_u^2} a, \quad \bar{b} = \frac{1}{\phi_u}(b - \xi), \quad \bar{c} = \frac{\psi_v}{\phi_u^2}(c + a\eta), \quad \bar{d} = \frac{1}{\phi_u^2} d, \\ \bar{b}' = \frac{1}{\psi_v} b', \quad \bar{c}' = \frac{1}{\phi_u} c', \quad \bar{d}' = \frac{1}{\phi_u \psi_v} d', \end{aligned}\right\} \tag{33}$$

where

$$\xi = \phi_{uu}/\phi_u, \quad \eta = \psi_{vv}/\psi_v.$$

We have also

$$\left.\begin{aligned}\bar{p}_u=\frac{1}{\phi_u}(p_u-\xi),\quad \bar{p}_v=\frac{1}{\psi_v}(p_v-3\eta),\\ \bar{p}_{uu}=\frac{1}{\phi_u^2}(p_{uu}-\xi p_u+\xi^2-\xi_u),\quad \bar{p}_{uv}=\frac{1}{\phi_u\psi_v}p_{uv},\\ \bar{p}_{vv}=\frac{1}{\psi_v^2}(p_{vv}-\eta p_v+3\eta^2-3\eta_v).\end{aligned}\right\}\tag{34}$$

Using (33) and (34), we find that the fundamental seminvariants A, D, B', C', D' are transformed into

$$\left.\begin{aligned}\bar{A}=\frac{\psi_v^2}{\phi_u^2}A,\quad \bar{B}'=\frac{1}{\psi_v}(B'+\frac{3}{4}\eta),\quad \bar{C}'=\frac{1}{\phi_u}(C'+\frac{1}{4}\xi),\\ \bar{D}'=\frac{1}{\phi_u\psi_v}(D'-\frac{1}{4}B'\xi-\frac{3}{4}C'\eta-\frac{3}{16}\xi\eta),\\ \bar{D}=\frac{1}{\phi_u^2}(D-\frac{1}{4}B\xi-\frac{3}{4}C\eta-\frac{1}{16}\xi^2+\frac{9}{16}A\eta^2+\frac{1}{4}\xi_u-\frac{3}{4}A\eta_v).\end{aligned}\right\}\tag{35}$$

For the sake of symmetry, we have retained in the expression for $\bar{D}$ the quantities B and C, which may be expressed in terms of the fundamental seminvariants by means of (30). It will be convenient to have the transformed expressions for these quantities; they are

$$\bar{B}=\frac{1}{\phi_u}(B-\frac{1}{2}\xi),\quad \bar{C}=\frac{\psi_v}{\phi_u^2}(C-\frac{1}{2}A\eta).\tag{36}$$

We call *invariants* and *covariants* those seminvariants and semi-covariants which remain unchanged by a transformation of the form (17). By noting that

$$\left.\begin{aligned}\bar{A}_u=\frac{\psi_v^2}{\phi_u^3}(A_u-2A\xi),\quad \bar{A}_v=\frac{\psi_v}{\phi_u^2}(A_v+2A\eta),\\ \bar{B}'_v=\frac{1}{\psi_v^2}(B'_v-B'\eta-\frac{3}{4}\eta^2+\frac{3}{4}\eta_v),\quad \bar{C}'_u=\frac{1}{\phi_u^2}(C'_u-C'\xi-\frac{1}{4}\xi^2+\frac{1}{4}\xi_u),\end{aligned}\right\}\tag{37}$$

we may verify the invariance of

$$\left.\begin{aligned}\mathfrak{A}=A,\quad \mathfrak{B}'=B'-\frac{3}{8}\frac{A_v}{A},\quad \mathfrak{C}'=C'+\frac{1}{8}\frac{A_u}{A},\\ \mathfrak{D}'=D'+B'C',\quad \mathfrak{D}=D-(B'A_v-AB'_v)-C'_u+3(AB'^2-C'^2).\end{aligned}\right\}\tag{38}$$

These are *relative* invariants, satisfying the equations

$$\bar{\mathfrak{A}}=\frac{\psi_v^2}{\phi_u^2}\mathfrak{A},\quad \bar{\mathfrak{B}}'=\frac{1}{\psi_v}\mathfrak{B}',\quad \bar{\mathfrak{C}}'=\frac{1}{\phi_u}\mathfrak{C}',\quad \bar{\mathfrak{D}}'=\frac{1}{\phi_u\psi_v}\mathfrak{D}',\quad \bar{\mathfrak{D}}=\frac{1}{\phi_u^2}\mathfrak{D}.\tag{39}$$

By means of (38), we may express the fundamental seminvariants A, D, B', C', D' entirely in terms of the invariants $\mathfrak{A}$, $\mathfrak{D}$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}'$ and their first derivatives. It follows that *every invariant is a function of the invariants* (38) *and their derivatives*, since an invariant is a function of the fundamental seminvariants and their derivatives.

If we suppose that $\mathfrak{B}'$ and $\mathfrak{C}'$ are not zero, we may form from the five relative invariants (38) the three *absolute* invariants $\mathfrak{A}\mathfrak{B}'^2/\mathfrak{C}'^2$, $\mathfrak{D}'/\mathfrak{B}'\mathfrak{C}'$, $\mathfrak{D}/\mathfrak{C}'^2$. Moreover, it is not difficult to see that any relative invariant $I^{(l,m)}$ must be transformed by (17) into an expression of the form

$$\bar{I}^{(l,m)} = \phi_u^l \psi_v^m I^{(l,m)};$$

$\mathfrak{B}'$ and $\mathfrak{C}'$ may then be used to form from $I^{(l,m)}$ an absolute invariant $\mathfrak{B}'^m \mathfrak{C}'^l I^{(l,m)}$, or again to form two invariants which satisfy the relations

$$\bar{\mathfrak{B}}'^m \bar{I}^{(l,m)} = \phi_u^l \mathfrak{B}'^m I^{(l,m)}, \quad \bar{\mathfrak{C}}'^l \bar{I}^{(l,m)} = \psi_v^m \mathfrak{C}' \bar{I}^{(l,m)}.$$

We call an invariant a ϕ-invariant, or a ψ-invariant, if its transform is multiplied by a power of ϕ_u alone, or by a power of ψ_u alone. Thus, from (39) we have the

$$\phi\text{-invariants } \mathfrak{C}', \mathfrak{D}, \mathfrak{B} = \mathfrak{A}\mathfrak{B}'^2, \quad \psi\text{-invariants } \mathfrak{B}', \mathfrak{C} = \mathfrak{D}'/\mathfrak{C}'. \tag{40}$$

Consider the operators

$$U = \frac{1}{\mathfrak{C}'}\frac{\partial}{\partial u}, \quad V = \frac{1}{\mathfrak{B}'}\frac{\partial}{\partial v}.$$

Let Φ be a ϕ-invariant, Ψ a ψ-invariant. Then $U(\Psi)$ is a new ψ-invariant, and $V(\Phi)$ a new ϕ-invariant.

Again, from two ϕ-invariants $\Phi^{(l)}$ and $\Phi^{(m)}$, for which $\bar{\Phi}^{(l)} = \phi_u^l \Phi^{(l)}$, $\bar{\Phi}^{(m)} = \phi_u^m \Phi^{(m)}$, we may form a new ϕ-invariant, its Wronskian with respect to u:

$$(\Phi^{(m)}, \Phi_u^{(l)}) \equiv l\,\Phi^{(l)}\Phi_u^{(m)} - m\,\Phi^{(m)}\Phi_u^{(l)}.$$

Starting with the invariants (40)—the three ϕ-invariants $\mathfrak{C}', \mathfrak{D}, \mathfrak{B}$, and the two ψ-invariants $\mathfrak{B}', \mathfrak{C}$—we may construct an infinite number of ϕ- and ψ-invariants by combining the Wronskian process with the U- and V-processes. It may be shown, by an argument parallel to that employed by Wilczynski in his first memoir on curved surfaces,* that *by means of the Wronskian process and the U- and V-processes all invariants may be deduced from the five fundamental invariants $\mathfrak{C}', \mathfrak{D}, \mathfrak{B}, \mathfrak{B}', \mathfrak{C}$. Of the complete system, there are $(n+1)(5n+6)/2$ absolute, or two more relative, invariants which contain derivatives of the fundamental invariants of orders up to and including n, and which are independent of each other and the five fundamental invariants.*

We must, however, note a restriction on the independence of the complete system of invariants. In the presence of the integrability conditions (23), they are redundant. In fact, equations (23) must be invariant under the transformations (8) and (17), and therefore expressible in terms of invariants. The first of (23) has already been used to give the relations (30); it may be verified without difficulty that the other three integrability conditions give three relations connecting the five fundamental invariants (40) and their first and second derivatives.

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), pp. 250–255.

If we have given the fundamental invariants (40) (or what is the same thing, the invariants (38)), satisfying the integrability conditions (23), we may always find the five fundamental seminvariants and therefore the seven coefficients of the canonical form of the system of differential equations. A transformation of the form

$$\bar{u} = \phi\,(u), \qquad \bar{v} = \psi\,(v) \tag{17}$$

will put the system into the intermediate form; and more generally still, a transformation

$$\bar{u} = U\,(u, v), \qquad \bar{v} = V\,(v) \tag{9}$$

will throw the system into the most general form (4). *Thus the system of curves C_u is completely determined, except for a projective transformation, by the fundamental invariants* (40), *or at any rate by the invariants* (38).

Geometrically it is evident that the invariants of the conjugate net are invariants of the family of curves C_u. It may be expected, therefore, that the invariants, as we have calculated them for the intermediate form of the system of differential equations, are in some way or other expressible in terms of the coefficients of the original system (4). It turns out, in fact, that to obtain these expressions it is not necessary to integrate the differential equation (14), although such integration is necessary in obtaining the coefficients of the intermediate form. Equations (19) give the coefficients of the intermediate form in terms of the coefficients of system (4), and in these expressions appear the quantities U_u and U_{uu}, the values of which may be found only on integrating (14). But if the values for the quantities $\bar{a}$, $\bar{b}$, etc., be substituted in (38), and if use be made of equations (14) and (18) to express all derivatives of U in terms of U_u and U_{uu}, it will be found that U_{uu} disappears, and that the invariants have the values

$$\mathfrak{A} = \frac{1}{U_u^2}\mathsf{A}, \quad \mathfrak{B}' = \mathsf{B}', \quad \mathfrak{C}' = \frac{1}{U_u}\mathsf{C}', \quad \mathfrak{D}' = \frac{1}{U_u}\mathsf{D}', \quad \mathfrak{D} = \frac{1}{U_u^2}\mathsf{D}, \tag{41}$$

where $\mathsf{A}, \mathsf{B}', \mathsf{C}', \mathsf{D}', \mathsf{D}$ are expressions in a, b, etc. (the coefficients of system (4)) and their derivatives, which expressions take the form of the quantities $\mathfrak{A}, \mathfrak{B}', \mathfrak{C}', \mathfrak{D}', \mathfrak{D}$ if we put $a' = 0$. We get therefore for the ϕ- and ψ-invariants (40):

$$\phi\text{-invariants,} \quad \mathfrak{C}' = \frac{1}{U_u}\mathsf{C}', \quad \mathfrak{D} = \frac{1}{U_u^2}\mathsf{D}, \quad \mathfrak{B} = \frac{1}{U_u^2}\mathsf{B},$$

$$\psi\text{-invariants,} \quad \mathfrak{B}' = \mathsf{B}', \quad \mathfrak{C} = \mathsf{C},$$

where $\mathsf{B} = \mathsf{A}/\mathsf{B}'^2$, $\mathsf{C} = \mathsf{D}'/\mathsf{C}'$. The fundamental invariants are therefore expressible explicitly in terms of the coefficients of system (4) and their derivatives. Only for the the three ϕ-invariants is there an extraneous factor, some power of U_u.

We may prove by induction that the invariants derived from the fundamental invariants by means of the U-, V- and Wronskian processes are similarly expressible in terms of the coefficients of (4). For the transformation

$$\bar{u} = U(u, v), \quad \bar{v} = v \tag{10}$$

we have (using (14))

$$\frac{\partial}{\partial \bar{u}} = \frac{1}{U_u}\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \bar{v}} = \frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u},$$

where $\bar{u}$ and $\bar{v}$ are the variables in the intermediate form, and u, v the variables in (4). Consequently, the operators U and V become

$$\mathbf{U} = \frac{1}{\mathbf{C}'}\frac{\partial}{\partial u}, \quad \mathbf{V} = \frac{1}{\mathbf{B}'}\left(\frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u}\right).$$

Assume that every ϕ-invariant $\mathfrak{M}$ up to a certain order is such that on transformation by (19) it becomes

$$\mathfrak{M} = U_u^l\,\mathbf{M}.$$

We then have

$$V(\mathfrak{M}) = \mathbf{V}(\mathbf{M}) = \frac{l\,U_u^{l-1}}{a\mathbf{B}'}\mathbf{M}(aU_{uv} - a'U_{uu}) + \frac{U_u^l}{\mathbf{B}'}\left(\frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u}\right)\mathbf{M}.$$

But by (18) we have

$$aU_{uv} - a'U_{uu} = a'_u U_u - a_u U_v = \frac{U_u}{a}(a a'_u - a' a_u),$$

so that

$$\mathbf{V}(\mathbf{M}) = \frac{U_u^l}{\mathbf{B}'}\left[\frac{l(a a'_u - a' a_u)}{a^2}\mathbf{M} + \left(\frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u}\right)\mathbf{M}\right].$$

The new invariant is therefore expressible in the required way, and is multiplied again by the factor U_u^l.

Assume further that up to a certain order every ψ-invariant $\mathfrak{N}$ is transformed by (19) in such a way that

$$\mathfrak{N} = \mathbf{N},$$

there being no extraneous factor, and $\mathbf{N}$ being the expression in the coefficients of (4). Then

$$U(\mathfrak{N}) = \mathbf{U}(\mathbf{N}) = \frac{1}{\mathbf{C}'}\mathbf{N}_u,$$

and is also expressible in the required way.

Now, the hypotheses as to the nature of the transforms of $\mathfrak{M}$ and $\mathfrak{N}$ are satisfied by the transforms (41) of the five fundamental invariants. The induction therefore follows, as far as the U- and V-processes are concerned. The proof as regards the Wronskian process is essentially the same. We may therefore state the important theorem:

All the invariants of the conjugate net of curves C_u and C_v are expressible explicitly in terms of the coefficients of system (4), *the integration of the differential equation* (14) *not being necessary even though such integration is required to put system* (4) *into the intermediate form.*

§ 6. *The Covariants and their Geometric Interpretation. Laplace Transformations.*

The semi-covariants (31) are transformed by (17) into

$$\bar{y} = y, \quad \bar{\rho} = \frac{1}{\phi_u}\rho, \quad \bar{\sigma} = \frac{1}{\psi_v}\sigma, \quad \bar{\tau} = \frac{1}{\psi_v^2}\ \tau - \eta\sigma),$$

where $\eta = \psi_{vv}/\psi_v$. The quantities y, ρ, σ are therefore covariants; in giving their geometric interpretation we shall find a fourth covariant to replace the quantity τ, which is not itself a covariant.

Analytically, the covariants

$$\rho = y_u - c' y, \quad \sigma = y_v - b' y$$

are transformations introduced by Laplace* in his study of the differential equation

$$y_{uv} = b' y_u + c' y_v + d' y.$$

Darboux† has given a very elegant geometric interpretation of the Laplace transformations, which we now outline.

If the four coordinates

$$y^{(k)} = f^{(k)}(u, v), \qquad (k = 1, 2, 3, 4), \tag{1 bis}$$

of a point y (or P_y) be substituted successively in the expressions for ρ and σ, we obtain the coordinates of two points, P_ρ and P_σ. Thus, for given values of u and v we get a point y and two associated points ρ and σ. To the surface S_0 given by equations (1 *bis*) correspond therefore two surfaces S_{-1} and S_1 given by the covariants ρ and σ. For $v =$ const., we get a curve C_u on S_0, and corresponding curves on S_{-1} and S_1.

We suppose that $y^{(k)} = f^{(k)}(u, v)$ are a fundamental system of solutions of the system of differential equations in the intermediate form,‡ so that the curves

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* "Recherches sur le Calcul intégral aux différences partielles," *Œuvres de Laplace*, t. IX, pp. 29 *et seq.*

† "Théorie Générale des Surfaces," t. II, livre IV, chap. I, II.

‡ Darboux does not consider a system of two differential equations, but only a single equation of the Laplace type:

$$y_{uv} = b' y_u + c' y_v + d' y.$$

For the geometric interpretation of the covariants, this single equation is sufficient; but the results have a deeper significance if with it we associate another equation to form the system (16). In particular, a projective theory is not possible with a single differential equation; but more than this, the introduction of a second equation not only makes the theory projective, but enables us to study the single system of curves C_u apart from the conjugate net. The force of this last remark will appear in the sequel.

u=const. and v=const. form a conjugate net on the surface S_0. The tangents to the curves v=const. form a congruence Γ_{-1}, one sheet of whose focal surface is S_0. The other sheet will be found by determining the edges of regression of the developables generated by the lines of the congruence which meet fixed curves of the family u=const. conjugate to the family v=const. Any point on the tangent at a point P_y to the curve v=const. which passes through P_y will have coordinates

$$Y^{(k)} = y_u^{(k)} + \lambda y^{(k)}, \qquad (k = 1, 2, 3, 4).$$

Suppose λ to be any function of u, v, and let P_y trace a curve u=const. Then the point P_Y traces a curve, and a point on the tangent to this curve at P_Y is given by

$$\begin{aligned}\frac{\partial Y^{(k)}}{\partial v} &= y_{uv}^{(k)} + \lambda y_v^{(k)} + \lambda_v y^{(k)} \\ &= b' y_u^{(k)} + (\lambda + c')\, y_v^{(k)} + (\lambda_v + d')\, y^{(k)}.\end{aligned}$$

If $\lambda = -c'$, the point Y becomes the point ρ, and we have

$$\frac{\partial \rho}{\partial v} = b' y_u + (d' - c'_v)\, y.$$

But then the point ρ_v lies on the line of the congruence Γ_{-1}, and the point ρ traces the required edge of regression. Therefore, *the covariant*

$$\rho = y_u - c' y$$

gives the second sheet S_{-1} of the congruence Γ_{-1} of tangents to the curves v=const. on the surface S_0.

Similarly, *the congruence Γ_1 of tangents to the curves u=const. on S_0 has as the second sheet S_1 of its focal surface the points given by the covariant*

$$\sigma = y_v - b' y.$$

We call ρ and σ respectively the *minus first* and *first Laplace transforms* of y, and correspondingly the surfaces S_{-1} and S_1 the *minus first* and *first Laplace transforms* of S_0. The first Laplace transform of S_{-1} is S_0, and its minus first Laplace transform is a surface S_{-2}, which is given by a covariant ρ_{-1}. We easily find the expression for ρ_{-1}. We have

$$\begin{aligned}\rho_v &= b' y_u + (d' - c'_v)\, y \\ &= b' \rho + (d' + b' c' - c'_v)\, y,\end{aligned}$$

The parenthesis is an invariant; we write

$$K = d' + b' c' - c'_v = D' + B' C' - C'_v, \tag{42}$$

and get

$$\left.\begin{aligned}\rho_v &= b' \rho + K y, \\ \rho_{uv} &= b' \rho_u + b'_u \rho + K y_u + K_u y,\end{aligned}\right\} \tag{43}$$

or, using (43),

$$\rho_{uv} = b'\rho_u + \left(c' + \frac{K_u}{K}\right)\rho_v + \left[b'_u + K - b'\left(c' + \frac{K_u}{K}\right)\right]\rho. \tag{44}$$

Consequently, the minus first Laplace transform of ρ is

$$\rho_{-1} = \rho_u - \left(c' + \frac{K_u}{K}\right)\rho. \tag{45}$$

It is of course a covariant of the original conjugate net on S_0, as may be verified by calculation.

The first Laplace transform of ρ is $\rho_v - b'\rho$, which by (43) is simply Ky, so that as stated above the first Laplace transform of S_{-1} is S_0.

Similarly, the minus first Laplace transform of S_1 is S_0, and its first Laplace transform is a new surface S_2 given by the covariant

$$\sigma_1 = \sigma_v - \left(b' + \frac{H_v}{H}\right)\sigma, \tag{46}$$

where

$$H = d' + b'c' - b'_u = D' + B'C' - B'_u \tag{47}$$

is an invariant.

Equation (44), and the analogous equation for σ,

$$\sigma_{uv} = \left(b' + \frac{H_v}{H}\right)\sigma_u + c'\sigma_v + \left[c'_v + H - c'\left(b' + \frac{H_v}{H}\right)\right]\sigma, \tag{48}$$

show that the curves $u=$const. and $v=$const. are conjugate on the surfaces S_{-2} and S_2. By repeating the Laplace transformations in the two directions, we obtain an infinite sequence of surfaces $\ldots, S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2, \ldots$; and on each the families of curves $u=$const. and $v=$const. will be conjugate. For the surface S_k, there will be a pair of congruences $\Gamma_{k+1}, \Gamma_{k-1}$, formed respectively by the tangents to the curves $u=$const., $v=$const.

The above is Darboux's very elegant geometric interpretation of the Laplace transformations of the partial differential equation

$$y_{uv} = b'y_u + c'y_v + d'y.$$

But for our projective theory we have added another partial differential equation of the second order to form the completely integrable system (16). We recall that system (16) is the intermediate form of the system of differential equations (4), and requires for its determination the integration of a partial differential equation of the first order, viz., equation (14). This, as we know, is equivalent to the determination of the family of curves conjugate to the family of curves C_u, or $v=$const. This is necessary for the application of Darboux's theory, since the surface is there referred to a conjugate net of parameter curves. However, it is evident geometrically that the Laplace

transforms of the surface S_0 are determined by the single system of curves $v=$const. *By the association of a second differential equation with Darboux's single equation, the Laplace transforms of a surface defined by a one-parameter family of curves may be found without the determination of the conjugate family of curves.*

The proof is very simple. Let us return to the notation of § 2, at the end of which we denoted the coefficients of the intermediate form by barred letters. In this notation the covariants ρ and σ are

$$\rho = \bar{y}_u - \bar{c}'\,\bar{y}, \quad \sigma = \bar{y}_v - \bar{b}'\,\bar{y}.$$

But we may express these quantities in terms of the variables and coefficients of system (4), which we denote by unbarred letters. From (11), we have

$$\bar{y}_u = \frac{1}{U_u}\,y_u, \quad \bar{y}_v = y_v - \frac{a'}{a}\,y_u,$$

in the latter of which we have made use of (14). From (19) we may take the values for $\bar{b}'$ and $\bar{c}'$, and thus obtain

$$\rho = \frac{1}{U_u}\left[y_u - \frac{a\,c' - a'\,c}{a}\,y\right], \tag{49}$$

$$\sigma = y_v - \frac{a'}{a}\,y_u - \frac{a(a b' - a' b) + a'(a c' - a' c) + a' a_u - a a'_u}{a^2}\,y. \tag{50}$$

The covariants ρ and σ are therefore expressed in terms of the variables and coefficients of system (4): σ entirely so, and ρ except for the factor U_u which can not be determined without the integration of equation (14). But geometrically only the ratios of the homogeneous coordinates of a point are significant; hence the first and minus first Laplace transforms of S_0 are indeed determined without the integration of any differential equation. We may say, in the language of Lie, that *the Laplace transforms of a one-parameter family of space curves given by equations* (1) *may be found from these equations by "performable operations" alone*; and not only are these operations "performable" in the very general sense of Lie, but also in the sense that *all of the algebraic processes required to obtain the explicit expressions may actually be carried out in practice.*

We may now set up our complete system of covariants. Since all derivatives of y are expressible in terms of y, y_u, y_v, y_{vv}, we need seek only covariants which contain these four quantities. We have already found these covariants; they are:

$$\left.\begin{array}{c} y, \quad \rho = y_u - c'\,y, \quad \sigma = y_v - b'\,y, \\ \sigma_1 = \sigma_v - \left(b' + \dfrac{H_v}{H}\right)\sigma = y_{vv} - \left(2\,b' + \dfrac{H_v}{H}\right)y_v + b'\left(b' + \dfrac{H_v}{H} - \dfrac{b'_v}{b'}\right)y. \end{array}\right\} \tag{51}$$

We may take these as our fundamental system of covariants. *Every covariant is a function of these fundamental covariants and of invariants.*

It is easily verified that σ_1 is, like ρ and σ, expressible explicitly in terms of the coefficients and variables of system (4). We recall that the complete system of invariants are also expressible in this way. It is evident geometrically that every invariant and covariant of a one-parameter family of curves is an invariant or covariant of the conjugate net of which this family is a part, and conversely. Also, analytically, every invariant and covariant of system (4) under the transformations of the group

$$y = \lambda(u, v)\,\bar{y}, \quad \bar{u} = U(u, v), \quad \bar{v} = V(v)$$

is certainly an invariant or covariant of system (16) under the transformations of the smaller group

$$y = \lambda(u, v)\,\bar{y}, \quad \bar{u} = U(u), \quad \bar{v} = V(v).$$

From these considerations we may conclude that *in the projective study of a one-parameter family of space curves there is no loss of generality, from an analytic point of view,* in considering instead the conjugate net of which the given system of curves is a component family.*

Since the quantities ρ and σ define two surfaces referred to conjugate nets as parameter curves, each of them must satisfy a system of differential equations of form (16). Equations (44) and (48) are respectively the second equations of the systems for ρ and σ. We shall merely indicate how the first equation of each system is found. We have

$$\left.\begin{aligned} \rho = y_u - c'y, \quad \rho_u &= ay_{vv} + (b-c')y_u + cy_v + (d-c'_u)y, \\ \rho_v = b'y_u + (d'-c'_v)y, \quad \rho_{vv} &= (b'^2+b'_v)y_u + Ky_v + (d'_v+b'd'-c'_{vv})y, \end{aligned}\right\} \tag{52}$$

and an expression for ρ_{uu} linear in y_{vv}, y_u, y_v, y. From this last and equations (52) we may eliminate the quantities y_{vv}, y_u, y_v, y, and obtain the required equation for ρ. This elimination will be impossible, however, if and only if the four equations (52) are linearly dependent in y_{vv}, y_u, y_v, y, in which case there will be a relation of the form

$$\alpha\rho_{vv} + \beta\rho_u + \gamma\rho_v + \delta\rho = 0,$$

excluded in § 1. The condition for this is the vanishing of the determinant

$$\begin{vmatrix} 0 & 1 & 0 & -c' \\ a & b-c' & c & d-c'_u \\ 0 & b' & 0 & d'-c'_v \\ 0 & b'^2+b'_v & K & d'_v+b'd'-c'_{vv} \end{vmatrix} = aK^2.$$

Since we have supposed $a \neq 0$, we need consider only the vanishing of K. We shall soon show that *if $K=0$, the minus first Laplace transform of S_0 is degenerate.*

* Nor, obviously, from a geometric point of view.

For σ, we have

$$\left.\begin{array}{c}\sigma = y_v - b'y, \quad \sigma_u = c'y_v + (d'-b'_u)y, \quad \sigma_v = y_{vv} - b'y_v - b'_v y, \\ \sigma_{vv} = (\alpha^{(4)} - b')y_{vv} + \beta^{(4)}y_u + (\gamma^{(4)} - 2b'_v)y_v + (\delta^{(4)} - b'_{vv})y,\end{array}\right\} \tag{53}$$

in the last of which $\alpha^{(4)}$, $\beta^{(4)}$, $\gamma^{(4)}$, $\delta^{(4)}$ are the coefficients in the equation

$$y_{vvv} = \alpha^{(4)}y_{vv} + \beta^{(4)}y_u + \gamma^{(4)}y_v + \delta^{(4)}y, \tag{54}$$

and are given by (21). Between (53) and a similar expression for σ_{uu}, the quantities y_{vv}, y_u, y_v, y may in general be eliminated, and the result will be the required equation for σ. However, as in the above, the elimination is impossible if the determinant of the coefficients of y_{vv}, y_u, y_v, y in (53) vanishes. This determinant is

$$\begin{vmatrix} 0 & 0 & 1 & -b' \\ 0 & 0 & c' & d'-b'_u \\ 1 & 0 & -b' & -b'_v \\ \alpha^{(4)}-b' & \beta^{(4)} & \gamma^{(4)}-2b'_v & \delta^{(4)}-b'_{vv} \end{vmatrix} = \beta^{(4)}H.$$

Obviously $\beta^{(4)}$ is an invariant; in fact, its value in (24) is easily reduced to

$$\beta^{(4)} = \frac{1}{A}(H + 2B'_u + 2C'_v). \tag{55}$$

Hence, *if either $\beta^{(4)} = (H + 2B'_u + 2C'_v)/A$ or $H = d' + b'c' - b'_u$ vanishes, the first Laplace transform of the surface S_0 is developable or degenerate.* We shall see that if $H=0$, the first Laplace transform is not developable, but degenerate.

There is a certain dissymmetry in the statement of the last two theorems for ρ and σ, which may easily be removed. From (54), we see that if $\beta^{(4)}=0$, we have

$$y_{vvv} = \alpha^{(4)}y_{vv} + \gamma^{(4)}y_v + \delta^{(4)}y;$$

in other words, *the vanishing of the invariant $\beta^{(4)} = (H + 2B'_u + 2C'_v)/A$ is the condition that the curves u=const. be plane curves* (not of course lying in the same plane).

It is easy to see geometrically that if the curves u=const. be plane, the first Laplace transform is developable or degenerate. In fact, this transform is the developable (which may be degenerate) enveloped by the family of planes which cut out the curves u=const. on the surface S_0.

It should be noted that in Darboux's theory the case $\beta^{(4)}=0$ is not exceptional, since the single equation (48) can always be set up if H is not zero. For the minus first Laplace transform, ρ, it is always possible to find both the equation in ρ_{uu} and the equation in ρ_{uv}, provided only that K does not vanish. But the curves v=const. may be plane, and the surface S_{-1} developable or degenerate. Then we should expect the coefficient of ρ_{vv} in the ρ_{uu}-equation to be zero. This is easily seen to be the case. We have

$$y_{uu} = a y_{vv} + b y_u + c y_v + d y,$$
$$y_{uuu} = \alpha^{(1)} y_{vv} + \beta^{(1)} y_u + \gamma^{(1)} y_v + \delta^{(1)} y,$$

where $\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, \delta^{(1)}$ are given by (21). *The condition that the curves $v=const.$ be plane is therefore the vanishing of the determinant*

$$\begin{vmatrix} a & c \\ \alpha^{(1)} & \gamma^{(1)} \end{vmatrix} = A^2[H + 3\mathfrak{B}'_u + \mathfrak{C}'_v], \tag{56}$$

which is of course an invariant.

Now,

$$\begin{aligned} \rho_{uu} &= y_{uuu} - c'_{uu} y - 2c'_u y_u - c' y_{uu} \\ &= (\alpha^{(1)} - ac') y_{vv} + (\beta^{(1)} - 2c'_u - bc') y_u \\ &\quad + (\gamma^{(1)} - cc') y_v + (\delta^{(1)} - c'_{uu} - dc') y, \end{aligned}$$

and if the determinant $a\gamma^{(1)} - c\alpha^{(1)} = 0$, we may eliminate, from this equation for ρ_{uu} and the equations for ρ, ρ_u, ρ_v given by (52), the quantities y_{vv}, y_u, y_v, y. The result is an equation of the form

$$\rho_{uu} = \beta \rho_u + \gamma \rho_v + \delta \rho,$$

which associated with equation (44) can define only a developable or degenerate surface.

We may therefore state the completed theorem, the last part of which is still to be proved: *If the curves $u=const.$ are plane, the first Laplace transform of S_0 is developable or degenerate. If the curves $v=const.$ are plane, the minus first Laplace transform is developable or degenerate. The only other cases of a similar nature are those in which the invariant H (or the invariant K) vanishes, in which event the first (or minus first) Laplace transform is degenerate.*

To prove the last part of this theorem, it is only necessary to note that for $K=0$ we have $\rho_v = b'\rho$. This shows that only one point ρ corresponds to all the points of a curve $u=$const. on S_0, and that corresponding to the family of curves $u=$const. we have the points of a curve, say C_ρ. This is the degenerate surface S_1. In the same way we get a degenerate surface S_{-1} if $H=0$. We may therefore describe these degenerate cases as follows in terms of the original one-parameter family of curves. *If the invariant H vanishes, the curves $C_u(v=const.)$ are the curves along which the surface S_0 is touched by a family of cones enveloping the surface and having their vertices on a curve in space. If K vanishes, the curves C_u are conjugate to a family of curves of contact of cones enveloping the surface and having their vertices on a curve in space.*

The invariants H and K, and the similar quantities for the different Laplace transforms, are important analytically in the study of the differential equation

$$y_{uv}=b'y_u+c'y_v+d',$$

as was first pointed out by Laplace. In fact, if either H or K vanishes, the differential equation is soluble by quadratures, and the same is true for the i-th Laplace transform (i positive or negative) of the differential equation, if either of the corresponding invariants H_i, K_i vanishes. This theory is well known through Darboux's researches, and need not be gone into here. It should be noted, however, that in our theory we deal with two partial differential equations, so that a number of new analytical problems arise in this connection. For instance, if one of the equations may be solved by quadratures, the solutions must still be further restricted so as to satisfy the other differential equation.*

§ 7. *The Surface Referred to its Asymptotic Lines.*

In the first three of his five memoirs on curved surfaces,† Wilczynski studied non-developable surfaces referred to their asymptotic curves. We shall now show how the results of his investigations may be made available for the theory of a one-parameter family of space curves.

The equations in the intermediate form (16) have S_0 as an integral surface. This surface will be left unchanged by any transformation of the form

$$\bar{u}=\phi\,(u,v),\qquad \bar{v}=\psi\,(u,v). \tag{57}$$

This transformation refers S_0 to a different net of parameter curves, and we seek the particular transformation which converts the conjugate net on S_0 into the net of asymptotic curves.

Let us carry out the transformation (57) on system (16). We obtain a system of two equations of the second order, with derivatives of y with respect to $\bar{u}$ and $\bar{v}$. Denote these derivatives by $\bar{y}_u$, $\bar{y}_v$, etc., and solve for the derivatives $\bar{y}_{uu}$ and $\bar{y}_{vv}$ in terms of $\bar{y}_{uv}$, $\bar{y}_u$, $\bar{y}_v$, $\bar{y}$. The result is the two equations

$$\left.\begin{aligned}\bar{y}_{uu}&=\alpha\,\bar{y}_{uv}+\beta\,\bar{y}_u+\gamma\,\bar{y}_v+\delta\,\bar{y},\\ \bar{y}_{vv}&=\alpha'\,\bar{y}_{uv}+\beta'\,\bar{y}_u+\gamma'\,\bar{y}_v+\delta'\,\bar{y},\end{aligned}\right\} \tag{58}$$

where

$$\left.\begin{aligned}\Delta\alpha\ &=-(\phi_u\psi_v-\phi_v\psi_u)\,(\psi_u^2+a\psi_v^2),\\ \Delta\alpha'&=(\phi_u\psi_v-\phi_v\psi_u)\,(\phi_u^2+a\phi_v^2),\\ \Delta\gamma\ &=-\psi_v(\psi_u\psi_{uu}+a\psi_v\psi_{uv})+\psi_u(\psi_u\psi_{uv}+a\psi_v\psi_{vv})\\ &\quad+\psi_v[b\psi_u^2-c'(\psi_u^2-a\psi_v^2)]+\psi_u[c\psi_v^2-b'(\psi_u^2-a\psi_v^2)],\\ \Delta\beta'&=\phi_v(\phi_u\phi_{uu}+a\phi_v\phi_{uv})-\phi_u(\phi_u\phi_{uv}+a\phi_v\phi_{vv})\\ &\quad-\phi_v[b\phi_u^2-c'(\phi_u^2-a\phi_v^2)]-\phi_u[c\phi_v^2-b'(\phi_u^2-a\phi_v^2)],\end{aligned}\right\} \tag{59}$$

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* The same remark is made by Wilczynski in a similar connection, in his memoir "One-parameter Families and Nets of Plane Curves," *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 493.

† *Transactions of the American Mathematical Society*, Vols. VIII, IX, X (1907–09).

$$\Delta = (\phi_u \psi_v - \phi_v \psi_u)(\phi_u \psi_u + a\phi_v \psi_v), \tag{60}$$

and the remaining coefficients, whose values do not concern us, are fractions with the common denominator Δ.

We may make α and α' vanish, by taking for $\phi_u : \phi_v$ and $\psi_u : \psi_v$ the two roots of the quadratic

$$t^2 + a = 0,$$

or more definitely, by taking for ϕ and ψ solutions of the partial differential equations of the first order:

$$\left.\begin{aligned} \phi_u + \sqrt{-a}\,\phi_v &= 0, \\ \psi_u - \sqrt{-a}\,\psi_v &= 0. \end{aligned}\right\} \tag{61}$$

If this be done, we see from the resulting form of (58) that the curves $\bar{u}=$const. and $\bar{v}=$const. are asymptotic.*

By differentiation of $\phi_u^2 + a\phi_v^2 = 0$, we have

$$\phi_u \phi_{uu} + a\phi_v \phi_{uv} = -\frac{a_u}{2}\phi_v^2,$$

$$\phi_u \phi_{uv} + a\phi_v \phi_{vv} = -\frac{a_v}{2}\phi_v^2,$$

with analogous equations for ψ. Using these and (61), we find for the quantities (59) and (60):

$$\Delta = -4a\sqrt{-a}\,\phi_v^2 \psi_v^2,$$

which we now see is not zero, and

$$\left.\begin{aligned} \gamma &= -\frac{\psi_v}{\sqrt{-a}\,\phi_v^2}\left[c' - \frac{b+2c'}{4} + \frac{1}{8}\frac{a_u}{a} + \sqrt{-a}\left(b' - \frac{2ab' - c - a_v}{4a} - \frac{3}{8}\frac{a_v}{a}\right)\right], \\ \beta' &= \frac{\phi_v}{\sqrt{-a}\,\psi_v^2}\left[c' - \frac{b+2c'}{4} + \frac{1}{8}\frac{a_u}{a} - \sqrt{-a}\left(b' - \frac{2ab' - c - a_v}{4a} - \frac{3}{8}\frac{a_v}{a}\right)\right]. \end{aligned}\right\} \tag{62}$$

Thus, except for the factors outside the square brackets, both γ and β' are expressible explicitly in terms of the coefficients of (16). The remaining coefficients of (58) will also be thus expressible, except that from the expressions for β and γ' one of the second derivatives of ϕ and ψ will not be removable. However, it may be verified that *all the invariants and covariants of the surface referred to its asymptotic curves are expressible explicitly in terms of the coefficients and variables which appear in equations* (16). The quantities ϕ

* Darboux, "Théorie Générale des Surfaces," t. I, p. 145. The above transformation is essentially that made by Wilczynski in his first memoir on curved surfaces, *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 243.

and ψ will be found to occur only as first derivatives ϕ_v and ψ_v, but always as extraneous factors.*

The quantities γ and β' are invariants. If we discard the factors outside the brackets, and write $\bar{\gamma}$, $\bar{\beta}'$ for the brackets themselves, we find that

$$\bar{\gamma} = \mathfrak{C}' + \sqrt{-a}\,\mathfrak{B}', \quad \bar{\beta}' = \mathfrak{C}' - \sqrt{-a}\,\mathfrak{B}', \tag{63}$$

where $\mathfrak{B}'$ and $\mathfrak{C}'$ are two of the fundamental invariants (38).

We are now able to give the geometric significance of these invariants. If, in the equations

$$\left.\begin{aligned} y_{uu} &= \beta y_u + \gamma y_v + \delta y, \\ y_{vv} &= \beta' y_u + \gamma' y_v + \delta' y, \end{aligned}\right\} \tag{64}$$

the coefficient γ is zero, then the curves v=const. are straight lines, and S_0 is a ruled surface. Similarly, if $\beta'=0$, the curves u=const. are straight lines. If both γ and β' are zero, the surface is a quadric, since quadrics are the only ruled surfaces having two families of straight-line generators. We may say, then, that *the surface S_0 is ruled if and only if the invariant*

$$\bar{\beta}'\bar{\gamma} = a\mathfrak{B}'^2 + \mathfrak{C}'^2$$

is zero. If $\bar{\beta}'$ and $\bar{\gamma}$ are both zero, that is, if $\mathfrak{B}'$ and $\mathfrak{C}'$ are both zero, the surface is a quadric.

We shall not consider in this paper the many problems which here suggest themselves. In particular, it would be important to put into relation with the various congruences and ruled surfaces associated with the asymptotic net certain similar configurations connected with the conjugate net which we have been considering. The nature of the surface in the neighborhood of a point has been studied in detail by Wilczynski in the five memoirs on curved surfaces already quoted, and the bearing of these results on the conjugate net of curves has still to be investigated. We leave these and similar questions open for the present.

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* These remarks, and the similar ones at the end of § 5, are important, though they seem to have escaped notice until now. In many of the problems in projective differential geometry already studied by Wilczynski and others, it has been found convenient to reduce the differential equations to certain simplified forms, or, we should say, to suppose them to be so reduced. This reduction generally requires the integration of differential equations, which of course is generally impossible. Nevertheless, it will be found, as we have already seen, that *if the reduced form bears an intrinsic geometric relation to the original form*, then the invariants and covariants of the reduced form are expressible in terms of the coefficients of the original form, without the integration of any differential equations. This remark has in fact enabled the present writer to set up a complete system of invariants for the general theory of curved surfaces. Cf. G. M. Green, "On the Theory of Curved Surfaces, and Canonical Systems in Projective Differential Geometry," *Transactions of the American Mathematical Society*, Vol. XVI (1915), pp. 1–12.

§ 8. *The Differential Equations for the General Theory of Congruences of Straight Lines.*

In his prize memoir* of 1909, Wilczynski developed a projective theory of congruences of straight lines, basing it upon the consideration of a completely integrable system of partial differential equations. Evidently, the theory of a one-parameter family of curves is geometrically equivalent to the theory of the congruence of tangents to these curves, so that it will be of interest to connect the two theories analytically by setting up the completely integrable system for the theory of congruences, and relating it to the system of equations (16). Wilczynski, however, assumes the focal surface of the congruence to be known. Since in a later development we shall need the equations for the more general case, we shall set up the completely integrable system for a congruence neither sheet of whose focal surface is known.

Let the congruence consist of the ∞^2 lines joining corresponding points† on the surfaces S_y and S_z given by the equations

$$y^{(k)} = f^{(k)}(u, v), \quad z^{(k)} = g^{(k)}(u, v), \qquad (k = 1, 2, 3, 4). \tag{65}$$

These surfaces may evidently always be chosen so that the determinant

$$\Delta = \begin{vmatrix} y^{(1)} & z^{(1)} & y_u^{(1)} & z_v^{(1)} \\ y^{(2)} & z^{(2)} & y_u^{(2)} & z_v^{(2)} \\ y^{(3)} & z^{(3)} & y_u^{(3)} & z_v^{(3)} \\ y^{(4)} & z^{(4)} & y_u^{(4)} & z_v^{(4)} \end{vmatrix} \tag{66}$$

is not identically zero. Then we may always determine the coefficients in the partial differential equations

$$\begin{aligned} y_v &= a^{(1)} y + b^{(1)} z + c^{(1)} y_u + d^{(1)} z_v, \\ z_u &= a^{(2)} y + b^{(2)} z + c^{(2)} y_u + d^{(2)} z_v, \\ y_{uu} &= \alpha^{(1)} y + \beta^{(1)} z + \gamma^{(1)} y_u + \delta^{(1)} z_v, \\ z_{vv} &= \alpha^{(2)} y + \beta^{(2)} z + \gamma^{(2)} y_u + \delta^{(2)} z_v, \end{aligned} \tag{67}$$

so that the quantities (65) may be solutions of (67), just as the coefficients of system (4) were found in § 1. System (67) is the completely integrable system we seek; conversely, *any* system of form (67), whose coefficients satisfy suitable integrability conditions, will be completely integrable and may be taken to define two surfaces (65) and the associated congruence. The most general system of solutions of form (65) will be a projective transformation of any particular fundamental system of solutions, so that we have to study the invariants and covariants of (67) under the most general transformations of the variables which leave fixed the congruence of lines. The transformation

* "Sur la Théorie Générale des Congruences," *Académie Royale de Belgique. Classe des Sciences*, Mémoires in 4°, 2. série, t. III (1910–1912).

† *I. e.*, points corresponding to the same values of the parameters u, v.

$$y = \alpha \bar{y} + \beta \bar{z}, \qquad (\alpha\delta - \beta\gamma \neq 0), \tag{68}$$
$$z = \gamma \bar{y} + \delta \bar{z},$$

where $\alpha, \beta, \gamma, \delta$ are functions of u and v, is the most general transformation which leaves fixed each line of the congruence, but changes the surfaces S_y, S_z in an arbitrary way. Similarly, the transformation

$$\bar{u} = U(u, v), \quad \bar{v} = V(u, v) \tag{69}$$

interchanges among themselves the lines of the congruence. Transformations (68) and (69) together form the most general group of point transformations which leave fixed the congruence of lines.

We now seek a transformation of form (68) which changes the surfaces S_y and S_z into the two sheets of the focal surface of the congruence. Substituting (68) in the first two of equations (67), we obtain two equations in the new dependent variables $\bar{y}$, $\bar{z}$. Solving these for $\bar{y}_v$, $\bar{z}_u$, we obtain two equations of the same form as the first two of (67), and for which the new coefficients $\bar{d}^{(1)}$ and $\bar{c}^{(2)}$ have the values

$$\left.\begin{aligned} \bar{d}^{(1)} &= \frac{c^{(2)}\beta^2 + (c^{(1)}d^{(2)} - d^{(1)}c^{(2)} - 1)\beta\delta + d^{(1)}\delta^2}{(\alpha - d^{(1)}\gamma)(\delta - c^{(2)}\beta) - c^{(1)}d^{(2)}\beta\gamma}, \\ \bar{c}^{(2)} &= \frac{c^{(2)}\alpha^2 + (c^{(1)}d^{(2)} - d^{(1)}c^{(2)} - 1)\alpha\gamma + d^{(1)}\gamma^2}{(\alpha - d^{(1)}\gamma)(\delta - c^{(2)}\beta) - c^{(1)}d^{(2)}\beta\gamma}. \end{aligned}\right\} \tag{70}$$

We can therefore make $\bar{d}^{(1)}$ and $\bar{c}^{(2)}$ both vanish by taking for β/δ and α/γ the two roots of the quadratic

$$c^{(2)}t^2 + (c^{(1)}d^{(2)} - d^{(1)}c^{(2)} - 1)\,t + d^{(1)} = 0, \tag{71}$$

provided the roots be distinct. The differential equations then take the forms

$$\bar{y}_v = \bar{a}^{(1)}\bar{y} + \bar{b}^{(1)}\bar{z} + \bar{c}^{(1)}\bar{y}_u,$$
$$\bar{z}_u = \bar{a}^{(2)}\bar{y} + \bar{b}^{(2)}\bar{z} + \bar{d}^{(2)}\bar{z}_v,$$

which show that the line $\bar{y}\bar{z}$ is tangent to the surfaces $S_{\bar{y}}$ and $S_{\bar{z}}$. From (68) we have

$$\bar{y} = \frac{\delta y - \beta z}{\alpha\delta - \beta\gamma}, \quad \bar{z} = \frac{-\gamma y + \alpha z}{\alpha\delta - \beta\gamma}. \tag{72}$$

Therefore, *the two sheets of the focal surface of the congruence are given by* (72), *in which β/δ and α/γ are the roots (supposed distinct) of the quadratic* (71).

If the discriminant of the quadratic (71) vanishes, it is easily seen that the two sheets of the focal surface coincide. This being impossible for the congruence of tangents to our one-parameter family of curves, we shall discard the case of coincident roots.

We turn now to a certain congruence associated with our family of curves C_u. If we join the points ρ and σ corresponding to a point y, and do the same for every point y of the surface S_0, we obtain a congruence of lines, whose focal surface we proceed to determine. With the aid of (52) and (53) it is easily found that

$$\left.\begin{aligned}\rho_v &= a^{(1)}\rho + b^{(1)}\sigma + c^{(1)}\rho_u + d^{(1)}\sigma_v,\\ \sigma_u &= a^{(2)}\rho + b^{(2)}\sigma + c^{(2)}\rho_u + d^{(2)}\sigma_v,\end{aligned}\right\} \tag{73}$$

where

$$\left.\begin{aligned}a^{(1)} &= b' - \frac{K}{\mathfrak{D}}(b-c'), \quad b^{(1)} = -\frac{K}{\mathfrak{D}}(ab'+c), \quad c^{(1)} = \frac{K}{\mathfrak{D}}, \quad d^{(1)} = -\frac{K}{\mathfrak{D}}a,\\ a^{(2)} &= -\frac{H}{\mathfrak{D}}(b-c'), \quad b^{(2)} = c' - \frac{H}{\mathfrak{D}}(ab'+c), \quad c^{(2)} = \frac{H}{\mathfrak{D}}, \quad d^{(2)} = -\frac{H}{\mathfrak{D}}a.\end{aligned}\right\} \tag{74}$$

The quadratic (71) is for this case

$$Ht^2 - \mathfrak{D}t - KA = 0,$$

whose roots are

$$t_1,\, t_2 = \frac{\mathfrak{D} \pm \sqrt{\mathfrak{D}^2 + 4AHK}}{2H}.$$

Substituting these roots for β/δ, α/γ in (72), and discarding certain factors, *we find the covariants*

$$\left.\begin{aligned}R &= 2H\rho - (\mathfrak{D} + \sqrt{\mathfrak{D}^2 + 4AHK})\,\sigma,\\ S &= 2H\rho - (\mathfrak{D} - \sqrt{\mathfrak{D}^2 + 4AHK})\,\sigma,\end{aligned}\right\} \tag{75}$$

which give the two sheets of the focal surface of the congruence of lines joining corresponding points ρ and σ.

The four points ρ, σ, R, S lie on a line; their anharmonic ratio must be an absolute invariant. It is, in fact,

$$\frac{\left(\dfrac{\mathfrak{D} + \sqrt{\mathfrak{D}^2 + 4AHK}}{2H}\right)(\infty)}{\left(\dfrac{\mathfrak{D} - \sqrt{\mathfrak{D}^2 + 4AHK}}{2H}\right)(\infty)} = -1 - \frac{\mathfrak{D}}{2AHK}(\mathfrak{D} + \sqrt{\mathfrak{D}^2 + 4AHK}).$$

In the above it has been supposed that $\mathfrak{D} \neq 0$. But taking the quantities (75) as they stand, regardless of the method of their derivation, we easily find that they actually satisfy two equations of the form

$$\begin{aligned}R_v &= a^{(1)}R + b^{(1)}S + c^{(1)}R_u,\\ S_u &= a^{(2)}R + b^{(2)}S + d^{(2)}S_v,\end{aligned}$$

and hence give the two sheets of the focal surface, provided only that $\mathfrak{D}$ and H do not both vanish. We may then state the following simple theorems in regard to the congruence of lines joining corresponding points ρ, σ:

The sheets of the focal surface coincide if $\mathfrak{D}^2 + 4\,AHK = 0$.

On every line $\rho\sigma$ *of the congruence, the pair of points* ρ, σ *is separated harmonically by the pair of focal points* R, S *if and only if* $\mathfrak{D} = 0$.

This last gives a geometric interpretation for the vanishing of the fundamental invariant $\mathfrak{D}$.

We return now to the general theory of congruences. The surfaces S_y, S_z have been transformed into the two sheets of the focal surface of the congruence, but the differential equations are not yet in the form used by Wilczynski. By a suitable transformation of the form

$$\bar{u} = U\,(u, v), \quad \bar{v} = V\,(u, v), \tag{69}$$

the differential equations of the first order may be transformed into

$$y_v + \alpha\,y = \omega\,z, \quad z_u + \beta\,z = \omega'\,y. \tag{76}$$

We shall not carry out this transformation, which is equivalent to the determination of the two systems of developables of the congruence, and requires the integration of a partial differential equation of the first order. These developables are known for the congruences to which we intend to apply Wilczynski's results.

We consider now the congruence of tangents to the one-parameter family of curves C_u. The two sheets of the focal surface are given by the covariants ρ and y, which we know satisfy the differential equations

$$\rho_v - b'\,\rho = K\,y, \quad y_u - c'\,y = \rho, \tag{77}$$

which are of the form (76). Instead of ρ and y we shall take as dependent variables the quantities

$$\eta = \lambda\,\rho, \quad \zeta = \mu\,y, \tag{78}$$

where λ and μ satisfy the relations

$$\lambda_v + b'\,\lambda = 0, \quad \mu_u + c'\,\mu = 0. \tag{79}$$

The new variables are easily seen to satisfy the differential equations

$$\eta_v = m\,\zeta, \quad \zeta_u = n\,\eta, \tag{80}$$

where

$$m = \frac{\lambda}{\mu}K, \quad n = \frac{\mu}{\lambda}. \tag{80a}$$

But the completely integrable system of differential equations contains besides (80) two equations of the second order satisfied by η and ζ. These equations are found most easily as follows. We have from (52)

$$a\,y_{vv} = \rho_u - (b - c')\,\rho - c\,y_v - [d - c'_u + c'\,(b - c')]\,y,$$

and find also that

$$\begin{aligned}\rho_{uu} = {} & (\alpha^{(1)} - a\,c')\,y_{vv} + (\beta^{(1)} - 2\,c'_u - b\,c')\,y_u \\ & + (\gamma^{(1)} - c\,c')\,y_v + (\delta^{(1)} - c'_{uu} - d\,c')\,y,\end{aligned}$$

$\alpha^{(1)}$, $\beta^{(1)}$, $\gamma^{(1)}$, $\delta^{(1)}$ being given by (21). The first of these may be transformed immediately by means of (78) into the last equation of system (81) below. The expression for ρ_{uu} may be transformed likewise into the third equation of system (81), after y_{vv} and y_u have been replaced by equivalent expressions in ρ_u, ρ, y_v, y. We thus obtain, after some calculation, the last two of the following completely integrable system of differential equations:

$$\left.\begin{aligned} \eta_v = m\zeta, \quad \zeta_u = n\eta, \\ \eta_{uu} = \alpha\eta + \beta\zeta + \gamma\eta_u + \delta\zeta_v, \\ \zeta_{vv} = \alpha'\eta + \beta'\zeta + \gamma'\eta_u + \delta'\zeta_v, \end{aligned}\right\} \tag{81}$$

where

$$\left.\begin{aligned} m &= \frac{\lambda}{\mu}K, & n &= \frac{\mu}{\lambda}, \\ \gamma &= b + \frac{a_u}{a} + 2\frac{\lambda_u}{\lambda}, & \delta &= \frac{\lambda}{\mu}\left[\gamma^{(1)} - cc' - c\left(b + \frac{a_u}{a}\right)\right], \\ \gamma' &= \frac{\mu}{\lambda}\frac{1}{a}, & \delta' &= -\frac{c}{a} + 2\frac{\mu_v}{\mu}, \end{aligned}\right\} \tag{81a}$$

and the values of α, β, α', β' do not concern us.

System (81) is the completely integrable system (D) which Wilczynski takes as the basis for his theory of congruences. To avoid confusion in notation, we have changed all of Wilczynski's letters, except u, v, m and n, into the corresponding Greek letters.

We find very easily, by using the expression for $\gamma^{(1)}$ from (21), that

$$\left.\begin{aligned} \delta &= \frac{\lambda}{\mu}\left[\gamma^{(1)} - cc' - c\left(b + \frac{a_u}{a}\right)\right] \\ &= \frac{\lambda}{\mu}\left[c_u + ac'_v + ad' + ab'c' - \frac{ca_u}{a}\right] \\ &= A\frac{\lambda}{\mu}\left[H + 3\mathfrak{B}'_u + \mathfrak{C}'_v\right] \\ &= A\frac{\lambda}{\mu}\left[K + 2(B'_u + C'_v) - \frac{\partial^2}{\partial u\partial v}\log A\right]. \end{aligned}\right\} \tag{81b}$$

Except for the factor $\frac{\lambda}{\mu}$, this is an invariant of the one-parameter family of curves.

It is readily verified that

$$\gamma_v = \delta'_u. \tag{82}$$

In fact, on reduction this is found to be equivalent to the relation

$$\frac{\partial}{\partial v}(b + 2c') = \frac{\partial}{\partial u}\left(\frac{2ab' - c - a_v}{a}\right).$$

We may therefore put, with Wilczynski,

$$f_u = \gamma, \quad f_v = \delta'. \tag{83}$$

Then the four quantities $\overset{(\eta)}{\mathfrak{B}}, \overset{(\eta)}{\mathfrak{C}}{}'', \overset{(\zeta)}{\mathfrak{B}}, \overset{(\zeta)}{\mathfrak{C}}{}''$ given by

$$\left.\begin{aligned} 4\overset{(\eta)}{\mathfrak{B}} &= f_u - \frac{1}{2}\frac{\delta_u}{\delta} - \frac{3}{2}\frac{m_u}{m}, \quad 4\overset{(\eta)}{\mathfrak{C}}{}'' = f_v + \frac{3}{2}\frac{\delta_v}{\delta} + \frac{1}{2}\frac{m_v}{m}, \\ 4\overset{(\zeta)}{\mathfrak{B}} &= f_u + \frac{3}{2}\frac{\gamma'_u}{\gamma'} + \frac{1}{2}\frac{n_u}{n}, \quad 4\overset{(\zeta)}{\mathfrak{C}}{}'' = f_v - \frac{1}{2}\frac{\gamma'_v}{\gamma'} - \frac{3}{2}\frac{n_v}{n} \end{aligned}\right\} \tag{84}$$

are invariants of the congruence, and these, with the invariants m, n, γ', δ, form a system of eight invariants such that any six may be taken as a fundamental system in the sense that any invariant is a function of these six and of their derivatives.*

The quantities (84) are invariants of the one-parameter family of curves, and are expressible in terms of the coefficients of (16). Upon calculation we find in fact that

$$\left.\begin{aligned} 4\overset{(\eta)}{\mathfrak{B}} &= -4\mathfrak{C}' + \frac{A_u}{A} - \frac{3}{2}\frac{K_u}{K} - \frac{1}{2}\frac{H_u + 3\mathfrak{B}'_{uu} + \mathfrak{C}'_{uv}}{H + 3\mathfrak{B}'_u + \mathfrak{C}'_v}, \\ 4\overset{(\eta)}{\mathfrak{C}}{}'' &= -4\mathfrak{B}' + \frac{A_v}{A} + \frac{1}{2}\frac{K_v}{K} + \frac{3}{2}\frac{H_v + 3\mathfrak{B}'_{uv} + \mathfrak{C}'_{vv}}{H + 3\mathfrak{B}'_u + \mathfrak{C}'_v}, \\ 4\overset{(\zeta)}{\mathfrak{B}} &= -4\mathfrak{C}', \quad 4\overset{(\zeta)}{\mathfrak{C}}{}'' = -4\mathfrak{B}'. \end{aligned}\right\} \tag{85}$$

We may choose as the six fundamental invariants these four and

$$mn = K, \quad \gamma'\delta = H + 3\mathfrak{B}'_u + \mathfrak{C}'_v.$$

Then it is immediately evident that all invariants of the congruence, being functions of these six and of their derivatives, are expressible in terms of the coefficients of (16) and their derivatives, the knowledge of integrals of equations (79) being unnecessary. Any invariant equation of the theory of congruences will therefore express a property of a one-parameter family of curves. Thus, *the vanishing of the invariant*

$$\left.\begin{aligned} W &= mn - \gamma'\delta \\ &= K - (H + 3\mathfrak{B}'_u + \mathfrak{C}'_v) \\ &= -2(B'_u + C'_v) + \frac{\partial^2}{\partial u\,\partial v}\log A \end{aligned}\right\} \tag{86}$$

is the condition that the congruence of tangents to our one-parameter family of curves be a W-congruence, i. e., a congruence in which asymptotic lines on the two sheets of the focal surface correspond.†

Again, we may express the conditions that the congruence of tangents to the one-parameter family of curves belong to a linear complex, or, as we may say, the conditions that the one-parameter family of curves belong to a linear

* E. J. Wilczynski, *loc. cit.*, pp. 20, 23. † E. J. Wilczynski, *loc. cit.*, p. 46.

complex. The conditions, as given by equations (73a) of Wilczynski's memoir, are

$$\left.\begin{array}{c} W=0, \quad \gamma'\beta+m n_v=0, \quad \alpha' m+\gamma' m_u=0, \\ \begin{vmatrix} m_u-m\gamma & \gamma'_u \\ m & \gamma' \end{vmatrix}=0, \quad \begin{vmatrix} m_v & \gamma'_v-\gamma'\delta' \\ m & \gamma' \end{vmatrix}=0. \end{array}\right\} \tag{87}$$

But by the integrability conditions (12) of that memoir we have

$$\beta=-\delta_v-\delta f_v, \quad \alpha'=-\gamma'_u-\gamma' f_u,$$

so that the second and third of equations (87) become

$$m n_v-\gamma'\delta_v-\gamma'\delta\delta'=0, \quad \gamma' m_u-m\gamma'_u-m\gamma\gamma'=0,$$

or, making use of the relation

$$W\equiv m n-\gamma'\delta=0,$$

$$m\gamma'_v-m_v\gamma'-m\gamma'\delta'=0, \quad \gamma' m_u-m\gamma'_u-m\gamma\gamma'=0. \tag{88}$$

These are exactly equivalent to the fifth and fourth, respectively, of equations (87). Equations (88) are invariant equations, and when expressed in terms of the coefficients of (16) they become, after a somewhat lengthy reduction,

$$\left.\begin{array}{l} 4\mathfrak{B}'-\dfrac{K_v}{K}-\dfrac{A_v}{2A}=0, \\ 4\mathfrak{C}'+\dfrac{K_u}{K}-\dfrac{A_u}{2A}=0. \end{array}\right\} \tag{89}$$

If the first of these be differentiated with respect to u, the second with respect to v, and the results added, we obtain the equation

$$4(\mathfrak{B}'_u+\mathfrak{C}'_v)-\frac{\partial^2}{\partial u\,\partial v}\log A=0,$$

which is the same as $2W=0$. We have then the theorem: *The one-parameter family of curves belongs to a linear complex if and only if*

$$4\mathfrak{B}'-\frac{K_v}{K}-\frac{A_v}{2A}=0, \quad 4\mathfrak{C}'+\frac{K_u}{K}-\frac{A_u}{2A}=0.$$

Enough has been indicated to show how the results of Wilczynski's theory of congruences may be applied directly to the study of a one-parameter family of curves. In fact, the two theories are practically identical geometrically, so that either may be used to approach the theory of congruences. Which is to be employed in any particular case must be decided by considerations of economy in calculation.

COLLEGE OF THE CITY OF NEW YORK, *October*, 1913.

Linear Combinants of Systems of Binary Forms, with the Syzygies of the Second Degree Connecting Them.

BY WALTER FRANCIS SHENTON.

The large use that has been made of the theory of combinants in the realm of the invariant theory of rational curves has made the present research seem well worth while. It has been the purpose of this investigation to set forth a comparatively simple and orderly method for obtaining the linear and quadratic combinants of a given system of two or three binary forms of the same order.

After having followed out the work sufficiently far to cover all the cases of rational plane and space curves as far as the sextic, we have arranged these explicit forms in a series of tables for easy reference. By this means we believe that they will be valuable to any person who desires to find the corresponding combinants for any canonical forms.

I. *Combinant Defined.*

A combinant of a system of quantics of the same order is an invariant or covariant of this system which remains unchanged, to within a constant factor, not only when the variables are linearly transformed, but also when for this system of quantics there is substituted another system of quantics which are linear combinations of the original quantics.

In particular, a combinant of a number of binary quantics $f_1, f_2, f_3, \ldots, f_n$, in the same variables, is an invariant or covariant of these quantics which differs only by a power of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & c_2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n & b_n & c_n & \ldots \end{vmatrix}$$

from the same invariant or covariant of $F_1, F_2, F_3, \ldots, F_n$, where

$$F_i = a_i f_1 + b_i f_2 + c_i f_3 + \ldots, \qquad (i = 1, 2, 3, \ldots, n).$$

II. *A combinant may be expressed in terms of coefficients which are determinants formed from the matrix of the coefficients of the quantics from which the combinant is derived. Conversely, a covariant or invariant of a system of quantics is a combinant of the system if its coefficients may be expressed in terms of such determinants.*

The first of these theorems is an immediate outgrowth of the fundamental theorem for invariants. A more general statement of this theorem is given

by Gordan.* To develop a proof for the converse, let us consider two binary forms; say, $(at)^n$ and $(bt)^n$. The matrix of their coefficients is

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & b_n \end{vmatrix}.$$

If now we look upon (a_0, b_0), (a_1, b_1), etc., as coordinates of points along a line, the combinants, which may be regarded as associated with the whole line rather than the individual points upon it, will remain invariant under the transformations

$$a_i' = \alpha a_i + \beta b_i, \quad b_i' = \gamma a_i + \delta b_i, \qquad (i = 0, 1, 2, \dots, n);$$

that is,

$$f(a_0', \dots, a_n', b_0', \dots, b_n') = (\alpha\delta - \beta\gamma)^w f(a_0, \dots a_n, b_0, \dots b_n).$$

From this we can see that an invariant or covariant with coefficients in terms of determinants like $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ also satisfies the further test and is a combinant. This same argument can be extended for three binary n-ics by regarding the elements of the matrix as coordinates of points in a plane.

III. *Operators: Their Application to Invariants and Covariants; Combinants Formed from Their Leading Coefficients by the Use of Operators.*

We shall be making constant use of two differential operators:†

$$\Omega = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \dots + p a_{p-1} \frac{\partial}{\partial a_p}$$

and

$$O = p a_1 \frac{\partial}{\partial a_0} + (p-1) a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}}.$$

For an invariant, I, of a single quantic, they are called "annihilators," since $\Omega I = 0$ and $OI = 0$; that is, the action of either operator on I causes it to vanish.

On the other hand, these two operators enable us to form all the coefficients of a covariant of a single quantic from its end coefficient. If the end coefficient is that of the first term, we call it the *source* or *leading coefficient*. If we represent this source by C_0, then

$$\Omega C_0 = 0.$$

This is the first requirement for a source; namely, that it must vanish under Ω. However, the operator O behaves much differently, for $OC_0 = C_1$, $OC_1 = \frac{1}{2} C_2$,

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* *Math. Ann.*, Vol. V. † Cf. Elliott, "Algebra of Quantics," p. 112 *et seq.*

$\ldots\ldots, OC_n=0$, where C_n is the last coefficient. In general, we may write

$$C_k=\frac{1}{k!}(O)^k C_0,$$

where C_k is the coefficient of the $(k+1)$-th term.

For the covariants and invariants of a system of quantics of the same order p, we may apply the same arguments as before if we use for Ω and O the sum of the operators formed for each of the quantics; thus,

$$\begin{aligned}\Omega=&\left(a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\ldots\ldots+pa_{p-1}\frac{\partial}{\partial a_p}\right)\\&+\left(b_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial b_2}+\ldots\ldots+pb_{p-1}\frac{\partial}{\partial b_p}\right)\\&+\left(c_0\frac{\partial}{\partial c_1}+2c_1\frac{\partial}{\partial c_2}+\ldots\ldots+pc_{p-1}\frac{\partial}{\partial c_p}\right)+\ldots\ldots\end{aligned}$$

and

$$\begin{aligned}O=&\left(pa_1\frac{\partial}{\partial a_0}+(p-1)a_2\frac{\partial}{\partial a_1}+\ldots\ldots+a_p\frac{\partial}{\partial a_{p-1}}\right)\\&+\left(pb_1\frac{\partial}{\partial b_0}+(p-1)b_2\frac{\partial}{\partial b_1}+\ldots\ldots+b_p\frac{\partial}{\partial b_{p-1}}\right)\\&+\left(pc_1\frac{\partial}{\partial c_0}+(p-1)c_2\frac{\partial}{\partial c_1}+\ldots\ldots+c_p\frac{\partial}{\partial c_{p-1}}\right)+\ldots\ldots\end{aligned}$$

A combinant of several forms of the same order has already been defined as an invariant or covariant of this system of forms. Hence, if we have a proper leading coefficient, we may determine the others according to the method just outlined. In the last section we also showed that an n-rowed determinant from the matrix of the coefficients of our system of n forms would be a proper coefficient for a combinant. Hence, if we choose such a combination of determinants of one weight* as will vanish under Ω, it will be a proper leading coefficient, and the remaining coefficients may be easily calculated.

IV. *The effect of operating with* O *on a determinant is to replace an entire column of the determinant by a new column.*

Take an n-rowed determinant of the type we are considering, say

$$\Delta=\begin{vmatrix}a_0 & a_1 & a_2 & \ldots\ldots & a_{n-1}\\ b_0 & b_1 & b_2 & \ldots\ldots & b_{n-1}\\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots\\ n_0 & n_1 & n_2 & \ldots\ldots & n_{n-1}\end{vmatrix}.$$

* We define the weight of the determinant $\begin{vmatrix}a_i & a_j\\ b_i & b_j\end{vmatrix}$, which we write symbolically $|ij|$, as $i+j-1$; and the weight of $\begin{vmatrix}a_i & a_j & a_k\\ b_i & b_j & b_k\\ c_i & c_j & c_k\end{vmatrix}$, written $|ijk|$, as $i+j+k-3$. This method is arbitrarily chosen so that the determinant of lowest weight shall be of weight zero.

/ ? ?

If A_i is the minor of a_i, and B_i is the minor of b_i, etc., we have the following expansion of Δ in terms of the elements of the r-th column and its minors:

$$\Delta=(-1)^r(a_r A_r+b_r B_r+c_r C_r+\ldots.+n_r N_r).$$

Now the terms of 0 that would operate on the elements of the r-th column are

$$(n-r)\left(a_{r+1}\frac{\partial}{\partial a_r}+b_{r+1}\frac{\partial}{\partial b_r}+\ldots.\right),$$

and when we operate with them we get

$$(n-r)(-1)^r(a_{r+1}A_r+b_{r+1}B_r+\ldots.+n_{r+1}N_r),$$

where the only effect on the determinant has been to replace the entire r-th column by a new one. We have considered *any* column; hence the total effect of the operator 0 is to form a sum of such determinants in which one column of the original determinant is replaced by a new column.

In our notation for these determinants, the elements i, j, k of the symbol stand for the common subscripts of the letters in the column they represent. For convenient writing we consider the summed operator 0 as

$$0=pa_1\frac{\partial}{\partial a_0}+(p-1)a_2\frac{\partial}{\partial a_1}+\ldots.,$$

and consider the effect on subscripts rather than on letters, really counting the symbols as being subscripts of the single letter a.

Example: For three cubics,

$$0=3a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+a_3\frac{\partial}{\partial a_2},$$

and the effect of operating on $|012|$, say, becomes

$$0|012|=3|112|+2|022|+|013|,$$

where the symbols $|112|$ and $|022|$ vanish because they represent determinants with two columns identical.

V. *Notation for Linear Combinants and the Syzygies.*

In our actual practice we have been led to adopt the following uniform notation for the linear combinants of systems of two and three binary forms and the syzygies of the second degree.

For two binary forms $(\alpha t)^n$ and $(\beta t)^n$, the linear combinants are named:

$$a=(at)^{2(n-1)}=(\text{symbolically})\,|\alpha\beta|\,(\alpha t)^{n-1}(\beta t)^{n-1},$$
$$b=(bt)^{2(n-3)}=(\text{symbolically})\,|\alpha\beta|^3(\alpha t)^{n-3}(\beta t)^{n-3},$$
$$c=(ct)^{2(n-4)},\ \text{etc.}$$

For the linear combinants of three binary forms, $(\alpha t)^n$, $(\beta t)^n$, $(\gamma t)^n$, we assign the following names:

$$A=(At)^{3n-6},\ B=(Bt)^{3n-10},\ C=(Ct)^{3n-12},\ D=(Dt)^{3n-14},\ E=(Et)^{3n-16}, \text{ etc.},$$

where the ordinary symbolic notation is too cumbersome to note here for the general case.

The syzygies in both cases have been numbered with Roman numerals, that of the lowest weight and highest order being called I, the numbers being used only when the syzygies actually occur. In case there are two or more syzygies of the same order in any set of forms, they are distinguished by an Arabic numeral suffix; thus, IV–1, IV–2, IV–3.

VI. *Computation of the Linear Combinants of a System of Two Binary Quartics $(\alpha t)^4$ and $(\beta t)^4$, Written with Binomial Coefficients.*

For this case the operators are

$$\Omega= a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+4a_3\frac{\partial}{\partial a_4},$$

$$O=4a_1\frac{\partial}{\partial a_0}+3a_2\frac{\partial}{\partial a_1}+2a_3\frac{\partial}{\partial a_2}+a_4\frac{\partial}{\partial a_3}.$$

The only determinant of weight zero that we have is $|01|$. $\Omega|01|=0$; hence it may be used as a leading coefficient. Call it a_0; then

$$\begin{aligned}
a_0&=|01|,\\
a_1&=Oa_0=3|02|,\\
a_2&=\frac{1}{2}Oa_1=3|03|+6|12|,\\
a_3&=\frac{1}{3}Oa_2=|04|+8|13|,\\
a_4&=\frac{1}{4}Oa_3=3|14|+6|23|,\\
a_5&=\frac{1}{5}Oa_4=3|24|,\\
a_6&=\frac{1}{6}Oa_5=|34|,\\
a_7&=\frac{1}{7}Oa_6=0.
\end{aligned}$$

Since a_7 is zero, we evidently have a form of order 6 in the variables, whence

$$a=a_0t^6+a_1t^5+a_2t^4+a_3t^3+a_4t^2+a_5t+a_6,$$

where the coefficients are computed as above.

Consider now determinants of weight 1. There is but one, $|02|$. $\Omega|02|\neq 0$; therefore it cannot be a leading coefficient of a combinant.

We have two determinants, $|03|$ and $|12|$, of weight 2. Let us find a linear combination of them, say $\lambda_1|03|+\lambda_2|12|$, such that $\Omega(\lambda_1|03|+\lambda_2|12|)=0$.

$$\Omega(\lambda_1|03|+\lambda_2|12|)=3\lambda_1|02|+\lambda_2|02|.$$

Evidently the condition that this vanish is $3\lambda_1+\lambda_2=0$. This occurs most simply when $\lambda_1=1$ and $\lambda_2=-3$; whence $|03|-3|12|$ is a proper leading coefficient of weight 2. Call it b_0; then we have

$$\begin{aligned} b_0 &= |03|-3|12|, \\ b_1 &= Ob_0=|04|-2|13|, \\ b_2 &= \frac{1}{2}Ob_1=|14|-3|23|, \\ b_3 &= \frac{1}{3}Ob_2=0. \end{aligned}$$

Hence our second combinant is of order 2 and is

$$b=b_0t^2+b_1t+b_2.$$

The determinants of next weight are $|04|$ and $|13|$. A linear combination of these, say $\lambda_1|04|+\lambda_2|13|$, will not vanish under Ω unless $\lambda_1=\lambda_2=0$; hence we have no leading coefficient of weight 3. In this same way we can show that there are no other leading coefficients of greater weight.

VII. *Computation of the Linear Combinants of a System of Three Binary Quartics, $(\alpha t)^4$, $(\beta t)^4$, $(\gamma t)^4$, Written with Binomial Coefficients.*

The operators are the same as in Section VI.

The determinant of weight zero is $|012|$ and vanishes under Ω; hence it may form a leading coefficient. The coefficients are thus:

$$\begin{aligned} A_0 &= |012|, \\ A_1 &= OA_0=2|013|, \\ A_2 &= \frac{1}{2}OA_1=|014|+3|023|, \\ A_3 &= \frac{1}{3}OA_2=2|024|+4|123|, \\ A_4 &= \frac{1}{4}OA_3=|034|+3|124|, \\ A_5 &= \frac{1}{5}OA_4=2|134|, \\ A_6 &= \frac{1}{6}OA_5=|234|, \\ A_7 &= \frac{1}{7}OA_6=0. \end{aligned}$$

From these we can see that we have a combinant of order 6. It is

$$A = A_0 t^6 + A_1 t^5 + A_2 t^4 + A_3 t^3 + A_4 t^2 + A_5 t + A_6.$$

There is only one determinant of weight 1; it is $|013|$. $\Omega|013| \neq 0$; therefore we can have no combinant beginning with a coefficient of weight 1.

Of weight 2 we have the determinants $|014|$ and $|023|$. Let us find a linear combination of these two, say $\lambda_1|014| + \lambda_2|023|$, which will vanish under Ω.

$$\Omega(\lambda_1|014| + \lambda_2|023|) = 4\lambda_1|013| + 2\lambda_2|013|,$$

which will vanish if $\lambda_1 = 1$ and $\lambda_2 = -2$, whence $|014| - 2|023|$ is a proper leading coefficient of weight 2. From it we obtain, as before, the coefficients of the combinant

$$B = (|014| - 2|023|)t^2 + (|024| - 8|123|)t + |034| - 2|124|.$$

The determinants of weight greater than 2 will not form linear combinations which will vanish under Ω; hence there are no further linear combinants for three quartics.

VIII. *Determinant Identities.*

There exist among the coefficients of the linear combinants certain quadratic identities. The types we shall use are:

(1). Like $|01||23| + |02||31| + |03||12| = 0$, where one column is fixed and the others are cyclically permuted.

(2). Like $|012||034| + |013||042| + |014||023| = 0$, which is an extended form of type (1). It is carried over from the binary to the ternary domain by prefixing a common column to each determinant and extending the columns from two to three rows.

(3). Like $|123||456| - |234||156| + |341||256| - |412||356| = 0$, where two columns are fixed and the others are permuted cyclically.

All three types may be shown to vanish identically by expansion. However, it is interesting to see how they may be explained geometrically. The first is merely the linear condition existing between four collinear points. It is the only identity that concerns us in the binary domain. From it we get, by the Clebsch "principle of transference," the identity of type (2) for the ternary domain. We may consider 1, 2, 3, 4 as the names of four points in a plane. If we connect them with an arbitrary point 0, we have in the identity the coincidence relation of the lines $\overline{01}, \overline{02}, \overline{03}, \overline{04}$.

Type (3) is indigenous with the ternary domain. Starting with any six numbers, they may be arranged to satisfy this identity in fifteen different ways, but these represent only five independent linear relations among the six numbers.

IX. *Syzygies of Degree* 2 *Connecting the Linear Combinants.*

There are certain identities of degree 2 existing among the transvectants of the linear combinants of any system of quantics. We shall now show how to determine these identities.

Let us consider, for instance, the linear combinants of the three quartics as given in Section VII. They are A, of order 6 and weight zero,* and B, of order 2 and weight 2.

The transvectants of second degree arising from these two forms—counting their squares and product as transvectants—are given by the table:

	Transvectants.	Order.	Weight.
I.	A^2	12	0
II.	$\|A, A\|^2,\ AB$	8	2
III.	$\|A, B\|$	6	3
IV.	$\|A, A\|^4,\ B^2,\ \|A, B\|^2$	4	4
V.	$\|A, A\|^6,\ \|B, B\|^2$	0	6

The determinant identities are all of type (2), formed from the numbers 0, 1, 2, 3, 4. We have the following identities, one term being given for the whole identity:

$|012|\,|034|$ of weight 4, $|304|\,|312|$ of weight 7,
$|104|\,|123|$ of weight 5, $|403|\,|412|$ of weight 8.
$|204|\,|213|$ of weight 6,

The syzygies depend upon the identities of the same weight to such an extent that if we do not have any identity of the desired weight, we can not have any syzygy. As the weight of the lowest identity is 4, the transvectants of the rows I, II and III can not give rise to any syzygies. Let us now consider the row IV.

$|A, A|^4$, the fourth transvectant of A on itself, is obtained by taking the fourth polar of A and forming its bilinear invariant. Since $A=(At)^6$ is written without the binomial coefficients, we have this fourth polar as

$$\begin{aligned}(A_0 t_1^2+2A_1 t_1 t_2+A_2 t_2^2)\cdot T_1^4+(A_1 t_1^2+A_2 t_1 t_2+A_3 t_2^2)\cdot 4T_1^3 T_2 \\ +(A_2 t_1^2+A_3 t_1 t_2+A_4 t_2^2)\cdot 6T_1^2 T_2^2+(A_3 t_1^2+A_4 t_1 t_2+A_5 t_2^2)\cdot 4T_1 T_2^3 \\ +(A_4 t_1^2+A_5 t_1 t_2+A_6 t_2^2)\cdot T_2^4,\end{aligned}$$

whence

$$|A, A|^4=2(A_0 A_4-4A_1 A_3+3A_2 A_2)t_1^4+\ldots\ldots$$

The second form, B^2, is merely the square of $(Bt)^2$, whence

$$B^2=|(Bt)^2|^2=B_0^2 t_1^4+\ldots\ldots$$

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*In the succeeding discussion we shall repeatedly refer to the weight of a linear combinant or syzygy as identical with the weight of the leading coefficient.

The third form, $|A, B|^2$, is the second transvectant of A on B. It is then the bilinear invariant of the second polar of A and the second polar of B.

The second polar of A is

$$T_1^2(A_0 t_1^4+4A_1 t_1^3 t_2+6A_2 t_1^2 t_2^2+4A_3 t_1 t_2^3+A_4 t_2^4)$$
$$+2T_1 T_2(A_1 t_1^4+4A_2 t_1^3 t_2+6A_3 t_1^2 t_2^2+4A_4 t_1 t_2^3+A_5 t_2^4)$$
$$+T_2^2(A_2 t_1^4+4A_3 t_1^3 t_2+6A_4 t_1^2 t_2^2+4A_5 t_1 t_2^3+A_6 t_2^4),$$

and the second polar of B is

$$B_0 T_1^2+2B_1 T_1 T_2+B_2 T_2^2,$$

whence

$$|A, B|^2=(A_0 B_2-2A_1 B_1+A_2 B_0)t^4+\ldots\ldots$$

We have in each case written only the leading coefficients of the transvectants, since these determine all the others in such a way that whatever linear relations connect the leading coefficients of these transvectants will also hold for any other set of coefficients. Now in terms of $(At)^6$ and $(Bt)^2$ expressed with binomial coefficients, the A's and B's have the following values:

$$A_0=|012|,\qquad B_0=|014|-2|023|,$$
$$A_1=\frac{1}{3}|013|,\qquad B_1=\frac{1}{2}(|024|-8|123|),$$
$$A_2=\frac{1}{15}(|014|+3|023|),\qquad B_2=|034|-2|124|,$$
$$A_3=\frac{1}{10}(|024|+2|123|),$$
$$A_4=\frac{1}{15}(|034|+3|124|),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Using these values, we set down the actual values of the leading coefficients of $|A, A|^4$, B^2, and $|A, B|^2$.

	$\|012\|\|124\|$	$\|012\|\|034\|$	$\|013\|\|123\|$	$\|013\|\|024\|$	$\|023\|^2$	$\|023\|\|014\|$	$\|014\|^2$
$\|A, A\|^4$	2/5	2/15	—8/15	—4/15	6/25	4/25	2/75
B^2	....	....	....	....	4	—4	1
$\|A, B\|^2$	—2	1	8/3	—1/3	—2/5	1/15	1/15
	(1)	I	(2)	I	(3)	I	(4)

The determinants marked with the Roman I are connected by the identity

$$|012||034|+|013||024|+|014||023|=0.$$

If now the entire first row be multiplied by λ_1, the second by λ_2, and the third by λ_3, the λ's may, as we shall show, be so determined that

$$\lambda_1 | A, A |^4 + \lambda_2 B^2 + \lambda_3 | A, B |^2 = 0.$$

To accomplish this, we must satisfy the equations formed by equating to zero the sum of the coefficients of each of the columns marked with an Arabic numeral. Thus we have the four simultaneous equations in λ_1:

$$\begin{array}{ll} (1) & \lambda_1 \qquad\qquad -5\lambda_3 = 0, \\ (2) & -\lambda_1 \qquad\quad +5\lambda_3 = 0, \\ (3) & 3\lambda_1 + 50\lambda_2 - 5\lambda_3 = 0, \\ (4) & 2\lambda_1 + 75\lambda_2 + 5\lambda_3 = 0, \end{array}$$

from which, since not all the equations are independent, we can determine the ratios of the λ's. We have, for the simplest integral values,

$$\lambda_1 = 25, \quad \lambda_2 = -1, \quad \lambda_3 = 5.$$

These values will make each of the four columns from which the equations were obtained vanish. For the columns marked I their behavior is much different. When they are substituted in these latter columns, they give in each case the absolute value $\frac{25}{3}$. Hence, for these three columns taken together, we have the value

$$\frac{25}{3} \left(|012| \, |034| + |013| \, |042| + |014| \, |023| \right),$$

which vanishes independently of the λ's, since the quantity in the parentheses is one of the determinant identities of the second type mentioned in the last section.

We have, therefore, for our syzygy,

$$25 | A, A |^4 - B^2 + 5 | A, B |^2 = 0.$$

This is an expression of order 4 in the variables and will therefore, when expanded, have five terms. The weights of these terms are 4, 5, 6, 7 and 8, respectively. We have already shown (p. 254) that there is just one determinant identity of each of these weights, so that there are just enough identities to account for all the terms of the syzygy, and there are none left over for the formation of any other syzygies; hence we can have no other syzygies connecting the linear combinants of three quartics. There is, then, a unique syzygy of degree 2 connecting the linear combinants of three binary quartics, itself a quartic syzygy.

X. *Tables of Linear Combinants of Systems of Binary Forms of the Same Order, with the Syzygies of the Second Degree Connecting Them.*

Following the methods just briefly illustrated, we have computed the linear combinants for systems of two and three binary forms, up to and including the case of binary septimics, and have either computed or enumerated the syzygies of the second degree connecting them. The results of these computations are presented in the following set of tables.

The works of a number of investigators who have conducted researches along these lines previous to this paper have been consulted in the preparation of this work for corroborative evidence. Those papers of particular value in this instance are one by Stroh* in which he gives a splendid enumeration of the linear combinants of systems of two, three and four binary forms, and two by Berzolari† in which are given the linear and quadratic combinants of systems of two and three quartics and quintics. Both of these authors have obtained their results through considerations other than those which are presented here.

Linear Combinants of Two Binary Forms, $(\alpha t)^n$ and $(\beta t)^n$, and the Syzygies of the Second Degree Connecting Them.

Two Quadratics. Linear combinant:

$$a=|01|t^2+|02|t+|12|.$$

Two Cubics. Linear combinants:

$$a=|01|t^4+2|02|t^3+(|03|+3|12|)t^2+2|13|t+|23|,$$
$$b=|03|-3|12|.$$

Syzygy:

$$\text{I.}\quad 6|a,\ a|^4-b^2=0.$$

Two Quartics. Linear combinants:

$$a=|01|t^6+3|02|t^5+(3|03|+6|12|)t^4+(|04|+8|13|)t^3$$
$$+(3|14|+6|23|)t^2+3|24|t+|34|,$$
$$b=(|03|-3|12|)t^2+(|04|-2|13|)t+(|14|-3|23|).$$

Syzygy:

$$\text{I.}\quad |a,\ a|^4-|a,\ b|^2-b^2=0.$$

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* E. Stroh, "Zur Theorie der Combinanten," *Math. Ann.*, Vol. XXII (1893), p. 404.

† L. Berzolari, "Combinanti dei sistemi lineari di quintiche binarie," *Cir. Mt. Rd. Palermo*, Vol. VII (1893), pp. 5–18; "Binary Combinants Associated with Curves of the Fourth Order," *Annali di Matematica*, Vol. XX (1883), pp. 101–105.

Two Quintics. **Linear combinants:**

$$\begin{aligned}a=&|01|t^8+4|02|t^7+2(3|03|+5|12|)t^6+4(|04|+5|13|)t^5\\&+(|05|+15|14|+20|23|)t^4+4(|15|+5|24|)t^3\\&+2(3|25|+5|34|)t^2+4|35|t+|45|,\end{aligned}$$

$$\begin{aligned}b=&(|03|-3|12|)t^4+2(|04|-2|13|)t^3+(|05|+|14|-8|23|)t^2\\&+2(|15|-2|24|)t+|25|-3|34|,\end{aligned}$$

$$d=|05|-5|14|+10|23|.$$

Syzygies (formed with $210a$, $30b$ and $14d$):

I. $2|a, a|^4-15b^2+8|a, b|^2+2ad=0.$

II. $2|a, a|^6-50|b, b|^2-15|a, b|^4-35bd=0.$

III. $|a, a|^8+35|b, b|^4-30d^2=0.$

Two Sextics. **Linear combinants:**

$$\begin{aligned}a=&|01|t^{10}+5|02|t^9+5(2|03|+3|12|)t^8+10(|04|+4|13|)t^7\\&+5(|05|+9|14|+10|23|)t^6+(|06|+24|15|+75|24|)t^5\\&+5(|16|+9|25|+10|34|)t^4+10(|26|+4|35|)t^3\\&+5(2|36|+3|45|)t^2+5|46|t+|56|,\end{aligned}$$

$$\begin{aligned}b=&(|03|-3|12|)t^6+3(|04|-2|13|)t^5+3(|05|-5|23|)t^4\\&+(|06|+6|15|-15|24|)t^3+3(|16|-5|34|)t^2\\&+3(|26|-2|35|)t+(|36|-3|45|),\end{aligned}$$

$$\begin{aligned}d=&(|05|-5|14|+10|23|)t^2+(|06|-4|15|+5|24|)t\\&+(|16|-5|25|+10|34|).\end{aligned}$$

Syzygies:

I. $140b^2-945|a, a|^4+630|a, b|^2-81|a, d|=0.$

II. $14|a, b|^5-3|b, d|=0.$

III. $252|a, a|^6-140|b, b|^2+210|a, b|^4-54|a, d|^2+33|b, d|=0.$

IV. $243d^2-1890|a, a|^8-1400|b, b|^4+840|a, b|^6+540|b, d|^2=0.$

V. $126|a, a|^{10}-280|b, b|^6+27|d, d|^2=0.$

Two Septimics. **Linear combinants:**

$$\begin{aligned}a=&|01|t^{12}+6|02|t^{11}+3(5|03|+7|12|)t^{10}+10(2|04|+7|13|)t^9\\&+15(|05|+7|14|+7|23|)t^8+6(|06|+14|15|+35|24|)t^7\\&+(|07|+35|16|+189|25|+175|34|)t^6\\&+6(|17|+14|26|+35|35|)t^5+15(|27|+7|36|+7|45|)t^4\\&+10(2|37|+7|46|)t^3+3(5|47|+7|56|)t^2+6|57|t+|67|,\end{aligned}$$

$$\begin{aligned}b=&(|03|-3|12|)t^8+4(|04|-2|13|)t^7+2(3|05|-|14|-12|23|)t^6\\&+4(|06|+3|15|-9|24|)t^5+(|07|+13|16|-9|25|-45|34|)t^4\\&+4(|17|+3|26|-9|35|)t^3+2(3|27|-|36|-12|45|)t^2\\&+4(|37|-2|46|)t+(|47|-3|56|),\end{aligned}$$

$$d=(|05|-5|14|+10|23|)t^4+2(|06|-4|15|+5|24|)t^3$$
$$+(|07|-|16|-9|25|+25|34|)t^2+2(|17|-4|26|+5|35|)t$$
$$+(|27|-5|36|+10|45|),$$

$$f=|07|-7|16|+21|25|-35|34|.$$

Syzygies (formed with $924a$, $70b$, $6d$ and f):

I. $$15|a, a|^4-378b^2-144|a, b|^2+220ad=0.$$

II. $$5|a, a|^6-504|b, b|^2+45|a, b|^4-220|a, d|^2-1155bd+165af=0.$$

III. $$3|a, b|^5-5|a, d|^3-98|b, d|=0.$$

IV–1 and IV–2. Terms involving the following transvectants:

$$|a, a|^8,\ |b, b|^4,\ d^2,\ |a, b|^6,\ |a, c|^4,\ |b, d|^2,\ bf,\ |a, b|^7,\ |b, d|^3.$$

V–1 and V–2. Terms involving the following transvectants:

$$|a, a|^{10},\ |b, b|^6,\ |d, d|^2,\ |a, b|^8,\ |b, d|^4,\ df.$$

VI. $$5|a, a|^{12}+1386|b, b|^8+59,290|d, d|^4-457,380f^2=0$$

Linear Combinants of Three Binary Forms, $(\alpha t)^n$, $(\beta t)^n$, $(\gamma t)^n$, and the Syzygies of the Second Degree Connecting Them.

Three Cubics. Linear combinant:

$$A=|012|t^3+|013|t^2+|023|t+|123|.$$

Three Quartics. Linear combinants:

$$A=|012|t^6+2|013|t^5+(|014|+3|023|)t^4+(2|024|+4|123|)t^3$$
$$+(|034|+3|124|)t^2+2|134|t+|234|,$$

$$B=(|014|-2|023|)t^2+(|024|-8|123|)t+(|034|-2|124|).$$

Syzygy:

I. $$25|A, A|^4-B^2+5|A, B|^2=0.$$

Three Quintics. Linear combinants:

$$A=|012|t^9+3|013|t^8+(3|014|+6|023|)t^7,$$
$$+(|015|+8|024|+10|123|)t^6+(3|025|+6|034|+15|124|)t^5$$
$$+(3|035|+6|125|+15|134|)t^4+(|045|+8|135|+10|234|)t^3$$
$$+(3|145|+6|235|)t^2+3|245|t+|345|,$$

$$B=(|014|-2|023|)t^5+(|015|-10|123|)t^4+(2|025|-10|124|)t^3$$
$$+(2|035|-10|134|)t^2+(|045|-10|234|)t+(|145|-2|235|),$$

$$C=(2|015|-5|024|+20|123|)t^3+(3|025|-15|034|+15|124|)t^2$$
$$-(3|035|-15|125|+15|134|)t-(2|045|-5|135|+20|234|).$$

Syzygies (formed with $84A$, $5B$ and C):

I. $25|A, A|^4-294B^2-100|A, C|=0.$

II. $5|A, B|^3-8BC=0.$

III. $5|A, A|^6+294|B, B|^2+192C^2+40|A, C|^3=0.$

IV. $|A, B|^5-16|B, C|^2=0.$

V. $5|A, A|^8-196|B, B|^4+112|C, C|^2=0.$

Three Sextics. **Linear combinants:**

$$\begin{aligned}
A=&|012|t^{12}+4|013|t^{11}+(6|014|+10|023|)t^{10}\\
&+(4|015|+20|024|+20|123|)t^9\\
&+(|016|+15|025|+20|034|+45|124|)t^8\\
&+(4|026|+20|035|+36|125|+60|134|)t^7\\
&+(6|036|+10|045|+10|126|+64|135|+50|234|)t^6\\
&+(4|046|+20|136|+36|145|+60|235|)t^5\\
&+(|056|+15|146|+20|236|+45|245|)t^4\\
&+(4|156|+20|246|+20|345|)t^3\\
&+(6|256|+10|346|)t^2+4|356|t+|456|,
\end{aligned}$$

$$\begin{aligned}
B=&(|014|-2|023|)t^8+(2|015|-|024|-12|123|)t^7\\
&+|(016|+4|025|-2|034|-21|124|)t^6\\
&+(3|026|+4|035|-6|125|-32|134|t^5\\
&+(4|036|+3|045|+3|126|-16|135|-40|234|)t^4\\
&+(3|046|+4|136|-6|145|-32|235|)t^3\\
&+(|056|+4|146|-2|236|-21|245|)t^2\\
&+(2|156|-|246|-12|345|)t+|256|-2|346|,
\end{aligned}$$

$$\begin{aligned}
C=&(2|015|-5|024|+20|123|)t^6+(2|016|-20|034+30|124|)t^5\\
&+(5|026|-20|035|+30|125|)t^4-(20|045|-20|126|)t^3\\
&-(5|046|-20|136|+30|145)t^2-(2|056|-20|236|+30|245|)t\\
&-(2|156|-5|246|+20|345|),
\end{aligned}$$

$$\begin{aligned}
D=&(|016|-3|025|+5|034|)t^4\\
&+(2|026|-2|035|-18|125|+30|134|)t^3\\
&+(3|036|-3|045|-3|126|-12|135|+75|234|)t^2\\
&+(2|046|-2|136|-18|145|+30|235|)t\\
&+(|056|-3|146|+5|236|),
\end{aligned}$$

$$E=|036|-3|045|-3|126|+6|135|-15|234|.$$

Syzygies (formed with $6930A$, $560B$, $3C$, $2D$ and E):

I. $64|A, A|^4-405B^2-72|A, B|^2-7392|A, C|+3520AD=0,$

II. $|A, B|^3-14|A, C|^2-42BC-16|A, D|=0,$

III–1 and III–2. Involving the following transvectants:

$|A, A|^6$, $|B, B|^2$, C^2, $|A, B|^4$, $|A, C|^2$, $|B, C|$, AF, $|A, D|^2$, AD.

IV–1 and IV–2. Involving the following transvectants:

$|A, B|^5$, $|A, C|^4$, $|B, C|^2$, $|A, D|^3$ $|B, D|$, CD.

V–1, V–2 and V–3. Involving the following transvectants:

$|A, A|^8$, $|B, B|^4$, $|C, C|^2$, $|A, B|^6$, $|A, C|^5$, $|B, C|^3$, $|A, D|^4$, BF, $|C, D|^2$, D^2, $|C, D|$.

VI–1 and VI–2. Involving the following transvectants:

$|A, B|^7$, $|A, C|^6$, $|B, C|^4$, $|B, D|^3$, CF, $|C, D|^2$.

VII–1, VII–2 and VII–3. Involving the following transvectants:

$|A, A|^{10}$, $|B, B|^6$, $|C, C|^4$, $|A, B|^8$, $|B, C|^5$, $|B, D|^4$, $|C, D|^3$, DF, $|D, D|^2$.

VIII. $80|B, C|^6 - |C, D|^4 = 0$.

IX. Involving $|A, A|^{12}$, $|B, B|^8$, $|C, C|^6$, $|D, D|^4$, F^2.

Three Septimics. Linear combinants:

$$\begin{aligned}
A = {} & |012|t^{15} + 5|013|t^{14} + (10|014| + 15|023|)t^{13} \\
& + (10|015| + 40|024| + 35|123|)t^{12} \\
& + (5|016| + 45|025| + 50|034| + 105|124|)t^{11} \\
& + (|017| + 24|026| + 75|035| + 126|125| + 175|134|)t^{10} \\
& + (5|027| + 45|036| + 50|045| + 70|126| + 280|135| + 175|234|)t^{9} \\
& + (10|037| + 40|046| + 15|127| + 175|136| + 210|145| + 315|235|)t^{8} \\
& + (10|047| + 40|137| + 15|056| + 175|146| + 210|236| + 315|245|)t^{7} \\
& + (5|057| + 45|147| + 50|237| + 70|156| + 280|246| + 175|345|)t^{6} \\
& + (|067| + 24|157| + 75|247| + 126|256| + 175|346|)t^{5} \\
& + (5|167| + 45|257| + 50|347| + 105|356|)t^{4} \\
& + (10|267| + 40|357| + 35|456|)t^{3} \\
& + (10|367| + 15|457|)t^{2} + 5|467|t + |567|,
\end{aligned}$$

$$\begin{aligned}
B = {} & (|014| - 2|023|)t^{11} + (3|015| - 2|024| - 14|123|)t^{10} \\
& + (3|016| + 6|025| - 5|034| - 35|124|)t^{9} \\
& + (|017| + 10|026| + 5|035| - 21|125| - 70|134|)t^{8} \\
& + (4|027| + 15|036| + 5|045| + 7|126| - 70|135| - 105|234|)t^{7} \\
& + (7|037| + 14|046| + 7|127| - 49|145| - 147|235|)t^{6} \\
& + (7|047| + 14|137| + 7|056| - 49|236| - 147|245|)t^{5} \\
& + (4|057| + 15|147| + 5|237| + 7|156| - 70|246| - 105|345|)t^{4} \\
& + (|067| + 10|157| + 5|247| - 21|256| - 70|346|)t^{3} \\
& + (3|167| + 6|257| - 5|347| - 35|356|)t^{2} \\
& + (3|267| - 2|357| - 14|456|)t + (|367| - 2|457|),
\end{aligned}$$

$$\begin{aligned}C=&(2|015|-5|024|+20|123|)t^9+(4|016|-3|025|-25|034|+45|124|)t^8\\&+(2|017|+9|026|-45|035|+57|125|+25|134|)t^7\\&+(7|027|-15|036|-60|045|+59|126|+15|135|+50|234|)t^6\\&+(5|037|-45|046|+27|127|+55|136|-90|145|+60|235|)t^5\\&-(5|047|-45|137|+27|056|+55|146|-90|236|+60|245|)t^4\\&-(7|057|-15|147|-60|237|+59|156|+15|246|+50|345|)t^3\\&-(2|067|+9|157|-45|247|+57|256|+25|346|)t^2\\&-(4|167|-3|257|-25|347|+45|356|)t-(2|267|-5|357+20|456|),\end{aligned}$$

$$\begin{aligned}D=&(|016|-3|025|+5|034|)t^7+(|017|-21|125|+35|134|)t^6\\&+(3|027|-21|126|+105|234|)t^5+(5|037|-35|136|+105|235|)t^4\\&+(5|047|-35|146|+105|245|)t^3+(3|057|-21|156|+105|345|)t^2\\&+(|067|-21|256|+35|346|)t+(|167|-3|257|+5|347|),\end{aligned}$$

$$\begin{aligned}E=&(2|017|-7|026|+7|035|+21|125|-35|134|)t^5\\&+(5|027|-21|036|+28|045|-7|126|+49|135|-210|234|)t^4\\&+(2|037|-14|046|+14|127|-42|136|+196|145|-168|235|)t^3\\&-(2|047|-14|137|+14|056|-42|146|+196|236|-168|245|)t^2\\&-(5|057|-21|147|+28|237|-7|156|+49|246|-210|345|)t\\&-(2|067|-7|157|+7|247|+21|256|-35|346|),\end{aligned}$$

$$\begin{aligned}F=&(|036|-3|045|-3|126|+6|135|-15|234|)t^3\\&+(|037|-2|046|-3|127|+4|136|+3|145|-9|235|)t^2\\&+(|047|-2|137|-3|056|+4|146|+3|256|-9|245|)t\\&+(|147|-3|237|-3|156|+6|246|-15|345|).\end{aligned}$$

Syzygies. In the following table we have enumerated the syzygies and indicated the transvectants which they will involve:

I. $|A,A|^4$, B^2, $|A,B|^2$, $|A,C|$, AD.

II. $|A,B|^3$, $|A,C|^2$, $|A,D|$, AE, BC.

III–1 and 2. $|A,A|^6$, $|B,B|^2$, C^2, $|A,B|^4$, $|A,C|^3$, $|A,D|^2$, $|A,E|$, $|B,C|$, AF, BD.

IV–1 to 3. $|A,B|^5$, $|A,C|^4$, $|A,D|^3$, $|A,E|^2$, $|A,F|$, $|B,C|^2$, $|B,D|$, BE, CD.

V–1 to 4. $|A,A|^8$, $|B,B|^4$, $|C,C|^2$, D^2 $|A,B|^6$, $|A,C|^5$, $|A,D|^4$, $|A,E|^3$, $|A,F|^2$, $|B,C|^3$, $|B,D|^2$, $|B,E|$, BF, $|C,D|$, CE.

VI–1 to 4. $|A,B|^7$, $|A,C|^6$, $|A,D|^5$, $|A,E|^4$, $|A,F|^3$, $|B,C|^4$, $|B,D|^3$, $|B,E|^2$, $|B,F|$, $|C,D|^2$, $|CE|$, CF, DE.

VII–1 to 6. $|A,A|^{10}$, $|B,B|^6$, $|C,C|^4$, $|D,D|^2$, E^2, $|A,B|^8$, $|A,C|^7$, $|A,D|^6$, $|A,E|^5$, $|B,C|^5$, $|B,D|^4$, $|B,E|^3$, $|B,F|^2$, $|C,D|^3$, $|C,E|^2$, $|C,F|$, $|D,E|$, DF.

VIII–1 to 5. $|A, B|^9$, $|A, C|^8$, $|A, D|^7$, $|B, C|^6$, $|B, D|^5$, $|B, E|^4$, $|B, F|^3$, $|C, D|^4$, $|C, E|^3$, $|C, F|^2$, $|D, E|^2$, $|D, F|$, EF.

IX–1 to 5. $|A, A|^{12}$, $|B, B|^8$, $|C, C|^6$, $|D, D|^4$, $|E, E|^2$, F^2, $|A, B|^{10}$, $|A, C|^9$, $|B, C|^7$, $|B, D|^6$, $|B, E|^5$, $|C, D|^5$, $|C, E|^4$, $|C, F|^3$, $|D, E|^3$, $|D, F|^2$, $|E, F|$.

X–1 to 4. $|A, B|^{11}$, $|B, C|^8$, $|B, D|^7$, $|C, D|^6$, $|C, E|^5$ $|D, E|^4$, $|D, F|^3$, $|E, F|^2$.

XI–1 to 3. $|A, A|^{14}$, $|B, B|^{10}$, $|C, C|^8$, $|D, D|^6$, $|E, E|^4$, $|F, F|^2$, $|B, C|^9$, $|C, D|^7$, $|D, E|^5$, $|E, F|^3$.

XI. *Various Methods for Checking the Computation of the Linear Combinants.*

(1). It has been shown by Clebsch* that the system of elementary covariants of a binary form with several sets of variables contains just as many linearly independent constants as the form itself. This theorem is modified by Stroh† to apply to the linear combinants of a system of p forms of order n. If the number of linearly independent constants—determinants from the matrix of the coefficients of the n-ics—be given by $\binom{n+1}{p}$, we must have the relation

$$\Sigma(\lambda_i+1)=\binom{n+1}{p},$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots.$ are the orders of the linear combinants belonging to the p forms. This says, in a word, that the total number of coefficients of the linear combinants of p forms of order n must be the same as the number of different p-rowed determinants which can be formed from the matrix of the coefficients of the n-ics.

(2). The symmetry of the linear combinants gives rise to a very convenient check on the accuracy of the work. The numerical multipliers of the various determinants are the same for terms the same distance from each end of the combinant, and the determinants having the same multipliers are complements of each other. By complementary determinants with reference to a system of p binary n-ics we mean two determinants of the same number of rows and columns so related that the one can be formed from the other by reversing its columns and then subtracting each integer from n. Thus, the complement of

$$|a, b, c, \ldots., p| \text{ is } |n-p, \ldots., n-c, n-b, n-a|.$$

In each case in the present paper, all the terms in the combinant were actually computed by the methods set down, and then these checks were applied.

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* Clebsch, "Binäre Formen," p. 39. † *Math. Ann.*, Vol. XXII, p. 404.

XII. *Methods for Checking the Computation of the Syzygies.*

For the syzygies we have two methods of checking:

(1). The total number of coefficients in all the forms of second degree that can be gotten from the linear combinants of any system of p binary n-ics is the same as the total number of products of p-rowed determinants, taken two at a time, which can be formed from the $(n+1)p$ coefficients of the matrix. This check enables us to be sure that we have made no omissions when we set down the transvectants of the linear combinants.

(2). An extension of the theorem stated for the number of coefficients in the linear combinants can be made for the syzygies. It is: *The number of coefficients in all the syzygies of a system of* p *binary* n*-ics is the same as the number of independent determinant identities of degree* 2.

The actual arithmetical operations of all the involved computations of the syzygies were checked on the "Millionaire" computing machine.

XIII. *The order of a linear combinant may be expressed in terms of the weight of the leading coefficient and the common order of the binary forms of the system.*

Consider a system of two binary n-ics. Call the leading coefficient of any one of its combinants w_0, and suppose that this leading coefficient involves the determinant $|a, b|$, whence $w_0 = a+b-1$; then the corresponding determinant in the last coefficient will be its complement, $|n-a, n-b|$, of weight $2n-a-b-1$. Since the weight increases by one in each coefficient after the first, we have for the number of coefficients, less one, in the combinant,

$$2n-a-b-1-a-b+1 = 2n-2w_0-2.$$

But the order r of the combinant is also a number one less than the number of coefficients, whence

$$r = 2n-2w_0-2.$$

This may easily be extended for the case of p binary n-ics. Suppose one of the determinants in the leading coefficient is $|a_1, a_2, \ldots, a_p|$, of weight

$$w_0 = \sum_{i=1}^{p} a_i - \sum_{s=0}^{p-1} (s+1);$$

then the corresponding determinant in the last coefficient is

$$|n-a_p, \ldots, n-a_2, n-a_1|,$$

of weight

$$p \cdot n - \sum_{i=1}^{p} a_i - \sum_{s=0}^{p-1} (s+1).$$

The number of coefficients in the combinant, less one, is then given by

$$r=p\cdot n-2w_0-2\sum_{s=0}^{p-1}(s+1).$$

This is then the formula for the order of the combinant, since the order is one less than the number of coefficients in the combinant. This formula gives a convenient method of determining the order of a linear combinant when the weight of the leading coefficient is known. The term $\sum_0^{p-1}(s+1)$ is introduced in the expression for the weight, so that the determinant $|0, 1, 2, \ldots, (p-1)|$ shall always be of weight zero.

For the special case of a system of three binary n-ics, the above general formula becomes

$$r=3n-2w_0-6.$$

XIV. *The Enumeration of the Quadratic Syzygies Connecting the Linear Combinants of Any System of Binary Forms of the Same Order, by Means of the Number and Weight of the Possible Determinant Identities of the Second Degree.*

It has already been shown (Section XI) that there are just as many different determinants which may be formed from the matrix of the coefficients of a system of binary forms of a common order as there are coefficients in all the linear combinants of that system. By means of this relation it is quite possible to make an enumeration of the linear combinants of a system by counting the number and weights of the determinants. However, the work of actually computing the explicit forms offers too few numerical difficulties to make such an enumeration of much profit.

On the other hand, in the case of the quadratic syzygies, where the computation is often much involved, there is much to be gained through a similar enumeration. The quadratic forms which must be accounted for are of two distinct classes. First, we have the odd transvectants of a linear combinant on itself. These odd transvectants vanish identically and so will not enter into the actual formation of the syzygies. Second, we have the even transvectants of a linear combinant on itself, the transvectants of a linear combinant on another linear combinant of the same system, the products of two linear combinants and their squares. The totality of members of this second class are the constituents of the syzygies, into which they are formed by means of the relations of the various coefficients, aided by the determinant identities of the forms given in Section VIII. We have shown that the number of coefficients in the quadratic syzygies of any system is the same as the number of quadratic

determinant identities. If, now, we determine the number of determinant identities of each possible weight, we have from their totality the number of coefficients in the complete system of syzygies. We have noticed, in the formation of the syzygies whose explicit forms we have developed, that each coefficient in the syzygy requires *one* identity of the same weight as that coefficient. Hence, from the weights of the determinant identities, we can easily predict the weights of all the coefficients which can occur in the syzygies, and thus, as we shall see presently, can make a complete enumeration of the syzygies. In the succeeding sections this method of enumeration is carried out for the case of two septimics, three sextics, and three septimics.

XV. *Enumeration of the Quadratic Syzygies of the System of Two Binary Septimics.*

The linear combinants are, $a\equiv(at)^{12}$ of weight 0, $b\equiv(bt)^{8}$ of weight 2, $d\equiv(dt)^{4}$ of weight 4, and $f\equiv(ft)^{0}$ of weight 6. From these we may form the following quadratic products and transvectants:

Table 1.

Forms.	Ord.	Wt.
a^2	24	0
$\|a, a\|^2$, ab	20	2
$\|a, b\|$	18	3
$\|a, a\|^4$, b^2, $\|a, b\|^2$, ad	16	4
$\|a, b\|^3$, $\|a, d\|$	14	5
$\|a, a\|^6$, $\|b, b\|^2$, $\|a, b\|^4$, $\|a, d\|^2$, af, bd	12	6
$\|a, b\|^5$, $\|a, d\|^3$, $\|b, d\|$	10	7
$\|a, a\|^8$, $\|b, b\|^4$, d^2, $\|a, b\|^6$, $\|a, d\|^4$, $\|b, d\|^2$, bf	8	8
$\|a, b\|^7$, $\|b, d\|^3$	6	9
$\|a, a\|^{10}$, $\|b, b\|^6$, $\|d, d\|^2$, $\|a, b\|^8$, $\|b, d\|^4$, df	4	10
$\|a, a\|^{12}$, $\|b, b\|^8$, $\|d, d\|^4$	0	12

In the case of two septimics, the numbers in our symbolic determinants are 0, 1, 2, 3, 4, 5, 6, 7, from which we can form 70 determinant identities of Type 1. They are distributed as follows:

Table 2.

Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
No.	1	1	2	3	5	5	7	7	8	7	7	5	5	3	2	1	1

Since we require one identity for each coefficient of the same weight as the identity, and our lowest identity is of weight 4, we can have no syzygies con-

taining coefficients of weight less than 4. We have one set of transvectants the weight of whose leading coefficients is 4. The forms in it are of order 16; hence the syzygy will be of order 16. As each term has a weight one greater than the preceding, the syzygy will require one identity of each weight from 4 to 20, inclusive. After deducting these from our total, 53 identities are left, the lowest one having a weight of 6. There can be no syzygy of weight 5. One set of transvectants, of order 12, contains a coefficient of weight 6. To admit one syzygy of this weight will require one identity of each weight from 6 to 18. Our list still contains enough identities for this one syzygy, but it exhausts our identities of weight 6 and 18; hence we can admit but one such syzygy. After it is formed, there are left 40 identities, in which there is *one* only of the lowest weight 7. We can therefore admit one syzygy of weight 7; it is of order 10, so that it uses up one identity of each weight from 7 to 17, leaving 29 identities, the lowest weight being 8, of which there are *two* identities. Since we have two identities of weight 8, we may have two syzygies of that weight. These are of order 8, and so will use up two identities of each weight from 8 to 16; and we have 11 identities remaining. Again there are two identities of the lowest weight 10, so that we may have two syzygies of weight 10. They are of order 4, and so use up ten of the identities, leaving one identity of weight 12, which is all we need to admit one syzygy of order zero and weight 12.

The distribution of the identities in the various terms of the syzygies, as well as the number of syzygies, shows up to advantage in Table 3. The numbers at the heads of the columns indicate the weights of the identities numbered. The footings of the columns must of necessity be identical with the number of identities of each weight given in Table 2.

Table 3.

Syz.	Ord.	Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I.	16	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
II.	12	6			1	1	1	1	1	1	1	1	1	1	1	1	1		
III.	10	7				1	1	1	1	1	1	1	1	1	1	1			
IV–1.	8	8					1	1	1	1	1	1	1	1	1				
IV–2.	8	8					1	1	1	1	1	1	1	1	1				
V–1.	4	10							1	1	1	1	1						
V–2.	4	10							1	1	1	1	1						
VI.	0	12									1								
Totals			1	1	2	3	5	5	7	7	8	7	7	5	5	3	2	1	1

XVI. *Enumeration of the Quadratic Syzygies for Three Binary Sextics.*

The linear combinants are:

	$(At)^{12}$,	$(Bt)^{8}$,	$(Ct)^{6}$,	$(Dt)^{4}$,	$(Ft)^{0}$,
of weight	0,	2,	3,	4,	6,

The enumeration of the transvectants is given in Table 1, which follows.

Table 1.

Transvectants.

No.	Ord.	Wt.
1	24	0
2	20	2
2	18	3
5	16	4
4	14	5
9	12	6
6	10	7
11	8	8
6	6	9
9	4	10
2	2	11
5	0	12
62	Total.	

Table 2.

Identities.

No.	Wt.
1	4
2	5
4	6
6	7
9	8
11	9
14	10
15	11
16	12
15	13
14	14
11	15
9	16
6	17
4	18
2	19
1	20
140	Total.

Table 3.

Syz.	Ord.	Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
I.	16	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
II.	14	5		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
III–1.	12	6			1	1	1	1	1	1	1	1	1	1	1	1	1		
III–2.					1	1	1	1	1	1	1	1	1	1	1	1	1		
IV–1.	10	7				1	1	1	1	1	1	1	1	1	1	1			
IV–2.						1	1	1	1	1	1	1	1	1	1	1			
V–1.	8	8					1	1	1	1	1	1	1	1	1				
V–2.							1	1	1	1	1	1	1	1	1				
V–3.							1	1	1	1	1	1	1	1	1				
VI–1.	6	9						1	1	1	1	1	1	1					
VI–2.								1	1	1	1	1	1	1					
VII–1.	4	10							1	1	1	1	1						
VII–2.									1	1	1	1	1						
VII–3.									1	1	1	1	1						
VIII.	2	11								1	1	1							
IX.	0	12									1								
Totals			1	2	4	6	9	11	14	15	16	15	14	11	9	6	4	2	1

In this case we have two types of determinant identities. There are 105 of Type 2, each counting for one condition, and seven of Type 3, each representing five independent identities. Hence, we have altogether 140 independent quadratic identities whose weights are distributed as in Table 2.

Without repeating such discussion as we gave in the previous section, we append herewith a Table 3, similar to that of Section XV, which shows the number of syzygies of each weight and order that can occur.

XVII. *Enumeration of the Quadratic Syzygies for Three Binary Septimics.*

The linear combinants are:

	$(At)^{15}$,	$(Bt)^{11}$,	$(Ct)^{9}$,	$(Dt)^{7}$,	$(Et)^{5}$,	$(Ft)^{3}$,
of weight	0,	2,	3,	4,	5,	6.

There are in all 128 transvectant forms from which to form the syzygies. They are enumerated in Table 1.

The 420 independent identities are noted in Table 2. There are 280 identities of Type 2, and 28 forms of Type 3, which are equivalent to 140 independent identities.

A Table 3, similar to that presented in the two sections immediately preceding this, is given here for the tabular enumeration of the syzygies.

Table 1.

Transvectants.

No.	Ord.	Wt.
1	30	0
2	26	2
2	24	3
5	22	4
5	20	5
10	18	6
9	16	7
15	14	8
13	12	9
18	10	10
13	8	11
17	6	12
8	4	13
10	2	14
128	Total.	

Table 2.

Identities.

No.	Wt.	No.	Wt.
1	4	1	26
2	5	2	25
4	6	4	24
7	7	7	23
11	8	11	22
15	9	15	21
21	10	21	20
26	11	26	19
31	12	31	18
35	13	35	17
38	14	38	16
38	15		

Total identities, 420.

Table 3.

Syz.	Ord.	Wt.	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
I.	22	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
II.	20	5		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
III–1.	18	6			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
III–2.					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
IV–1.	16	7				1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
IV–2.						1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
IV–3.						1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1			
V–1.	14	8					1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
V–2.							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
V–3.							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
V–4.							1	1	1	1	1	1	1	1	1	1	1	1	1	1	1				
VI–1.	12	9						1	1	1	1	1	1	1	1	1	1	1	1	1					
VI–2.								1	1	1	1	1	1	1	1	1	1	1	1	1					
VI–3.								1	1	1	1	1	1	1	1	1	1	1	1	1					
VI–4.								1	1	1	1	1	1	1	1	1	1	1	1	1					
VII–1.	10	10							1	1	1	1	1	1	1	1	1	1	1						
VII–2.									1	1	1	1	1	1	1	1	1	1	1						
VII–3.									1	1	1	1	1	1	1	1	1	1	1						
VII–4.									1	1	1	1	1	1	1	1	1	1	1						
VII–5.									1	1	1	1	1	1	1	1	1	1	1						
VII–6.									1	1	1	1	1	1	1	1	1	1	1						
VIII–1.	8	11								1	1	1	1	1	1	1	1	1							
VIII–2.										1	1	1	1	1	1	1	1	1							
VIII–3.										1	1	1	1	1	1	1	1	1							
VIII–4.										1	1	1	1	1	1	1	1	1							
VIII–5.										1	1	1	1	1	1	1	1	1							
IX–1.	6	12									1	1	1	1	1	1	1								
IX–2.											1	1	1	1	1	1	1								
IX–3.											1	1	1	1	1	1	1								
IX–4.											1	1	1	1	1	1	1								
IX–5.											1	1	1	1	1	1	1								
X–1.	4	13										1	1	1	1	1									
X–2.												1	1	1	1	1									
X–3.												1	1	1	1	1									
X–4.												1	1	1	1	1									
XI–1.	2	14											1	1	1										
XI–2.													1	1	1										
XI–3.													1	1	1										
Totals			1	2	4	7	11	15	21	26	31	35	38	38	38	35	31	26	21	15	11	7	4	2	1

XVIII. *Bibliography.*

We have appended here a list of the mathematical papers that have to do with the combinant theory. Much of great value aside from these papers has been found in the various standard texts on invariants and covariants, among which may be mentioned particularly: Grace and Young, "Algebra of Invariants"; Clebsch, "Binäre Formen"; Study, "Ternaere Formen"; and Gordan, "Invariantentheorie."

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BALTIMORE, MD., *April* 1, 1914.

Élimination d'une Inconnue entre Plusieurs Équations Algébriques.

PAR M. STUYVAERT (GAND).

1. Le problème de l'élimination d'une inconnue entre plusieurs équations algébriques se résout par la méthode de Kronecker. Ou plutôt l'artifice de Kronecker ramène le système de plusieurs équations $F_i=0$ à deux équations

$$\Sigma u_i F_i=0, \quad \Sigma v_i F_i=0,$$

où les u et les v sont des indéterminées; ainsi se trouve indiquée une marche à suivre pour résoudre la question.

Il restait à examiner quel résultat on obtient quand on applique effectivement le procédé. C'est ce qu'a fait M. Lloyd L. Dines dans la première partie de son intéressant mémoire "The Highest Common Factor of a System of Polynomials in One Variable" (AMERICAN JOURNAL OF MATHEMATICS, t. XXXV, p. 129); il résout même un problème plus général que celui de l'élimination, car il obtient les conditions d'éxistence d'un diviseur commun de degré quelconque.

Nous allons arriver aux mêmes résultats par une autre voie. S'il s'agissait seulement de démontrer d'une manière nouvelle ou même d'étendre les théorèmes de M. Lloyd L. Dines, le fait ne mériterait guère de retenir l'attention. Mais nous croyons qu'il y a plus.

L'artifice de Kronecker présente un inconvénient, non seulement pour la solution du problème qui nous occupe, mais aussi pour l'usage qu'on en fait dans les théories ultérieures.*

Voici cet inconvénient. Supposons que la première des équations proposées $F_1=0$ soit de degré supérieur aux autres. Le résultant de

$$\Sigma u_i F_i, \quad \Sigma v_i F_i$$

s'annule, quels que soient les u et les v, quand le premier coefficient a_0 de F_1 s'évanouit. Si l'on veut formuler des conclusions, il faut donc supposer les formes régulières, c'est-à-dire telles que le coefficient de la plus haute puissance de x n'est pas nul.

Mais si l'on opérait sur des polynomes dont les coefficients sont à leur tour fonctions de variables, il faudrait pouvoir admettre les cas d'évanouissement de ces coefficients.

* Cf. J. König, "Algebraischen Grössen," p. 102.

Ainsi J. König (*loc. cit.*, p. 97) observe que le résultant de deux polynomes s'annule quand ils ont un facteur commun en x ou quand leurs premiers coefficients s'annulent *tous deux*; que le discriminant (p. 120) d'un polynome s'évanouit ou que ses *deux* premiers coefficients s'annulent.

Il y a intérêt à conserver l'analogie et à chercher des conditions qui se réalisent, quand tous les polynomes F_i ont un diviseur commun en x, ou quand *tous* les premiers coefficients s'annulent. Ces deux alternatives se confondent quand on rend les polynomes homogènes, en particulier quand on cherche des conditions applicables à la géométrie projective.

Ces conditions sont précisément celles que M. Lloyd L. Dines a trouvées. Nous le montrons, dans les lignes suivantes, sans recourir à l'artifice de Kronecker et nos résultats dispensent de ce même artifice pour la théorie générale de l'élimination.

La marche que nous suivons s'arrête assez longuement au cas de deux polynomes en x et leur applique la méthode connue d'Euler, mais élargie et assouplie; on arrive ainsi à des résultats plus généraux et s'étendant sans peine à plusieurs polynomes.

2. Rappelons qu'une matrice M est dite de *rang* r lorsque tous les déterminants à $r+1$ lignes qu'on en peut extraire sont nuls, mais que l'un au moins de ses déterminants à r lignes diffère de zéro.

On vérifie immédiatement cette propriété: si une matrice M est de rang r, supposons qu'on la fasse précéder (ou suivre) d'une ligne ou d'une colonne d'éléments *arbitraires*; la nouvelle matrice est de rang $r+1$ en général, et exceptionnellement de rang r; mais si l'on a ajouté une rangée d'éléments *nuls*, le rang demeure inaltéré.

Voici une propriété tout aussi facile: on ne change pas le rang d'une matrice quand on ajoute, aux éléments d'une ligne ou colonne, les éléments correspondants d'une rangée parallèle multipliés par un facteur commun.

L'application la plus connue de la théorie des matrices concerne le système de m équations linéaires homogènes à n inconnues,

$$a_{i1}x_1+a_{i2}x_2+\ldots\ldots+a_{in}x_n=0, \qquad (i=1,2,\ldots\ldots,m).$$

On dit, pour abréger, que les solutions du système sont en nombre ∞^{n-r} quand on peut prendre à volonté $n-r$ des inconnues, convenablement choisies, et définir ainsi, sans ambiguité, un système de solutions non toutes nulles.

La propriété bien connue des systèmes linéaires peut s'énoncer: pour que le système ait ∞^{n-r} solutions, il faut et il suffit que la matrice des coefficients soit de rang r.

Lorsque le nombre m d'équations est inférieur au nombre n d'inconnues, le rang de la matrice ne peut dépasser m, donc on peut prendre au moins $n-m$ inconnues à volonté.

3. Considérons deux polynomes

$$\left.\begin{aligned} F&=a_0x^m+a_1x^{m-1}+\ldots.+a_m, \\ G&=b_0x^n+b_1x^{n-1}+\ldots.+b_n, \end{aligned}\right\} \qquad (m\geq n).$$

La méthode d'élimination d'Euler, étendue à la recherche des conditions pour l'existence d'un diviseur commun de degré α au moins, consiste à multiplier F et G par des polynomes inconnus, de degrés respectifs $n-\alpha$ et $m-\alpha$.

Pour généraliser davantage, multiplions F par un polynome *de degré arbitraire* h,

$$P=p_0x^h+p_1x^{h-1}+\ldots.+p_h,$$

et G par un polynome

$$Q=q_0x^k+q_1x^{k-1}+\ldots.+q_k,$$

de degré $k=m+h-n$. Ainsi les deux produits FP et GQ sont du même degré $m+h$, et l'on dispose des coefficients inconnus de P et de Q de façon que la somme

$$FP+GQ$$

soit identique à zéro.

Dans ce but il faut ecrire le système de $m+h+1$ relations

$$\left.\begin{array}{l} a_0p_0 \qquad\qquad\quad +b_0q_0 \qquad\qquad\quad =0, \\ a_1p_0+a_0p_1 \qquad +b_1q_0+b_0q_1 \qquad =0, \\ a_2p_0+a_1p_1+a_0p_2+b_2q_0+b_1q_1+b_0q_1=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (1)$$

linéaires et homogènes en p_i, q_i; ces coefficients sont en nombre

$$h+1+(m+h-n+1)=m-n+2h+2.$$

Formons la matrice des coefficients des équations (1),

$$A=\underbrace{\left\|\begin{array}{cccc} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & a_0 & \\ . & a_2 & a_1 & \ldots. \\ . & . & a_2 & \ldots. \\ \ldots & \ldots & \ldots & \ldots. \\ & & & \ldots. \end{array}\right.}_{h+1 \text{ col.}}\underbrace{\left.\begin{array}{cccc} b_0 & & & \\ b_1 & b_0 & & \\ b_2 & b_1 & b_0 & \\ . & b_2 & b_1 & \ldots. \\ . & . & b_2 & \ldots. \\ \ldots & \ldots & \ldots & \ldots. \\ & & & \ldots. \end{array}\right\|}_{m+h-n+1 \text{ col.}} (m+h+1 \text{ lignes}).$$

Si le nombre des relations (1) est au moins égal à celui des inconnues, c'est-à-dire si l'on a

$$h\leq n-1,$$

le système (1) n'admet, en général, que des solutions toutes nulles. Exceptionnellement il admet ∞^l solutions si la matrice M est de rang

$$(m-n+2h+2)-l.$$

Si le nombre des relations (1) est inférieur à celui des inconnues de λ unités,

$$\lambda=(m-n+2h+2)-(m+h+1)=h-n+1,$$

ce qui suppose $h > n-1$, le système (1) admet, en général, ∞^{λ} solutions, et exceptionnellement ∞^{l} $(l > \lambda)$ si la matrice M (qui a maintenant moins de lignes que de colonnes) est de rang

$$(m-n+2h+2)-l;$$

ce nombre est d'ailleurs inférieur au nombre $m+h+1$ de lignes, si l dépasse λ.

Ainsi nous avons obtenu la condition pour qu'il y ait, entre les polynomes F et G, ∞^{l} identités telles que

$$FP+GQ=0.$$

4. Si les polynomes F et G n'ont pas de diviseur commun en x, et si l'on n'a pas $a_0=b_0=0$, deux circonstances que l'on peut réunir en rendant les polynomes homogènes en x_1, x_2, nous allons voir que la matrice M a son rang maximé, en d'autres termes qu'elle est distincte de zéro, *quel que soit h.*

La chose va de soi si la matrice est carrée, car alors M est le résultant de F et G; s'il était nul, les polynomes rendus homogènes auraient un diviseur commun.

Si dans la matrice M, le nombre de lignes dépasse le nombre c de colonnes, ce qui suppose $h < n-1$, soit $c-\gamma$ son rang. Ajoutons une première ligne d'éléments nuls, ce qui conserve le rang, puis deux colonnes d'éléments a_0, a_1, a_2, ..., a_m, 0, 0, ... et b_0, b_1, b_2, ..., b_n, 0, 0, ...; nous formons une matrice analogue à M, à une ligne et deux colonnes de plus, et dont le rang ne peut dépasser $c+2-\gamma$. Après $n-1-h$ opérations pareilles, nous aboutissons au résultant, et nous voyons que son rang ne peut dépasser $m+n-\gamma$; si donc γ n'est pas nul, F et G rendus homogènes ont un diviseur commun, contrairement à l'hypothèse.

Enfin si la matrice M a moins de lignes que de colonnes, supposons, s'il est possible, qu'elle soit nulle. Ceci se réalise pour $a_0=b_0=0$, car a_0 et b_0 sont les seuls éléments de la première ligne; mais l'hypothèse $a_0=b_0=0$ a été écartée; nous pouvons effacer dans M la première ligne et les deux colonnes qui débutent par les éléments a_0, b_0 de la première ligne; en effet tous les déterminants extraits de la matrice restante par simple suppression de colonnes, multipliés par un nombre non nul a_0 ou b_0, donnent des résultants nuls, savoir des déterminants de la matrice initiale M. L'opération, poursuivie de proche en proche, mène encore au résultant qui est nul, donc F et G ont un diviseur commun, contrairement à l'hypothèse.

5. Mais considérons à présent le cas où les polynomes F, G ont un plus grand commun diviseur de degré α, en y comprenant le cas où l'on a $a_0=b_0=0$ et le p. g. c. d. de degré $\alpha-1$, etc. Bref les polynomes F et G sont les produits de deux polynomes f et g (sans diviseur commun en x, dont les premiers coefficients m sont pas tous deux nuls, de degrés $m-\alpha$ et $n-\alpha$) par un polynome δ de degré égal ou inférieur à α.

Choisissons h de façon que, pour ces quotients f et g, les relations (1) soient en nombre

$$m-\alpha+h+1$$

inférieur aux inconnues,

$$m-n+2h+2,$$

ce qui suppose $h>n-\alpha-1$. Alors, d'après ce qu'on vient de voir, les identités

$$fP+gQ\equiv 0$$

sont possibles exactement de

$$\infty^{h-(n-\alpha-1)}$$

manières (car il y a $h-(n-\alpha-1)$ inconnues de plus que d'équations, et la matrice des coefficients n'est pas nulle), et chacune de ces identités en donne une autre

$$\delta fP+\delta gQ\equiv FP+GQ\equiv 0,$$

et réciproquement, car δ n'est pas identique à zéro. Les polynomes F et G sont donc dans le cas exceptionnel, et la matrice M est de rang

$$(m-n+2h+2)-h+(n-\alpha-1)=m+h+1-\alpha,$$

et ce nombre est inférieur à celui $(m+h-1)$ des lignes, car α est positif.

Par suite évidemment, pour que les polynomes aient un p. g. c. d. de degré α *au moins*, il faut et il suffit que la matrice soit de rang inférieur ou égal à $m+h+1-\alpha$. En particulier, si l'on fait $h=n-\alpha$, le rang de la matrice M est tout au plus

$$(m-n+2h+2)-(n-\alpha)+(n-\alpha-1)=(m-n+2h+2)-1,$$

c'est-à-dire, tout simplement que la matrice M (à $m+n-2\alpha+2$ colonnes et $m+n-\alpha+1$ lignes) est nulle; c'est le résultat classique pour les conditions d'un p. g. c. d. de degré α au moins.

Comme autre cas particulier connu, si l'on fait $h=n-1$, la matrice M devient le résultant, et l'on voit que le rang du résultant est $m+n-\alpha$ quand α est le degré du p. g. c. d. Enfin pour $\alpha-1$, on retombe sur la propriété fondamentale du résultant, utilisée dans la démonstration ci-dessus.

On peut formuler le résultat de la manière suivante:

Si l'on donne à h successivement les valeurs 0, 1, 2, ..., *les matrices analogues à M ont chacune un échelon de plus que la précédente. La dernière d'entre elles qui n'est pas nulle donne, par le nombre $n-\alpha$ de ces colonnes qui contiennent des coefficients a, le degré du quotient g de G par le p. g. c. d. δ. La suivante est simplement nulle: son rang est inférieur d'une unité au nombre de ses colonnes, et de α unités à celui de ses lignes. Toutes les autres ont aussi leur rang inférieur de α au nombre de leurs lignes.*

6. Elucidons ce qui précède par un exemple. Soient

$$F=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4,$$
$$G=b_0x^3+b_1x^2+b_2x+b_3.$$

Pour fixer les idées, prenons P du troisième degré, donc Q du quatrième, ce qui donne huit relations,

$$\left.\begin{array}{l} a_0p_0 \qquad\quad +b_0q_0 \qquad\quad =0, \\ a_1p_0+a_0p_1 \quad +b_1q_0+b_0q_1=0, \\ \dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{1}$$

La matrice

$$M=\left\|\begin{array}{cccccccccc} a_0 & & & & b_0 & & & & \\ a_1 & a_0 & & & b_1 & b_0 & & & \\ a_2 & a_1 & a_0 & & b_2 & b_1 & b_0 & & \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & b_1 & b_0 & \\ a_4 & a_3 & a_2 & a_1 & & b_3 & b_2 & b_1 & b_0 \\ & a_4 & a_3 & a_2 & & & b_3 & b_2 & b_1 \\ & & a_4 & a_3 & & & & b_3 & b_2 \\ & & & a_4 & & & & & b_3 \end{array}\right\|$$

a huit lignes et neuf colonnes. Si F et G ont un p. g. c. d. quadratique δ, les quotients f et g admettent ∞^3 identités $fP+gQ\equiv 0$ (car les relations analogues à (1) pour f et g sont au nombre de six à neuf inconnues). Pour F et G, les huit relations (1) à neuf inconnues devraient avoir en général ∞^1 systèmes de solutions, mais ici elles en ont ∞^3, donc la matrice M est de rang $8-2=6$.

7. À propos de cet exemple, donnons deux corollaires; on verra sans peine qu'ils s'appliquent au cas le plus général.

I. Si dans la matrice M à huit lignes et neuf colonnes, on ajoute aux éléments des quatre premières colonnes respectivement ceux des colonnes 6, 7, 8, 9 multipliés par une indéterminée w_1, on obtient la même matrice M relative aux polynomes $F+w_1G$ et G. Cette opération ne change pas le rang de la matrice; la même opération où w_1 est remplacé par w_0 et les colonnes 6, 7, 8, 9 par les colonnes 5, 6, 7, 8, donne la matrice M relative à $F+w_0xG$ et G; les deux opérations simultanées donnent la matrice M relative à $F+wG$ et G, où w est un polynome du premier degré à coefficients indéterminés.

Cette remarque est générale, pourvu que F soit de degré égal ou supérieur à celui de G, et w est un polynome à coefficients indéterminés du degré $m-n$.

II. Si l'on avait pris $h=1$ (en général $h=n-\alpha$) on aurait trouvé, pour les conditions d'existence d'un p. g. c. d. quadratique,

$$\left\|\begin{array}{ccccc} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & b_3 & b_2 & b_1 \\ a_4 & a_3 & & b_3 & b_2 \\ & a_4 & & & b_3 \end{array}\right\| \text{ rang } 4;$$

donc simplement, cette dernière matrice est nulle, et c'est le résultat connu.

Mais si l'on avait fait $h=0$ (en général $h=n-\alpha-1$) on aurait trouvé

$$M' = \begin{Vmatrix} a_0 & b_0 & \\ a_1 & b_1 & b_0 \\ a_2 & b_2 & b_1 \\ a_3 & b_3 & b_2 \\ a_4 & & b_3 \end{Vmatrix} \text{ rang } 3,$$

ou simplement que cette nouvelle matrice *n'est pas nulle*, et c'est à cela qu'on doit reconnaitre que le p. g. c. d. n'est *que* quadratique.

Pour fixer les idées, supposons que le déterminant obtenu en supprimant les deux dernières lignes de la matrice ci-dessus diffère de zéro. Alors le p. g. c. d. est, à un facteur constant près,

$$\Delta = \begin{vmatrix} a_0 & b_0 & & & \\ a_1 & b_1 & b_0 & & \\ a_2 & b_2 & b_1 & 1 & \\ a_3 & b_3 & b_2 & -x & 1 \\ a_4 & & b_3 & & -x \end{vmatrix}.$$

En effet Δ est quadratique, car le coefficient de x^2 est le déterminant supposé non nul. De plus, en multipliant les quatre premières lignes de Δ respectivement par x^4, x^3, x^2, x et ajoutant à la dernière, celle-ci devient

$$F,\ Gx,\ G,\ 0,\ 0$$

donc $\Delta \equiv Fp+Gq$ et le p. g. c. d. quadratique de F et G doit diviser Δ; comme Δ est aussi quadratique, le quotient est indépendant de x.

Si dans la matrice M', il y a plus d'un déterminant non nul, il y a plus d'une manière de former ainsi le p. g. c. d. La règle actuelle est générale et s'étend même à plus de deux polynomes. Elle ne diffère pas au fond de celle qui a été donnée par M. Lloyd L. Dines (*loc. cit.*).

8. Soient trois polynomes

$$\left.\begin{aligned} F &= a_0x^m + a_1x^{m-1} + \dots + a_m, \\ G &= b_0x^n + b_1x^{n-1} + \dots + b_n, \\ G' &= c_0x^{n'} + c_1x^{n'-1} + \dots + c_{n'}, \end{aligned}\right\} \quad (m \geq n \geq n').$$

Appliquons textuellement la démonstration relative à deux formes, en prenant les polynomes auxiliaires P, Q, Q' de degrés respectifs h, $m+h-n$, $m+h-n'$. Les conditions analogues à (1) pour l'existence d'une identité

$$FP+GQ+G'Q' \equiv 0$$

sont en nombre $m+h+1$ entre

$$h+1+(m+h-n+1)+(m+h-n'+1) = 2m-n-n'+3h+3$$

inconnues homogènes, et la matrice des coefficients est

$$M = \left\| \begin{array}{cccccccccc} a_0 & & & b_0 & & & c_0 & & & \\ a_1 & a_0 & & b_1 & b_0 & & c_1 & c_0 & & \\ . & a_1 & \dots & . & b_1 & \dots & . & c_1 & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ a_m & \dots & & b_n & \dots & & c_{n'} & \dots & & \\ & a_m & \dots & & b_n & \dots & & c_{n'} & \dots & \\ & & \dots & & & \dots & & & \dots & \\ \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{h+1 \text{ col.}}} & \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{m+h-n+1}} & \multicolumn{3}{c}{\underbrace{\qquad\qquad}_{m+h-n'+1}} & \end{array} \right\| \quad (m+h+1 \text{ lignes}).$$

Si F, G, G' sont sans diviseur commun en x et si l'on n'a pas $a_0=b_0=c_0=0$, cette matrice a son rang maximé, en d'autres termes M n'est pas nulle.

En effet, choisissons deux polynomes

$$w = w_0 x^{m-n} + w_1 x^{m-n-1} + \dots,$$
$$w' = w_0' x^{m-n'} + w_1' x^{m-n'-1} + \dots,$$

tels que $a_0 + w_0 b_0 + w_0' c_0'$ soit différent de zéro et qu'en même temps le résultant de $F+wG$ et G' soit non nul. On soit qu'on peut toujours satisfaire à ces deux conditions, même par des valeurs entières des coefficients de w et w' (J. König, *loc. cit.*, p. 64), car par hypothèse, elles ne sont pas satisfaites identiquement.

Mais alors les polynomes $F+wG+w'G'$ et G' n'ont pas de diviseur commun en x et leurs premiers coefficients ne sont pas tous deux nuls.

Or si la matrice M est nulle, il en est de même de cette matrice transformée de manière qu'elle s'applique aux trois polynomes $F+wG+w'G'$, G, G', et aussi de la matrice partielle obtinue en effaçant, dans cette dernière, les colonnes relatives à G. Le résultat auquel nous arrivons est en contradiction avec ce que nous avons établi, dans les pages précédentes, pour deux polynomes $F+wG+w'G'$ et G'.

9. À présent si les polynomes F, G, G' rendus homogènes ont un p. g. c. d. de degré α, les quotients f, g, g' sont sans diviseur commun, n'ont pas leurs premiers coefficients tous nuls et sont de degrés $m-\alpha$, $n-\alpha$, $n'-\alpha$. Choisissons h supérieur à $n-\alpha-1$, donc aussi à $n'-\alpha-1$; alors le nombre $m+h+1-\alpha$ des équations (1) relatives à f, g, g' est inférieur au nombre des inconnues, de

$$l' = 2m-n-n'+3h-3-(m+h+1-\alpha) = m-n-n'+2h+2+\alpha,$$

nombre positif, parce que l'on a $h > n-\alpha-1$, $h > n'-\alpha-1$, $m \geq \alpha$. Donc les identités

$$fP + gQ + g'Q' \equiv 0$$

sont possibles de $\infty^{l'}$ manières, car il y a l' inconnues de plus que d'équations, et la matrice des coefficients n'est pas nulle.

Quant à F, G, G' ils sont dans le cas exceptionnel: la matrice M qui les concerne est de rang

$$2m-n-n'+3h+3-l' = m+h+1-\alpha.$$

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10. En particulier, si l'on fait $h=n-1$ (pour $h\geq n-1$, on a le résultat de M. Lloyd L. Dines), la matrice M_{m+n} a $m+n$ lignes. Plus particulièrement encore, pour que F, G, G' aient un p. g. c. d. de degré non assigné, en y comprenant le cas de $a_0=b_0=c_0=0$, il faut et il suffit que cette dernière matrice M_{m+n} soit simplement nulle.

On démontre, comme pour le résultant (J. König, *loc. cit.*, p. 100), que les déterminants extraits de cette dernière matrice sont des fonctions linéaires des polynomes donnés, et peuvent donc remplacer, dans l'exposé de la théorie du résolvant (*id.*, p. 102), les coefficients $R_0, R_1, \ldots.$ des produits de puissances des u et des v dans le résultant de $\Sigma u_i F_i$ et $\Sigma v_i F_i$.

La différence est que les coefficients $R_0, R_1, \ldots.$ de Kronecker s'annulent, ou bien quand il y a un p. g. c. d. en x, ou bien quand s'évanouissent les premiers coefficients des polynomes F_i de degré le plus élevé; tandis que, dans notre méthode (caractérisée par l'usage du polynomes w, w' au lieu des indéterminées u, v), les déterminants extraits de la matrice M_{m+n} s'annulent quand il y a un p. g. c. d. en x ou quand *tous* les premiers coefficients sont nuls. Dans la théorie du résultant, on doit supposer les formes rendues régulières, de sorte que les deux groupes de symboles sont équivalents; les nôtres, nous le répétons, seront utiles dans la théorie des polynomes homogènes et dans les applications géométriques.

11. L'éxtension à plus de trois polynomes est facile. Le noeud de la démonstration est toujours celui-ci. Si les polynomes n'ont aucun diviseur commun, ni leurs premiers coefficients tous nuls, la matrice analogue à M est non nulle.

Le cas de deux et trois polynomes ayant été élucidé, il reste à étendre la démonstration par induction complète.

Supposons donc le fait établi pour les $n+1$ polynomes $F, G, G', \ldots., G^{(n-1)}$ et soit $G^{(n)}$ un nouveau polynome. On peut choisir les polynomes $w, w', \ldots.,$ $w^{(n)}$ tels que $F+wG+w'G'+\ldots.+w^{(n)}G^{(n)}$ ait son premier coefficient non nul, et que le résultant de $F+wG$ et du p. g. c. d. de $G', \ldots., G^{(n)}$ diffère de zéro. Alors les polynomes

$$F+wG+w'G+\ldots.+w^{(n)}G^{(n)},\ G',\ G'',\ \ldots.,\ G^{(n)}$$

n'ont pas leurs premiers coefficients tous nuls et n'ont pas de diviseur commun en x.

Mais la matrice M relative aux $n+2$ polynomes $F, G, G', \ldots., G^{(n)}$ peut se transformer en la matrice analogue M' relative aux $n+2$ polynomes

$$F+wG+w'G'+\ldots.+w^{(n)}G^{(n)},\ G,\ G',\ G'',\ \ldots.,\ G^{(n)};$$

celle-ci est nulle en même temps que M, et de même la matrice partielle obtenue en effaçant de M' les colonnes relatives à G; mais alors on a une matrice nulle, analogue à M et relative à $n+1$ polynomes qui n'ont pas de diviseur commun, ce qui est supposé impossible.

GAND, le 17. *Mai*, 1914.

*Congruences Associated with a One-Parameter Family of Curves.**

By Ralph Dennison Beetle.

Introduction.

With a one-parameter family of curves may be associated a number of rectilinear congruences. The congruence formed by the tangents to the curves has been very extensively studied, and its properties have been found to be intimately connected with the nature of the surface on which the curves lie and with the relation of the curves to this surface. In this paper, we consider some of the other rectilinear congruences and also certain congruences of circles associated with the curves. Particular attention is given to the rectilinear congruences formed by the principal normals and binormals, and the congruence of circles formed by the osculating circles. The discussion in this preliminary paper is restricted to rather elementary properties of the congruences.

The paper is divided into four parts. In the first part, we state those general formulas relating to one-parameter families of curves which are necessary in the later portions of the paper. It is found advantageous to use the method recently suggested by Eisenhart.† In this method, the moving triedral formed by the tangent, the principal normal and the binormal serves as a frame of reference, and the treatment of problems relating to these lines is thereby essentially simplified.

The second part of the paper is devoted to rectilinear congruences. As is well known, it is a characteristic property of a system of geodesics that the congruence of tangents is normal, and of a system of asymptotic lines that the surface on which they lie is the middle surface of the congruence of tangents. When the congruence of principal normals or binormals is normal, or has the surface for its middle surface, the resulting geometric property of the curves is not so tangible, but a consideration of these properties of the congruences leads to a number of general theorems of interest.

* Presented to the American Mathematical Society, September 8, 1914.

† L. P. Eisenhart, "One-Parameter Families of Curves," American Journal of Mathematics, Vol. XXXVII (1915), p. 179.

Of the other rectilinear congruences discussed in the second part, we mention here only the congruence of polar lines. By considering this congruence, we find the following characterization of a system of lines of curvature. In order that a one-parameter family of curves be a system of lines of curvature of the surface on which the curves lie, it is necessary and sufficient that the point of meeting of the normal to the surface and the corresponding polar line be a focal point of the congruence of polar lines.

In the third part of the paper, we determine under what conditions the congruence of osculating circles is a cyclic system. We find that it is necessary and sufficient that the curves in question be lines of curvature of constant geodesic curvature. Hence, they are either plane geodesics or else spherical curves which lie on spheres orthogonal to the surface formed by the curves. In this part of the paper, we also discuss cyclic systems in which the circles lie on a single infinity of planes or of spheres.

In connection with the notions and results of the second part, it is interesting to consider the surfaces characterized by the fact that the asymptotic lines in one or both systems are geodesic parallels. These surfaces have apparently not previously been discussed. The fourth part of the paper deals with these surfaces. The determination of all such surfaces requires the solution of a rather complicated partial differential equation of the fourth order. A number of characteristic properties of these surfaces are found.

I. One-Parameter Families of Curves.

§ 1. *Equations of Condition.*

In the paper mentioned above, Eisenhart shows that, if p, q, r, t, ρ and τ satisfy the three conditions

$$\left.\begin{aligned} \frac{\partial q}{\partial u} - \frac{\partial p}{\partial v} - \frac{pr}{\rho} &= 0, \\ \frac{\partial A_2}{\partial u} - \frac{\partial}{\partial v}\left(\frac{p}{\rho}\right) + \frac{p}{\tau} A_3 &= 0, \\ \frac{\partial L_3}{\partial u} + \frac{\partial}{\partial v}\left(\frac{p}{\tau}\right) + \frac{p}{\rho} A_3 &= 0, \end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned} A_2 &= \frac{1}{p}\frac{\partial r}{\partial u} + \frac{q}{\rho} + \frac{t}{\tau}, \\ A_3 &= \frac{1}{p}\frac{\partial t}{\partial u} - \frac{r}{\tau}, \\ L_3 &= \frac{\rho}{p}\frac{\partial A_3}{\partial u} - \frac{\rho}{\tau} A_2, \end{aligned}\right\} \tag{2}$$

then the system of equations, consisting of the six equations

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial u} &= \frac{p}{\rho}l, & \frac{\partial\alpha}{\partial v} &= A_2 l + A_3\lambda, \\ \frac{\partial l}{\partial u} &= -\frac{p}{\rho}\alpha - \frac{p}{\tau}\lambda, & \frac{\partial l}{\partial v} &= -A_2\alpha + L_3\lambda, \\ \frac{\partial\lambda}{\partial u} &= \frac{p}{\tau}l, & \frac{\partial\lambda}{\partial v} &= -A_3\alpha - L_3 l, \end{aligned}\right\} \tag{3}$$

together with those obtained by replacing α, l, λ by β, m, μ, and by γ, n, ν, is completely integrable, and admits solutions such that the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$$

is orthogonal and positive. Under these conditions, the equations

$$\frac{\partial x}{\partial u} = p\alpha, \quad \frac{\partial x}{\partial v} = q\alpha + rl + t\lambda, \tag{4}$$

and the analogous ones for y and z, are consistent, and the locus of the point $P(x, y, z)$ is a surface S.

The surface S may be regarded as the locus of the one-parameter family of curves C obtained by assigning arbitrary constant values to v. Then α, β, γ; l, m, n; λ, μ, ν are, respectively, the direction-cosines of the tangent, principal normal and binormal of the curve C through the corresponding point P; ρ and τ are the radii of first and second curvature of the curve C.

Conversely, if $x, \alpha, l, \lambda, \rho, \tau$, etc., have the significance just indicated, and the curves C are not minimal or straight lines, the p, q, r and t defined by (4) and the analogous equations in y and z satisfy the equations (1).

§ 2. *Fundamental Quantities for the Surface S. Special Parametric Systems.*

If the linear element of the surface S is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

it follows from (4) that

$$E = p^2, \quad F = pq, \quad G = q^2 + r^2 + t^2, \tag{5}$$

whence

$$H = \sqrt{EG - F^2} = p\sqrt{r^2 + t^2}. \tag{6}$$

If X, Y, Z are the direction-cosines of the normal to the surface,

$$X = \frac{r\lambda - tl}{\sqrt{r^2 + t^2}} = \lambda\sin\omega + l\cos\omega, \tag{7}$$

where ω is the angle which the normal to the surface makes with the principal

normal of the curve C at the corresponding point. We note for reference that

$$\sin\omega=\frac{r}{\sqrt{r^2+t^2}},\quad \cos\omega=-\frac{t}{\sqrt{r^2+t^2}},\quad \tan\omega=-\frac{r}{t}. \tag{8}$$

If the second fundamental quadratic form* of the surface S is

$$-\Sigma dxdX=Ddu^2+2D'dudv+D''dv^2,$$

it is found that

$$\left.\begin{aligned} D&=-\frac{p^2t}{\rho\sqrt{r^2+t^2}}\\ D'&=\frac{p(rA_3-tA_2)}{\sqrt{r^2+t^2}}=\frac{pq}{\rho}\cos\omega+\left(\frac{\partial\omega}{\partial u}-\frac{p}{\tau}\right)\sqrt{r^2+t^2},\\ D''&=\frac{q(rA_3-tA_2)}{\sqrt{r^2+t^2}}+\left(L_3+\frac{\partial\omega}{\partial v}\right)\sqrt{r^2+t^2}.\end{aligned}\right\} \tag{9}$$

Since we have excluded the case in which the curves C are minimal, we may always assume that $p\neq 0$. If $p=1$, the parameter u is the arc of the curve C. Whenever p is a function of u alone, we may take $p=1$, since this result can be secured by a change of parameters which preserves the parametric curves.

We have excluded also the case in which the curves C are rulings. Then the necessary and sufficient condition that they be geodesics is that $r=0$. In order that they be asymptotic lines, it is necessary and sufficient that $t=0$.

If $q=0$, the parametric system is orthogonal, and conversely. The parametric system is conjugate if

$$rA_3-tA_2=0; \tag{10}$$

it consists of the lines of curvature if

$$q=0,\quad \frac{\partial\omega}{\partial u}-\frac{p}{\tau}=0, \tag{11}$$

in view of (5) and (9). In consequence of a well known property of the lines of curvature, the second of the conditions (11) is necessary and sufficient that the curves C be lines of curvature.†

§ 3. *The Quantities ω, r and t.*

From its definition, it is evident that the value of ω at a given point depends only on the curves C. It is readily proved that the values of r and t are independent of the choice of the curves $u=$const., but do depend on the particular distribution of the parameter v in the system of curves C. If, how-

* Cf. Eisenhart, "Differential Geometry," p. 114 (Ginn and Co., Boston, 1909). Hereafter, a reference to this book will be given in the form Eisenhart, p. 114.

† Eisenhart, p. 130.

ever, the parameter v is fixed, the values of ω, r and t are all uniquely determined. Since the system of curves C is arbitrary, except that we have excluded minimal and straight lines, it follows that we may associate with any system of curves defined by an equation of the form $\psi(u, v)=\text{const.}$ three functions ω_ψ, r_ψ and t_ψ.

Moreover, since any curve on the surface S, except a curve $u=\text{const.}$, can be defined by an equation of the form $v-\phi(u)=0$, and hence be regarded as belonging to the family given by $v-\phi(u)=\text{const.}$, we may define the three functions for any curve of the surface. In this case, we shall denote them by ω_ϕ, r_ϕ and t_ϕ. We proceed to find expressions for them.

From the well known formulas for normal curvature and geodesic curvature* it follows that

$$\tan\omega_\phi=\frac{E+2F\frac{\partial\phi}{\partial u}+G\left(\frac{\partial\phi}{\partial u}\right)^2}{H\left[D+2D'\frac{\partial\phi}{\partial u}+D''\left(\frac{\partial\phi}{\partial u}\right)^2\right]}\left\{\frac{\partial}{\partial u}\left[\frac{F+G\frac{\partial\phi}{\partial u}}{\sqrt{E+2F\frac{\partial\phi}{\partial u}+G\left(\frac{\partial\phi}{\partial u}\right)^2}}\right]\right.$$
$$\left.-\frac{\partial}{\partial v}\left[\frac{E+F\frac{\partial\phi}{\partial u}}{\sqrt{E+2F\frac{\partial\phi}{\partial u}+G\left(\frac{\partial\phi}{\partial u}\right)^2}}\right]\right\}. \tag{12}$$

In view of (5), (6) and (8),

$$r=\frac{H}{\sqrt{E}}\sin\omega \tag{13}$$

and

$$t=-\frac{H}{\sqrt{E}}\cos\omega. \tag{14}$$

Therefore we conclude that†

$$r_\phi=\frac{H\sin\omega_\phi}{\sqrt{E+2F\frac{\partial\phi}{\partial u}+G\left(\frac{\partial\phi}{\partial u}\right)^2}}, \tag{15}$$

and

$$t_\phi=-\frac{H\cos\omega_\phi}{\sqrt{E+2F\frac{\partial\phi}{\partial u}+G\left(\frac{\partial\phi}{\partial u}\right)^2}}. \tag{16}$$

§ 4. *The Curves $u=const.$*

If α_1, l_1, λ_1, p_1, q_1, r_1, t_1 ρ_1, τ_1, etc., are similarly defined for the curves $u=\text{const.}$, the following relations are easily deduced:

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* Eisenhart, pp. 117, 132, 136.

† Eisenhart, p. 74.

$$\alpha_1 = \frac{q\alpha + rl + t\lambda}{\sqrt{q^2+r^2+t^2}}; \tag{17}$$

$$l_1 = \frac{a_1\alpha + a_2 l + a_3\lambda}{\sqrt{a_1^2+a_2^2+a_3^2}}; \tag{18}$$

$$\lambda_1 = \frac{(ra_3 - ta_2)\alpha + (ta_1 - qa_3)l + (qa_2 - ra_1)\lambda}{\sqrt{q^2+r^2+t^2}\sqrt{a_1^2+a_2^2+a_3^2}}, \tag{19}$$

where

$$\left.\begin{aligned} a_1 &= \frac{\partial}{\partial v}\left(\frac{q}{\sqrt{q^2+r^2+t^2}}\right) - \frac{rA_2+tA_3}{\sqrt{q^2+r^2+t^2}}, \\ a_2 &= \frac{\partial}{\partial v}\left(\frac{r}{\sqrt{q^2+r^2+t^2}}\right) + \frac{qA_2-tL_3}{\sqrt{q^2+r^2+t^2}}, \\ a_3 &= \frac{\partial}{\partial v}\left(\frac{t}{\sqrt{q^2+r^2+t^2}}\right) + \frac{qA_3+rL_3}{\sqrt{q^2+r^2+t^2}}. \end{aligned}\right\} \tag{20}$$

We have also

$$\left.\begin{aligned} p_1 &= \sqrt{q^2+r^2+t^2}, \\ q_1 &= \frac{pq}{\sqrt{q^2+r^2+t^2}}, \\ r_1 &= \frac{pa_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \\ t_1 &= \frac{p(ra_3-ta_2)}{\sqrt{q^2+r^2+t^2}\sqrt{a_1^2+a_2^2+a_3^2}}, \end{aligned}\right\} \tag{21}$$

and

$$\left.\begin{aligned} \frac{1}{\rho_1} &= \frac{\sqrt{a_1^2+a_2^2+a_3^2}}{\sqrt{q^2+r^2+t^2}}, \\ \frac{1}{\tau_1} &= \frac{\sqrt{(b_1^2+b_2^2+b_3^2)-(a_1^2+a_2^2+a_3^2)}}{\sqrt{q^2+r^2+t^2}}, \end{aligned}\right\} \tag{22}$$

where the quantities b_1, b_2 and b_3 are obtained by replacing q, r and t by a_1, a_2 and a_3 in the expressions (20) for a_1, a_2 and a_3.

II. Rectilinear Congruences.

§ 5. *Congruences* Γ. *Notation.*

All of the rectilinear congruences studied in this paper are congruences generated by a line L whose direction, relative to the moving triedral of reference, remains invariable. Such a congruence we denote by the symbol Γ. If $P_0(x_0, y_0, z_0)$ is any point, fixed or variable, on the moving line L, and X_0, Y_0, Z_0 are the direction-cosines of the line, we have

$$x_0 = x + \xi_1 \alpha + \xi_2 l + \xi_3 \lambda, \quad X_0 = c_1 \alpha + c_2 l + c_3 \lambda, \tag{23}$$

where ξ_1, ξ_2, ξ_3 are functions of u and v, but c_1, c_2, c_3 are constants such that $c_1^2 + c_2^2 + c_3^2 = 1$.

We write

$$\left.\begin{aligned} &\mathcal{E}_0 = \Sigma\left(\frac{\partial X_0}{\partial u}\right)^2, \quad \mathcal{F}_0 = \Sigma \frac{\partial X_0}{\partial u}\frac{\partial X_0}{\partial v}, \quad \mathcal{G}_0 = \Sigma\left(\frac{\partial X_0}{\partial v}\right)^2, \\ &e_0 = \Sigma \frac{\partial x_0}{\partial u}\frac{\partial X_0}{\partial u}, \quad f_0 = \Sigma \frac{\partial x_0}{\partial v}\frac{\partial X_0}{\partial u}, \\ &\qquad\qquad f_0' = \Sigma \frac{\partial x_0}{\partial u}\frac{\partial X_0}{\partial v}, \quad g_0 = \Sigma \frac{\partial x_0}{\partial v}\frac{\partial X_0}{\partial v}, \end{aligned}\right\} \tag{24}$$

so that

$$\Sigma dX_0^2 = \mathcal{E}_0 du^2 + 2\mathcal{F}_0 du dv + \mathcal{G}_0 dv^2,$$

and

$$\Sigma dx_0 dX_0 = e_0 du^2 + (f_0 + f_0') du dv + g_0 dv^2.$$

Then the equation of the developable surfaces of the congruence is*

$$(\mathcal{E}_0 f_0' - e_0 \mathcal{F}_0) du^2 + (\mathcal{E}_0 g_0 - \mathcal{G}_0 e_0 + \mathcal{F}_0 f_0' - \mathcal{F}_0 f_0) du dv + (\mathcal{F}_0 g_0 - f_0 \mathcal{G}_0) dv^2 = 0. \tag{25}$$

If P_{01} and P_{02} are the focal points of the line L, their coordinates are of the form

$$x_{01} = x_0 + \rho_{01} X_0, \quad x_{02} = x_0 + \rho_{02} X_0, \tag{26}$$

where †

$$\left.\begin{aligned} \rho_{01} + \rho_{02} &= \frac{(f_0 + f_0')\mathcal{F}_0 - g_0 \mathcal{E}_0 - e_0 \mathcal{G}_0}{\mathcal{E}_0 \mathcal{G}_0 - \mathcal{F}_0^2}, \\ \rho_{01}\rho_{02} &= \frac{e_0 g_0 - f_0 f_0'}{\mathcal{E}_0 \mathcal{G}_0 - \mathcal{F}_0^2}. \end{aligned}\right\} \tag{27}$$

If the line L always passes through the corresponding point P, we may choose P for the point P_0. In this case, we shall speak of congruences Γ_P. The congruences Γ_P formed by the tangents, principal normals and binormals, we shall denote by Γ_t, Γ_n and Γ_b. Similarly, by Γ_N we indicate a congruence Γ_P such that the line L lies in the normal plane of the curve C.

§ 6. *Normal Congruences* Γ_P.

The condition that a congruence Γ be normal,‡ $f_0 = f_0'$, reduces to

$$\frac{\partial}{\partial u}[c_1(q - \xi_2 A_2 - \xi_3 A_3) + c_2(r + \xi_1 A_2 - \xi_3 L_3) + c_3(t + \xi_1 A_3 + \xi_2 L_3)]$$
$$= \frac{\partial}{\partial v}\left[c_1\left(p - \xi_2 \frac{p}{\rho}\right) + c_2\left(\xi_1 \frac{p}{\rho} + \xi_3 \frac{p}{\tau}\right) - c_3\left(\xi_2 \frac{p}{\tau}\right)\right], \tag{28}$$

* Eisenhart, p. 398. † Eisenhart, p. 399. ‡ Eisenhart, p. 393.

and becomes, in the case of congruences Γ_P,

$$c_1\left(\frac{\partial q}{\partial u}-\frac{\partial p}{\partial v}\right)+c_2\frac{\partial r}{\partial u}+c_3\frac{\partial t}{\partial u}=0,$$

or, in view of (1),

$$c_1\frac{pr}{\rho}+c_2\frac{\partial r}{\partial u}+c_3\frac{\partial t}{\partial u}=0. \tag{29}$$

The congruences Γ_t, Γ_n and Γ_b correspond to the choices $c_1=1$, $c_2=c_3=0$; $c_2=1$, $c_1=c_3=0$; $c_3=1$, $c_1=c_2=0$. We have already remarked that, if we rule out minimal and straight lines, the congruence Γ_t is normal if, and only if, $r=0$. We can now add

THEOREM 1. *The necessary and sufficient condition that the congruence of principal normals be normal is that r be constant along each curve C. The necessary and sufficient condition that the congruence of binormals be normal is that t be constant along each curve C.*

For the sake of brevity, we denote by *r-line* a curve along which r is constant, and by *t-line* a curve along which t is constant. Since geodesics are r-lines and asymptotic lines are t-lines, such lines exist on every surface. Moreover, the remarks of § 3 enable us to prove that, on every surface, there exist infinitely many r-lines and t-lines, other than geodesics and asymptotic lines. For the necessary and sufficient condition that the principal normals of a system of curves defined by

$$v-\phi(u)=\text{const.}$$

form a normal congruence is that there be a functional relation between

$$\frac{H^2\sin^2\omega_\phi}{E+2F\dfrac{\partial\phi}{\partial u}+G\left(\dfrac{\partial\phi}{\partial u}\right)^2}$$

and $v-\phi$. Equating to zero the Jacobian of these expressions and making use of (12), we find that ϕ satisfies a differential equation of the third order. A similar result is obtained in the case of t-lines.

Suppose now that the curves C are both r-lines and t-lines. Then

$$\frac{\partial r}{\partial u}=\frac{\partial t}{\partial u}=0,$$

and we can take $q=0$. In view of (5) and (8), we conclude that

$$\frac{\partial\omega}{\partial u}=\frac{\partial G}{\partial u}=0,$$

so that the curves C are geodesic parallels,* along each of which the osculating

* Eisenhart, p. 134.

plane meets the surface under constant angle. The converse theorem is easily established.

If we seek a system of curves such that all three of the congruences Γ_t, Γ_n and Γ_b are normal, it is therefore necessary and sufficient that they be geodesics whose orthogonal trajectories are geodesics. As is well known, orthogonal systems of geodesics exist only on developable surfaces; they correspond, of course, to orthogonal systems of straight lines in the plane.

From the form of the condition (29) there follows at once

THEOREM 2. *If two different lines, L_1 and L_2, generate normal congruences Γ_P, every congruence Γ_P generated by a line L in the plane of L_1 and L_2 is also normal, and is an associate* of the first two. If three non-coplanar lines generate normal congruences Γ_P, every congruence Γ_P is normal.*

§ 7. *Other Normal Congruences* Γ.

If the curves C are geodesics and the curves $u=$const. are their conjugates, we have by (10) $r=A_2=0$. If we take $c_1=c_3=0$, $c_2=1$, $\xi_2=\xi_3=0$, the condition (28) for normality becomes

$$\frac{\partial}{\partial v}\left(\xi_1\frac{p}{\rho}\right)=0, \tag{30}$$

so that we obtain a normal congruence by taking

$$\xi_1=\frac{\rho}{p}U_1,$$

where U_1 is a function of u alone. If $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ describes one of the surfaces normal to this congruence, we may write

$$\bar{x}=x+\frac{\rho}{p}U_1\alpha+\eta l,$$

where η is to be determined. Then, since we must have

$$\Sigma l\frac{\partial\bar{x}}{\partial u}=0,\quad \Sigma l\frac{\partial\bar{x}}{\partial v}=0,$$

we find that

$$\eta=-\int U_1du=-U,$$

so that

$$\bar{x}=x+\frac{\rho}{p}U'\alpha-Ul, \tag{31}$$

where the prime denotes differentiation with respect to u.

THEOREM 3. *If a geodesic system, $v=$const., and its conjugate system, $u=$const., are known on the surface S, each choice of a function of u deter-*

* Eisenhart, p. 402.

mines, without any integration, a surface $\bar{S}$ which corresponds to S with parallelism of tangent planes.

If S_1 is the other focal surface of the normal congruence of tangents to the geodesic system, the curves $u=$const. on S_1 are geodesics and the curves $v=$const. are their conjugates.* Hence, by Theorem 3, to each choice of a function of v will correspond a normal congruence in the tangent plane of S. Taking $c_1=c_2=0$, $c_3=1$, $\xi_2=\xi_3=0$, we find from (28) that the corresponding congruence is normal if

$$\xi_1=\frac{V_1-t}{A_3},$$

where V_1 is any function of v alone. The conditions $r=0$, $A_2=0$ are not needed in this case.

If $\bar{P}_1(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ describes a surface normal to this congruence, we have

$$\bar{x}_1=x+\frac{V'-t}{A_3}\alpha-V\lambda, \tag{32}$$

where $V=\int V_1 dv$.

THEOREM 4. *If any system of curves, $v=$const., is known on the surface S, each choice of a function of v determines, without any integration, a surface $\bar{S}_1$, the normal to which lies in the corresponding rectifying plane of the curve $v=$const. If the curves are geodesics, S and $\bar{S}_1$ correspond with orthogonality of tangent planes. If the curves are asymptotic lines, S and $\bar{S}_1$ correspond with parallelism of tangent planes.*

If we put $V=0$, $\bar{S}_1$ becomes S_1, the other focal surface of the congruence of tangent lines.

§ 8. *Congruences* Γ_N.

For congruences Γ_N, $c_1=0$, while c_2 and c_3 are any constants such that $c_2^2+c_3^2=1$. Writing $c_2=\cos\theta$, $c_3=\sin\theta$, we find that for a congruence Γ_N equations (27) become

$$\left.\begin{aligned}\rho_{01}+\rho_{02}&=\frac{L_3+\dfrac{q}{\tau}+\dfrac{\cos\theta}{\rho}(r\sin\theta-t\cos\theta)}{\dfrac{1}{\tau}(A_2\cos\theta+A_3\sin\theta)+\dfrac{L_3}{\rho}\cos\theta},\\ \rho_{01}\rho_{02}&=\frac{r\sin\theta-t\cos\theta}{\dfrac{1}{\tau}(A_2\cos\theta+A_3\sin\theta)+\dfrac{L_3}{\rho}\cos\theta}.\end{aligned}\right\} \tag{33}$$

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* Eisenhart, p. 403.

In order that S be the middle surface* of the congruence, it is necessary and sufficient† that

$$L_3+\frac{q}{\tau}+\frac{\cos\theta}{\rho}(r\sin\theta-t\cos\theta)=0. \tag{34}$$

Since the angle θ is constant, it is clear that, in general, there are no congruences Γ_N for which (34) is satisfied. If there are two such congruences, and θ_1 and θ_2 are the corresponding values of θ, we have

$$r(\sin\theta_1\cos\theta_1-\sin\theta_2\cos\theta_2)=t(\cos^2\theta_1-\cos^2\theta_2),$$

whence

$$\tan(\theta_1+\theta_2)=-\frac{r}{t}=\tan\omega.$$

Therefore the angles θ_1 and θ_2 can be so chosen that $\theta_1+\theta_2=\omega$, and hence the number of such congruences can not exceed two. It can be equal to two only in case ω is constant.

If ω is constant, the normals to the surface form a congruence Γ_N. In order that S be the middle surface of this congruence, it is necessary and sufficient that S be minimal.‡ We conclude that, when S is minimal and ω is constant, then S is the middle surface of the congruence of principal normals. For, in view of the preceding paragraph, if (34) is satisfied by $\theta=\omega$, it is also satisfied by $\theta=0$. This result is a special case of a theorem which we shall now prove.

When $q=0$, and (34) is satisfied for $\theta=0$, we find for the mean curvature of S the expression

$$K_m=\frac{ED''+GD-2FD'}{EG-F^2}=\frac{\dfrac{\partial\omega}{\partial v}}{\sqrt{r^2+t^2}}.$$

If, on the other hand, we assume that

$$q=\frac{\partial\omega}{\partial v}=K_m=0,$$

it is found that (34) is satisfied for $\theta=0$. We can therefore state

THEOREM 5. *For a system of curves C on a surface S, each of the following statements is a consequence of the two remaining:*

1°. *The surface S is minimal.*

2°. *The surface S is the middle surface of the congruence of principal normals.*

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* Eisenhart, p. 399.

† We do not consider the degenerate congruences for which $\mathfrak{E}_0\mathfrak{G}_0-\mathfrak{F}_0^2$ vanishes.

‡ Eisenhart, pp. 180, 251.

3°. *Along each of the orthogonal trajectories of the curves C, the angle* ω *is constant.*

It is easy to show, in view of (12), that an unlimited number of systems satisfying the condition 3° exist on every surface, independently of the cases in which ω is constant all over the surface. The condition 3° is satisfied by a system of geodesics or asymptotic lines; in each case, Theorem 5 states a well known characteristic property of minimal surfaces.

In like manner we prove

THEOREM 6. *For a system of curves C on a surface S, each of the following statements is a consequence of the two remaining:*

1°. *The orthogonal trajectories of the curves C are asymptotic lines.*

2°. *The surface S is the middle surface of the congruence of binormals.*

3°. *Along each of the orthogonal trajectories of the curves C, the angle* ω *is constant.*

In order that S be the middle surface of both the congruence of principal normals and the congruence of binormals, we must have

$$L_3+\frac{q}{\tau}-\frac{t}{\rho}=L_3+\frac{q}{\tau}=0.$$

Then $t=0$, and from (9) it is seen that the mean curvature of S is zero.

THEOREM 7. *The only curves C which have S for the middle surface of both the congruence of principal normals and the congruence of binormals are the asymptotic lines of a minimal surface.*

COROLLARY. *Whenever S is the middle surface of two of the congruences* Γ_t, Γ_n *and* Γ_b, *it is of the third also.*

Meusnier's theorem* states that all the osculating circles of the curves on S tangent to the curve C at the point P lie on a sphere M of radius $|\rho/\cos\omega|$ tangent to S at P. We shall now prove the following theorem with regard to this sphere:

THEOREM 8. *If the generating lines of two congruences* Γ_N *are constantly orthogonal, and the curves C bisect the angles between the curves cut out on S by the developables of one of the congruences, the focal points of the other congruence are harmonic with respect to the corresponding sphere M. If the focal points of one of the congruences are harmonic with respect to the sphere M, and the generating line of this congruence is not tangent to M, the curves C bisect the angles between the curves cut out on S by the developables of the other congruence.*

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* Eisenhart, p. 118.

For simplicity, we take $q=0$. The equation of the curves cut out by the developables will be identical with (25), and the angles between these curves are bisected by the parametric curves if, and only if,*

$$\mathcal{E}_0 g_0 - \mathcal{G}_0 e_0 + \mathcal{F}_0 f_0' - \mathcal{F}_0' f_0 = 0.$$

For the congruence Γ_N for which $\theta=\theta_1$, this condition takes the form

$$L_3 - \frac{\cos\theta_1}{\rho}(r\sin\theta_1 - t\cos\theta_1) = 0. \tag{35}$$

If the coordinates of the harmonic conjugate Q of P with respect to P_{01} and P_{02} are of the form

$$x + hX_0,$$

we find that

$$h = \frac{2\rho_{01}\rho_{02}}{\rho_{01}+\rho_{02}}. \tag{36}$$

Then, for the congruence Γ_N such that $\theta=\theta_2$, we have, by (33),

$$h\left[L_3 + \frac{\cos\theta_2}{\rho}(r\sin\theta_2 - t\cos\theta_2)\right] = 2(r\sin\theta_2 - t\cos\theta_2). \tag{37}$$

Since the sphere M is of radius $|\rho/\cos\omega|$, the point Q will lie on M if

$$h = \frac{2\rho}{\cos\omega}\cos(\omega-\theta_2) = -\frac{2\rho}{t}(r\sin\theta_2 - t\cos\theta_2). \tag{38}$$

When (38) is satisfied, and $h \neq 0$, (37) becomes

$$L_3 + \frac{\sin\theta_2}{\rho}(r\cos\theta_2 + t\sin\theta_2) = 0. \tag{39}$$

If $h=0$, the generating line of the congruence is tangent to the sphere M. Conversely, if (37) and (39) are satisfied, (38) follows. To complete the proof of Theorem 8, it is sufficient to note that (35) and (39) are equivalent if $\theta_1 = \theta_2 \pm \frac{\pi}{2}$.

COROLLARY. *If the curves C bisect the angles between the curves cut out on S by the developables of the congruence of binormals, the focal points of the congruence of principal normals are harmonic conjugates with respect to the osculating circle of the curve C, and conversely.*

For the congruence Γ_n, (36) becomes

$$h = \frac{2t}{\frac{t}{\rho} - \left(L_3 + \frac{q}{\tau}\right)}; \tag{40}$$

* Eisenhart, p. 76.

and for the congruence Γ_b, (34) reduces to

$$L_3+\frac{q}{\tau}=0. \tag{41}$$

Then $h=2\rho$ if, and only if,* (41) is satisfied. Hence we have

THEOREM 9. *In order that S be the middle surface of the congruence of binormals, it is necessary and sufficient that the focal points of the congruence of principal normals be harmonic conjugates with respect to the osculating circles of the curves C.*

Combining Theorem 8, corollary, and Theorem 9, we obtain

THEOREM 10. *When, and only when, the surface S is the middle surface of the congruence of binormals, do the curves C bisect the angles between the curves cut out on S by the developables of the congruence.*

Interpreted for asymptotic lines, this theorem states the well known fact that the surface is minimal if, and only if, the asymptotic lines form an orthogonal system.

If we put $q=0$, and require that $h=\rho$, it follows from (40) that

$$L_3+\frac{t}{\rho}=0. \tag{42}$$

Since (35) reduces to (42) when $\theta_1=0$, we have

THEOREM 11. *In order that the center of the osculating circle be the harmonic conjugate of P with respect to the focal points of the congruence of principal normals, it is necessary and sufficient that the curves C bisect the angles between the curves cut out on S by the developables of the congruence.*

If the curves C are geodesics, the curves cut out on S by the developables of the congruence of principal normals are the lines of curvature, so that we have the

COROLLARY. *In order that a system of geodesics be one system of the mean orthogonal lines, it is necessary and sufficient that the center of the osculating circle be the harmonic conjugate of P with respect to the surfaces of center of S.*†

This corollary is also a direct consequence of the fact that the normal curvature of the surface in the direction of the mean orthogonal lines is equal to one-half the mean curvature of the surface.

For the congruence Γ_b, (36) becomes, if we again take $q=0$,

$$h=\frac{2r}{L_3},$$

* Exception must be made of the case $t=0$, but Theorem 9 is easily verified directly in that case.

† Eisenhart, p. 179.

and for the congruence Γ_n (34) reduces to

$$L_3 - \frac{t}{\rho} = 0. \tag{43}$$

When (43) is satisfied, it is found that

$$h = -2\rho \tan \omega. \tag{44}$$

Conversely, if (44) is satisfied and $r \neq 0$, then (43) follows. If we denote by $\overline{M}$ the sphere obtained by reflecting the sphere M of Meusnier in the tangent plane of the surface, it is readily proved that (44) may be interpreted as follows:

THEOREM 12. *If S is the middle surface of the congruence of principal normals, the focal points of the congruence of binormals are harmonic conjugates with respect to the sphere $\overline{M}$. If the focal points of the congruence of binormals are harmonic conjugates with respect to the sphere $\overline{M}$, and the curves C are not geodesics, S is the middle surface of the congruence of principal normals.*

§ 9. *The Congruence of Polar Lines.*

If we take $\xi_1 = \xi_3 = c_1 = c_2 = 0$, $\xi_2 = \rho$, $c_3 = 1$, we obtain the congruence of polar lines of the curves C. If we assume that $q = 0$, we find that

$$\rho_{01} + \rho_{02} = -\frac{\frac{\partial \rho}{\partial u} A_3 + \frac{p\rho}{\tau} A_2}{\frac{p}{\tau} A_3}, \quad \rho_{01}\rho_{02} = \frac{A_2 \rho \frac{\partial \rho}{\partial u}}{\frac{p}{\tau} A_3}, \tag{45}$$

whence

$$\rho_{01} = -\frac{\tau}{p}\frac{\partial \rho}{\partial u}, \quad \rho_{02} = -\frac{A_2 \rho}{A_3}. \tag{46}$$

Thus the focal point P_{01} is the center of the osculating sphere of the curve, and it is easily proved that P_{02} is the point of contact of the normal plane of the curve C with the surface of which the double infinity of normal planes of the curves C are the tangent planes.

If the curves C are lines of curvature, we have $A_3 \sin \omega + A_2 \cos \omega = 0$, whence $\rho_{02} = \rho \tan \omega$. Therefore the point P_{02} is now the intersection of the polar line with the normal to the surface. Conversely, if P_{02} is this point, it follows from (46) that $A_3 \sin \omega + A \cos \omega = 0$.

THEOREM 13. *It is a characteristic property of the lines of curvature that the point of meeting of the normal to the surface and the corresponding polar line is a focal point of the congruence of polar lines.*

This theorem may be proved directly by purely geometric considerations. If the curves C are lines of curvature, so are their orthogonal trajectories C_1. Hence, the normal planes of the curves C are tangent to the developable surfaces formed by the normals to the surface S along the curves C_1. The point of contact Q of the normal plane with its envelope therefore lies on the normal to the surface. On the other hand, the polar lines are the characteristics of the normal planes of a curve C; hence, Q lies also on the polar line.

III. CONGRUENCES OF CIRCLES.

§ 10. *Cyclic Systems of Osculating Circles.*

The osculating circles of a one-parameter family of curves C, other than circles, form a congruence. We inquire under what conditions this congruence is a *cyclic system*,* that is, a congruence of circles which admit a one-parameter family of orthogonal surfaces. We consider first, however, a slightly more general problem.

At each point P, draw in the osculating plane a circle k tangent to the curve C at P. If $R(u, v)$ is the radius of this circle, M its center, Q any point on the circle, and θ the angle which the radius MQ makes with the tangent to the curve, the coordinates of Q and the direction-cosines of the tangent to the circle k at Q are of the form

$$\bar{x}=x+\alpha R\cos\theta+lR(1+\sin\theta), \tag{47}$$

and

$$\bar{X}=\alpha\sin\theta-l\cos\theta. \tag{48}$$

In order that the surface $\bar{S}$, locus of Q, be normal to the circles k, it is necessary that $\Sigma\bar{X}d\bar{x}=0$. This condition can be written

$$Rd\theta+Adu+Bdv=0, \tag{49}$$

where

$$A=\cos\theta\frac{\partial R}{\partial u}+p\left(\frac{R}{\rho}-1\right)\sin\theta+\frac{pR}{\rho},$$

$$B=\cos\theta\frac{\partial R}{\partial v}+(A_2R-q)\sin\theta+r\cos\theta+A_2R.$$

The condition that (49) admit a solution involving a parameter is that

$$R\left(\frac{\partial A}{\partial v}-\frac{\partial B}{\partial u}\right)+A\left(\frac{\partial B}{\partial\theta}-\frac{\partial R}{\partial v}\right)+B\left(\frac{\partial R}{\partial u}-\frac{\partial A}{\partial\theta}\right) \tag{50}$$

be identically zero. The expression (50) reduces to the form

$$\Phi_1\sin\theta+\Phi_2\cos\theta+\Phi_3,$$

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* Eisenhart, p. 426.

where the Φ's are functions of u and v. It is therefore identically zero if and only if

$$\Phi_1=\Phi_2=\Phi_3=0.$$

We thus obtain the three conditions

$$\left.\begin{aligned} r\left(1-\frac{R}{\rho}\right)=0, \\ (q-A_2R)\frac{\partial R}{\partial u}+p\left(\frac{R}{\rho}-1\right)\frac{\partial R}{\partial v}-\frac{p}{\tau}R^2A_3=0, \\ r\frac{\partial R}{\partial u}+p\frac{t}{\tau}R=0. \end{aligned}\right\} \tag{51}$$

Therefore, either $r=0$ or $R=\rho$. In the first case, the other two conditions reduce, in view of (1) and (2), to

$$\frac{\partial R}{\partial v}=\frac{1}{\tau}=0, \tag{52}$$

provided we take $q=0$, as we may without loss of generality. Hence, the curves C are the plane geodesics of a surface of Monge;* the only restriction on R is that it be constant along the orthogonal trajectories of the curves C. This result is included in a more general one, of which we give a geometric proof.

Every surface of Monge can be generated by a plane curve whose plane rolls, without slipping, over a developable surface.† The successive positions of the curve form the system of plane geodesics. Any single infinity of circles drawn in the plane of the curve will generate a congruence. Since the circles in the plane admit a one-parameter family of orthogonal curves, the congruence of circles admits as orthogonal trajectories the one-parameter family of surfaces generated by these curves, and is therefore a cyclic system. The normal surfaces are clearly also surfaces of Monge.

Conversely, Ribaucour‡ has shown that all cyclic systems of circles whose planes envelop a developable surface can be obtained in the manner just indicated.

If we do not have $r=0$, we must take $R=\rho$. If we again put $q=0$, the remaining conditions reduce to

$$\frac{\partial \omega}{\partial u}-\frac{p}{\tau}=0, \quad \frac{\partial}{\partial u}\left(\frac{\sin\omega}{\rho}\right)=0, \tag{53}$$

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* G. Monge, "Application de l'Analyse à la Géométrie, § 24, Paris (1849).

† Eisenhart, p. 306.

‡ A. Ribaucour, "Mémoire sur la théorie générale des surfaces courbes," *Journal de Mathématiques*, Ser. 4, Vol. VII (1891), p. 264.

The curves C must therefore be lines of curvature* of constant geodesic curvature.† Since the plane geodesics of a surface of Monge are of this type, we are able to state

THEOREM 14. *If the curves C are not plane geodesics, the circles k do not form a cyclic system unless they are the osculating circles. The necessary and sufficient condition that the osculating circles form a cyclic system is that the curves be lines of curvature of constant geodesic curvature.*

Lines of curvature of constant geodesic curvature are, except in the case of plane geodesics already discussed, necessarily spherical curves, and lie on spheres which meet the surface orthogonally.‡ Conversely, curves on a surface, which lie also on spheres orthogonal to the surface, are lines of curvature of constant geodesic curvature.

It is at once evident that the surfaces obtained by subjecting surfaces of Monge to an inversion will have a system of spherical lines of curvature of constant geodesic curvature. In this way it is possible to obtain all such surfaces for which the corresponding spheres have a point in common. We shall now show how those not having this property can then be obtained.

All surfaces Σ with a system of spherical lines of curvature fall into two classes:§

1°. The surfaces (Σ_1) obtained by transforming by inversion all surfaces with a system of plane lines of curvature.

2°. The surfaces (Σ_2) obtained by subjecting the surfaces Σ_1 to the Combescure transformation for surfaces Σ.

The Combescure transformation is the following. The coordinates of the centers of the one-parameter family of spheres on which lie the curves $v=$const. of a surface Σ are given by three functions of v alone

$$V_1,\ V_2,\ V_3.$$

If, as usual, X, Y, Z are the direction-cosines of the normal to the surface, and $\alpha_1, \beta_1, \gamma_1$ are the direction-cosines of the tangent to the curve $u=$const., and if we suppose the parametric system orthogonal, we have

$$x=\xi X+\eta\alpha_1+V_1, \tag{54}$$

and similar equations for y and z, where ξ and η are functions of v alone. If now we have any five other functions of v satisfying the equations

* See § 2.

† Eisenhart, p. 132.

‡ Darboux, "Leçons sur la théorie générale des surfaces," Vol. III, p. 121, Paris (1896).

§ Bianchi, "Lezioni di Geometria Differenziale," Vol. II, p. 305, Pisa (1903).

$$\frac{\bar{\xi}'}{\xi'}=\frac{\bar{\eta}}{\eta}=\frac{\overline{V}_1'}{V_1'}=\frac{\overline{V}_2'}{V_2'}=\frac{\overline{V}_3'}{V_3'}, \tag{55}$$

where the prime denotes differentiation with respect to v, the point whose coordinates are of the form

$$\bar{x}=\bar{\xi}X+\bar{\eta}\alpha_1+\overline{V}_1 \tag{56}$$

describes a surface $\overline{\Sigma}$, which has the same spherical representation of its lines of curvature as Σ, and of which the curves $v=$const. are spherical and lie on spheres with centers given by

$$\overline{V}_1,\ \overline{V}_2,\ \overline{V}_3.$$

When the spherical lines of curvature of Σ have constant geodesic curvature, $\xi=0$, and conversely. For then the centers of the spheres lie on the tangents to the curves $u=$const. When $\xi=0$, it does not necessarily follow that $\bar{\xi}=0$, but, if we define a *modified Combescure transformation* by requiring that $\bar{\xi}=0$ when $\xi=0$, we can prove

THEOREM 15. *All surfaces S with a system of lines of curvature of constant geodesic curvature consist of*

1°. *The surfaces (S_1) obtained by transforming by inversion all surfaces of Monge.*

2°. *The surfaces (S_2) obtained by subjecting the surfaces S_1 to the modified Combescure transformation.*

To prove this theorem, it must be shown that, if $\xi=\bar{\xi}=0$, and

$$\frac{\bar{\eta}}{\eta}=\frac{\overline{V}_1'}{V_1'}=\frac{\overline{V}_2'}{V_2'}=\frac{\overline{V}_3'}{V_3'}=x, \tag{57}$$

it is always possible so to determine x that, after the transformation, the spheres to which the surface is orthogonal have a point in common. This fact is readily established by a suitable modification of the discussion given by Darboux* to prove the corresponding fact for the general Combescure transformation.

It is at once evident that the surfaces normal to the osculating circles of the spherical lines of curvature of a surface of the class (S_1) belong themselves to that class. That the corresponding theorem for surfaces of the class (S_2) is true is easily proved, in view of the theorem that, when a surface cuts a sphere orthogonally, the intersection is a line of curvature of the surface.

§ 11. *Cyclic Systems of Circles on a One-Parameter Family of Spheres.*

The cyclic systems discussed in the preceding section had the property that the circles were on a one-parameter family of planes or of spheres. We

* Darboux, "Leçons," Vol. IV, p. 249.

showed how to obtain all cyclic systems for which the circles were on a single infinity of planes. In this section, we determine all cyclic systems for which the circles are on a single infinity of spheres. We must first demonstrate some properties of the modified Combescure transformation.

Consider any one-parameter family of spheres admitting a family of orthogonal surfaces S. These surfaces are necessarily of class (S_1) or (S_2). Let S_0 be one of the surfaces S, and subject it to the modified Combescure transformation given by (57). If

$$x_0=\eta\alpha_1+V_1, \quad y_0=\eta\beta_1+V_2, \quad z_0=\eta\gamma_1+V_3,$$

then, by (56),

$$\left.\begin{aligned}\bar{x}_0&=\bar{\eta}\alpha_1+\overline{V}_1\\&=x\eta\alpha_1+\int xV_1'dv\\&=x(\eta\alpha_1+V_1)-\int x'V_1dv\\&=xx_0-\int x'V_1dv,\end{aligned}\right\} \tag{58}$$

and, similarly,

$$\left.\begin{aligned}\bar{y}_0&=xy_0-\int x'V_2dv,\\ \bar{z}_0&=xz_0-\int x'V_3dv.\end{aligned}\right\} \tag{59}$$

Equations (58) and (59) define, for each value of v, a linear transformation with constant coefficients. This transformation is independent of the particular surface S which is transformed. It depends only on the quantities x, V_1, V_2, V_3, and may be regarded as a uniform magnification leaving the origin invariant, followed by a translation. Since the points (of the surfaces S) which correspond to a given value of v all lie on the same sphere, we observe that the transformation of the surfaces S sets up a one-to-one correspondence between the points of each of the old spheres and the points of the corresponding new sphere.* In view of these remarks, we are able to state

THEOREM 16. *A given modified Combescure transformation carries all the surfaces orthogonal to a given one-parameter family of spheres over into surfaces orthogonal to a common set of spheres. The points on each of the original spheres are carried over into the points of the corresponding new sphere by a linear transformation, which varies with the sphere, but has, in each case, the properties:*

1°. *Circles are carried into circles.*

2°. *Angles are preserved.*

If the circles of a cyclic system lie on a one-parameter family of spheres through a common point, the cyclic system can be obtained by inversion from

* It is immaterial whether all points of the spheres are considered, or only those which lie on one or more of the orthogonal surfaces.

one in which the circles lie on a one-parameter family of planes. If the spheres do not pass through a common point, it is evident that the orthogonal surfaces are of the type S_2. As was remarked in § 10, there exist modified transformations of Combescure which carry these orthogonal surfaces of type S_2 over into surfaces of type S_1 orthogonal to a family of spheres through a common point. We have just shown that the same transformation may be used for all the orthogonal surfaces, and, by Theorem 16, the congruence of circles orthogonal to the surfaces S_2 is carried over into a congruence of circles orthogonal to the surfaces S_1. We have thus proved:

THEOREM 17. *Every cyclic system such that the circles lie on a single infinity of spheres can be obtained by:*

1°. *Subjecting to inversion the cyclic systems of circles whose planes envelop a developable surface.*

2°. *Subjecting to the modified transformation of Combescure the orthogonal surfaces of the cyclic systems given by* 1°.

The circles of one of these cyclic systems may be regarded as forming a one-parameter family of tubular surfaces, which are easily proved to be of type (S_1) or (S_2). Hence, 2° may be replaced by

2°. *Subjecting to the modified transformation of Combescure the tubular surfaces generated by the circles of the cyclic systems given by* 1°, *as a single sphere is made to take on in succession the positions of the spheres of the given family.*

IV. O-SURFACES.

§ 12. *Definition. Differential Equation.*

The characteristic property of an orthogonal system is that, at their points of meeting, the curves have orthogonal tangents. If we also require the principal normals to be orthogonal, the system must consist of asymptotic lines and their orthogonal trajectories; if we, on the other hand, add the requirement that the binormals be orthogonal, the system must consist of geodesics and their orthogonal trajectories.* In order to obtain all three properties simultaneously, it is necessary to consider geodesics whose geodesic parallels are asymptotic lines. Such a system will be called an *O-system,* a surface with one such system an *O-surface,* and a surface with two such systems a *double O-surface.*

The consideration of O-surfaces is also suggested by the notions of the second part of this paper. For the principal normals of a system of asymptotic

* Cf. §§ 2, 4.

lines form a normal congruence only in case the asymptotic lines are geodesic parallels, and the congruence of binormals of a geodesic system on a surface S has S for its middle surface only in case the orthogonal trajectories of the geodesics are asymptotic lines.

In order to determine whether any O-systems can exist, we must seek solutions of the fundamental equations (1) such that either

$$q=r=t_1=0,$$

or

$$q=r_1=t=0.$$

Making the first choice, we have also

$$\frac{\partial p}{\partial v}=0,$$

and can put $p=1$.* The equations (1) and the condition $t_1=0$ now take the form

$$\left.\begin{aligned} \frac{\partial^2 t}{\partial u^2}-\frac{t}{\tau^2}&=0,\\ \frac{1}{\rho}\frac{\partial t}{\partial u}+\frac{\partial}{\partial v}\left(\frac{1}{\tau}\right)&=0,\\ \frac{2}{\tau}\frac{\partial t}{\partial u}+t\frac{\partial}{\partial u}\left(\frac{1}{\tau}\right)-\frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)&=0. \end{aligned}\right\} \tag{60}$$

Eliminating ρ and τ, it is found that t must satisfy the partial differential equation of the fourth order

$$\frac{\partial}{\partial v}\left[\frac{1}{\frac{\partial t}{\partial u}}\frac{\partial}{\partial v}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}\right]+t\frac{\partial}{\partial u}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}+2\frac{\partial t}{\partial u}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}=0, \tag{61}$$

and that ρ and τ are then given by

$$\frac{1}{\rho}=-\frac{1}{\frac{\partial t}{\partial u}}\frac{\partial}{\partial v}\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}, \tag{62}$$

and

$$\frac{1}{\tau}=\sqrt{\frac{\frac{\partial^2 t}{\partial u^2}}{t}}. \tag{63}$$

Thus far we have always excluded the possibility that $1/\rho=0$, since the equations (1) then become meaningless. However, a special investigation of

* Cf. § 2.

this case proves that equations (60)–(63) are still valid when $1/\rho=0$. They are then easily integrated, and lead, in view of the theorem of Catalan,* to the right helicoid. For the curves $v=$const. are now rulings and the surface is therefore minimal.

If $1/\tau=0$, $1/\rho\neq 0$, equations (60) become

$$\frac{\partial^2 t}{\partial u^2}=0, \quad \frac{\partial t}{\partial u}=0, \quad \frac{\partial}{\partial v}\left(\frac{1}{\rho}\right)=0.$$

We can then take $t=1$, and ρ an arbitrary function of u. Referring to (2) and (3), we find that λ, μ and ν are then constant; or, from (20) and (22), it follows that $1/\rho_1=0$. Either of these facts shows that the O-surfaces which correspond to the assumption $1/\tau=0$, $1/\rho\neq 0$ are all cylinders. Moreover, every cylinder is clearly an O-surface. If $1/\tau=1/\rho=0$, it is found that the direction-cosines of the normal to the surface are constant; hence the surface is a plane.

§ 13. *Fundamental Quantities. Special Parametric Systems.*

We find that

$$\left.\begin{array}{l} E=1, \quad F=0, \quad G=t^2, \quad H=\varepsilon t, \\ D=-\dfrac{\varepsilon}{\rho}, \quad D'=-\dfrac{\varepsilon t}{\tau}, \quad D''=0, \end{array}\right\} \tag{64}$$

where $\varepsilon=-\cos\omega$ and is therefore either 1 or -1. Then the total and mean curvatures are given by

$$K=-\frac{1}{\tau^2}, \quad K_m=-\frac{\varepsilon}{\rho}. \tag{65}$$

These values of K and K_m could have been determined without reference to (64). For Enneper's theorem† states that

$$K=-\frac{1}{\tau_1^2},$$

where τ_1 is the radius of torsion of the asymptotic line $u=$const., and, from the theorem that the geodesic torsions in two orthogonal directions differ only in sign,‡ follows

$$\frac{1}{\tau_1}=-\frac{1}{\tau}.$$

Moreover, it is well known that the mean curvature of any surface is equal to the normal curvature of the orthogonal trajectories of the asymptotic lines.

* Eisenhart, p. 148. † Eisenhart, p. 140. ‡ Eisenhart, p. 139.

Conversely, it is readily proved that, if any surface has a system of geodesics such that either

$$K=-\frac{1}{\tau^2}$$

or

$$K_m=\frac{\cos\omega}{\rho},$$

the surface is necessarily an O-surface.

From (65) and the results of the last section, it follows that the only developable O-surfaces are the plane and the cylinders; the only minimal O-surfaces are the right helicoids.

We seek now conditions under which a surface referred to its asymptotic lines or its lines of curvature is an O-surface. If S is referred to its asymptotic lines, the condition that it be an O-surface, with the curves $v=$const. geodesic parallels, is that the curves defined by

$$p\,du+q\,dv=0$$

be geodesics. This condition is most easily deduced from the considerations of § 6. We found there that the necessary and sufficient condition that the congruence of principal normals of the curves $v=$const. form a normal congruence is that r be a function of v alone. We also found that

$$r=\frac{H}{\sqrt{E}}\sin\omega,$$

whence, in this case,

$$r=\pm\frac{H}{\sqrt{E}}.$$

We can therefore state

THEOREM 18. *The necessary and sufficient condition that a surface referred to its asymptotic lines be an O-surface is that one of the two conditions*

$$\frac{\partial}{\partial u}\left(\frac{H^2}{E}\right)=0,\quad \frac{\partial}{\partial v}\left(\frac{H^2}{G}\right)=0 \tag{66}$$

be satisfied. If both conditions are satisfied, the surface is a double O-surface. When the first of (66) *is satisfied, the parameter v can be so chosen that the linear element is*

$$E\,du^2+2\sqrt{E(G-1)}\,du\,dv+G\,dv^2; \tag{67}$$

when both conditions are satisfied, the linear element can take the form

$$E\,du^2+2\sqrt{E(E-1)}\,du\,dv+E\,dv^2. \tag{68}$$

Moreover, these linear elements are characteristic of O-surfaces referred to their asymptotic lines.

If we transform the parameters of (68) by putting

$$u=u_1+v_1, \quad v=u_1-v_1,$$

the new parametric curves are the lines of curvature and the first and second fundamental forms become

$$2(E+\sqrt{E(E-1)})du_1^2+2(E-\sqrt{E(E-1)})dv_1^2=E_1du_1^2+G_1dv_1^2,$$

and

$$2D'du_1^2-2D'dv_1^2=D_1du_1^2+D_1''dv_1^2.$$

Thus

$$\frac{1}{E_1}+\frac{1}{G_1}=1, \quad D_1+D_1''=0, \tag{69}$$

Conversely, if the lines of curvature are parametric and equations (69) are satisfied, by the substitution

$$u_1=\frac{u+v}{2}, \quad v_1=\frac{u-v}{2},$$

the asymptotic lines are made parametric, and the linear element assumes the form (68). Thus we have

THEOREM 19. *In order that a surface referred to its lines of curvature be a double O-surface, it is necessary and sufficient that the parameters can be so chosen that*

$$\frac{1}{E}+\frac{1}{G}=1, \quad D+D''=0.$$

The lines of curvature of a double O-surface form an isothermal-conjugate system and are geodesic ellipses and hyperbolas.†*

§ 14. *O-Surfaces which are Surfaces of Weingarten. Double O-Surfaces.*

In view of (65), the necessary and sufficient condition that an O-surface be a surface of Weingarten‡ is that ρ be a function of τ. If, for brevity, we put

$$Q=\frac{\dfrac{\partial^2 t}{\partial u^2}}{t},$$

equations (61)–(63) become

$$\frac{\partial^2 Q}{\partial v^2}-\frac{\dfrac{\partial^2 t}{\partial u\partial v}}{\dfrac{\partial t}{\partial u}}\frac{\partial Q}{\partial v}+t\frac{\partial t}{\partial u}\frac{\partial Q}{\partial u}-\frac{1}{2Q}\left(\frac{\partial Q}{\partial v}\right)^2+4Q\left(\frac{\partial t}{\partial u}\right)^2=0, \tag{70}$$

$$\frac{1}{\tau}=\sqrt{Q}, \tag{71}$$

* Eisenhart, p. 198. † Eisenhart, p. 213. ‡ Eisenhart, p. 291.

$$\frac{1}{\rho}=-\frac{1}{2\sqrt{Q}}\frac{\frac{\partial Q}{\partial v}}{\frac{\partial t}{\partial u}}. \tag{72}$$

The condition that ρ be a function of τ then becomes

$$\frac{\partial^2 Q}{\partial u\partial v}-tQ\frac{\frac{\partial Q}{\partial v}}{\frac{\partial t}{\partial u}}+t\frac{\frac{\partial t}{\partial u}\left(\frac{\partial Q}{\partial u}\right)^2}{\frac{\partial Q}{\partial v}}-\frac{\frac{\partial Q}{\partial u}\frac{\partial Q}{\partial v}}{2Q}+4\frac{Q\left(\frac{\partial t}{\partial u}\right)^2\frac{\partial Q}{\partial u}}{\frac{\partial Q}{\partial v}}=0. \tag{73}$$

In order to determine all O-surfaces which are W-surfaces, it would be necessary to find all common solutions of (70) and (73). That they are consistent is seen by supposing that

$$\frac{\partial t}{\partial u}=\frac{\partial t}{\partial v}, \tag{74}$$

for they then become identical.

In view of a well known theorem,* every solution of (70) which satisfies (74) corresponds to a surface which is either a helicoid or a surface of revolution, since t, ρ and τ are then functions of $u+v$. When (74) is satisfied, (70) reduces to the ordinary differential equation of the fourth order

$$t''''-\frac{t'\,t'''}{t}-\frac{t''\,t'''}{t'}-\frac{1}{2}\frac{t'''^2}{t''}+t\,t'\,t'''+\frac{3}{2}\frac{t'^2\,t''}{t^2}+3t'^2\,t''=0. \tag{75}$$

In order that an O-surface be double,

$$\phi=\frac{2\rho}{t\tau}=-\frac{4Q}{t}\frac{\frac{\partial t}{\partial u}}{\frac{\partial Q}{\partial v}}$$

must satisfy the equation of geodesics

$$\frac{\partial\phi}{\partial u}+\phi\frac{\partial\phi}{\partial v}+t\frac{\partial t}{\partial u}\phi^3+\frac{\frac{\partial t}{\partial v}}{t}\phi^2+2\frac{\frac{\partial t}{\partial u}}{t}\phi=0, \tag{76}$$

so that

$$\frac{\partial^2 Q}{\partial u\partial v}-\frac{tQ\frac{\partial Q}{\partial v}}{\frac{\partial t}{\partial u}}+\frac{\frac{\partial t}{\partial u}\frac{\partial Q}{\partial v}}{t}-\frac{\frac{\partial Q}{\partial u}\frac{\partial Q}{\partial v}}{Q}+\frac{4Q\left(\frac{\partial t}{\partial u}\right)^2\frac{\partial Q}{\partial u}}{\frac{\partial Q}{\partial v}}=0. \tag{77}$$

As before, we merely show that (70) and (77) are consistent and do not attempt to find all their common solutions. If (74) is satisfied, (70) and (77) are consistent only if

* A. Enneper, *Göttingen Nachrichten*, 1870, p. 335.

$$\frac{\left(\frac{\partial Q}{\partial u}\right)^2}{2Q}-\frac{\frac{\partial t}{\partial u}\frac{\partial Q}{\partial u}}{t}+t\frac{\partial t}{\partial u}\frac{\partial Q}{\partial u}=0.$$

Disregarding the trivial case in which

$$\frac{\partial Q}{\partial u}=0,$$

we must take

$$\frac{\partial Q}{\partial u}=2Q\left(\frac{\frac{\partial t}{\partial u}}{t}-t\frac{\partial t}{\partial u}\right). \tag{78}$$

It is found that, when (74) is satisfied, every solution of (78) satisfies both (70) and (77). Hence the O-surfaces which correspond to solutions of the ordinary differential equation of the third order

$$t'''-3\frac{t'\,t''}{t}+2t\,t'\,t''=0 \tag{79}$$

are double O-surfaces and at the same time are W-surfaces. The solution of (79) can be reduced to a single quadrature, and is given by

$$\int\frac{dt}{\sqrt{c_2-c_1e^{-t^2}(1+t^2)}}=u+v+c_3.$$

§ 15. *Two Characteristic Properties of O-Surfaces.*

In case the curves C are geodesics, Theorem 9 affords the following characterization of O-surfaces, which is also a direct consequence of (65):

THEOREM 20. *In order that a surface S be an O-surface, it is necessary and sufficient that there be on it a system of geodesics, with respect to the osculating circles of which the corresponding points of the two surfaces of center of S are harmonic conjugates.*

The considerations of § 7 lead to another characteristic property of O-surfaces. When $t=0$, (32) becomes

$$\bar{x}=x+\frac{V'\tau}{r}\alpha-V\lambda.$$

If, furthermore, S is an O-surface with the curves $v=$const. geodesic parallels, we have by § 13

$$\frac{\partial r}{\partial u}=0,$$

whence, taking $r=1$, $V=v$, the point whose coordinates are of the form

$$\bar{x}=x+\tau\alpha-v\lambda$$

generates a surface with the same spherical representation of the parametric lines as S.

Conversely, if S is any surface with the curves $v=\text{const.}$ asymptotic lines, and the congruence Γ given by

$$x_0=x+\tau a, \quad X_0=\lambda$$

is normal, we must have

$$\frac{\partial}{\partial u}(\tau A_3)=-\frac{\partial r}{\partial u}=0,$$

so that the surface S is an O-surface. Thence follows

THEOREM 21. *Let S be any surface, and, on each asymptotic line of one system, mark, on the positive* half-tangent, the point P_1 whose distance from the point P of contact is equal to the radius of torsion of the asymptotic line, and hence equal, to within the sign, to*

$$\sqrt{-\frac{1}{K}},$$

where K is the total curvature of S. Then the necessary and sufficient condition that S be an O-surface, with the asymptotic lines in question as geodesic parallels, is that the parallel, through P_1, to the normal to the surface shall generate a normal congruence.

PRINCETON UNIVERSITY, *June*, 1914.

* The negative would do equally well.

Plane Sextic Curves Invariant under Birational Transformations.

By Anna Helen Tappan.

Introduction.

One of the fundamental problems in the theory of birational transformations is the reduction of an algebraic curve of any given genus to its canonical form. Clebsch* has shown that the non-hyperelliptic curve of genus p can always be birationally reduced to a curve of order $p-\pi+2$, where $p=3\pi+(0, 1, 2)$; thus a general curve of genus 5 can be reduced to a sextic with five double points, and one of genus 6 to a sextic with four double points.

The groups of birational transformations belonging to normal curves of genus $p=3$ have been studied by Wiman.† He has also treated algebraic curves of genus 4, 5, 6, briefly. Miss Van Benschoten,‡ however, has studied plane curves of genus 4 in more detail, and has determined their forms and properties. The groups of birational transformations of algebraic curves of genus 5 have been studied by McKelvey,§ and those belonging to the normal curves of genus 6 have been determined by Snyder.||

We have noted above that the canonical form of the general curve of genus either $p=5$ or $p=6$ is a sextic; moreover, the groups of birational transformations belonging to these two genera (and in particular to the canonical sextics) are especially interesting in that we are no longer concerned merely with linear transformations. It is the object of the present paper to discuss the various types of sextics which are invariant under any birational transformation, linear or non-linear.

* Clebsch, "Vorlesungen über Geometrie," Vol. I, Leipzig (1876), pp. 687, 710.

† Wiman, "Ueber die Hyperelliptischen Curven und diejenigen vom Geschlechte $p=3$," *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI (1895).

‡ A. L. Van Benschoten, "The Birational Transformations of Algebraic Curves of Genus 4," American Journal of Mathematics, Vol. XXXI (1909).

§ J. V. McKelvey, "Groups of Birational Transformations of Algebraic Curves of Genus 5," American Journal of Mathematics, Vol. XXXIV (1912).

|| V. Snyder, "Normal Curves of Genus 6, and their Groups of Birational Transformations," American Journal of Mathematics, Vol. XXX (1908).

The problem of finding all the algebraic plane curves invariant with respect to collineations of their plane has been solved up to and including all curves of the fifth order.* But the various types of sextics invariant under collineations of their plane have, so far, never been determined in detail. A few months ago, Vojtěch† published a memoir concerning the plane sextic curves that are invariant under periodic collineations which leave the sides of a triangle invariant. His results agree for the most part with my own, which are given in the first part of the first chapter of this paper. Dr. Vojtěch‡ omits one type invariant under a particular collineation of period 11 and includes a composite sextic among those invariant under a continuous linear group. His paper is written in Czechish and there is no abstract of it in any other language. My own results were obtained before seeing it, and, though I can not claim priority, I believe it is wisest to include this part of my results in the present paper. I am unable to comment on the methods employed by Dr. Vojtěch; our results agree with the exception of the two cases mentioned above.

After my paper was sent to the publisher, a second paper by Dr. Vojtěch appeared,§ which discusses the groups of classes (b) and (c). In a foot-note the error in the former paper is corrected. As our results agree, I believe the enumeration is complete.

In the present paper we shall discuss in the first chapter the various groups (cyclic and non-cyclic) of linear transformations which leave invariant plane sextic curves of their plane; in the second chapter we shall be concerned primarily with non-linear birational transformations, both Cremona and Riemann, which leave plane sextic curves invariant.

CHAPTER I.

LINEAR TRANSFORMATIONS.

Groups of linear transformations under which plane sextic curves remain invariant may be divided into the three following classes:

(a) Those which have as invariant points the vertices of the fundamental triangle.

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* V. Snyder, "Plane Quintic Curves which Possess a Group of Linear Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908); E. Ciani, "Le Quintiche Piane Autoproiettive," *Rend. Circ. Mat. Palermo*, Vol. XXXVI (1913).

† J. Vojtěch, "Rovinné Sextiky Invariantní při Periodických Kollineacích," Prague, 1913.

‡ Vojtěch, *loc. cit.*, p. 20, (10); p. 22, (23).

§ J. Vojtěch, "Konečné grupy kollineací a rovinné sextiky k sobě příslušne," *České akademie césare Frantеška Josefa* 42, XXII (1913), 29 pp.

(b) Those which are obtained by combining transformations of class (a) with one or more transformations of the permutation group of the three homogeneous point coordinates.

(c) Those which do not belong to class (a) or to class (b), such as the simple G_{168} (Klein), which leaves invariant the Hessian of Klein's quartic.

The most simple sextic invariant under any linear transformation is the binomial sextic, of which there is only one type. The equation of this sextic is $f_6 \equiv z^6 + x^5 y = 0$. The collineation belonging to it is $\begin{pmatrix} \alpha^6 x & y & \alpha^5 z \\ x & y & z \end{pmatrix}$, where α is arbitrary. This transformation need not be periodic.

We shall now consider what sextics are left invariant by homologies, represented by $\begin{pmatrix} x\, y\, \alpha z \\ x\, y\, z \end{pmatrix}$, $\alpha^r = 1$, where r is the period. If we write the general equation of a sextic as

$$f_6 \equiv az^6 + a_1 z^5 + a_2 z^4 + a_3 z^3 + a_4 z^2 + a_5 z + a_6 = 0,$$

where a is a constant and a_i is a binary in x, y of degree i, we find the five following types invariant under homologies of period $2 \leqq r \leqq 6$:

For $r=2$: $f_6 \equiv az^6 + a_2 z^4 + a_4 z^2 + a_6 = 0.$ (1_1)

For $r=3$: $f_6 \equiv az^6 + a_3 z^3 + a_6 = 0.$ (2_1)

For $r=4$: $f_6 \equiv a_2 z^4 + a_6 = 0.$ (Derived from (1_1).) (3_1)

For $r=5$: $f_6 \equiv a_1 z^5 + a_6 = 0.$ (4_1)

For $r=6$: $f_6 \equiv az^6 + a_6 = 0.$ (Derived from (2_1).) (5_1)

(A) *General Discussion of Collineations of Type* (a).

In our further discussion of collineations of this class we shall be concerned with those which are not homologies. For collineations whose period is a power of 2, we find nine types of sextics for $r=4$, and seven for $r=8$.

For $r=4$, $C \equiv \begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4(ax^2 + by^2) + z^2(cx^3y + dxy^3) + ex^6 + fx^4y^2 + gx^2y^4 + hy^6 = 0. \quad (1_1). \qquad (6_1)$$

$$f_6 \equiv az^6 + bz^4xy + z^2(cx^4 + dx^2y^2 + ey^4) + fx^5y + gx^3y^3 + hxy^5 = 0. \quad (1_1). \qquad (7_1)$$

For $r=8$, $C \equiv \begin{pmatrix} ax & a^3y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + dz^2x^2y^2 + x^5y + xy^5 = 0. \qquad ((7_1) \text{ if } b=c=e=g=0.) \qquad (8_1)$$

This sextic is invariant under $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that it belongs to the group ($n=16$) generated by A and C. (Cf. (5_2).)

$$f_6 \equiv z^4x^2 + dz^2xy^3 + x^4y^2 + y^6 = 0. \qquad ((6_1) \text{ if } b=c=e=g=0.) \qquad (9_1)$$

$$f_6 \equiv bz^4xy + z^2(x^4+y^4) + x^3y^3 = 0. \qquad ((7_1) \text{ if } a=d=f=h=0.) \qquad (10_1)$$

This sextic is invariant under $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that it belongs to the group $(n=16)$ generated by C and A. (Cf. (6_2).)

For $r=8$, $C \equiv \begin{pmatrix} \alpha x & \alpha^5 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4xy + x^6 + ax^4y^2 + bx^2y^4 + y^6 = 0. \qquad (3_1). \qquad (11_1)$$

$$f_6 \equiv z^4(x^2+y^2) + ax^5y + bx^3y^3 + cxy^5 = 0. \qquad (3_1). \qquad (12_1)$$

For $r=8$, $C \equiv \begin{pmatrix} \alpha x & \alpha^7 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4x^2 + cz^2x^3y + x^4y^2 + y^6 = 0. \qquad ((6_1) \text{ if } b=d=e=g=0.) \qquad (13_1)$$

$$f_6 \equiv z^2(x^4+y^4) + fx^5y + xy^5 = 0. \qquad ((7_1) \text{ if } a=b=d=g=0.) \qquad (14_1)$$

There are seven types of sextics which remain invariant under collineations whose period is a power of 3: two types for $r=3$, and five for $r=9$.

For $r=3$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^6 + bz^4xy + z^3(cx^3+dy^3) + ez^2x^2y^2 + z(fx^4y+gxy^4) + hx^6 + jx^3y^3 + ky^6 = 0. \qquad (15_1)$$

$$f_6 \equiv az^5y + bz^4x^2 + cz^3xy^2 + z^2(dx^3y+ey^4) + z(fx^5+gx^2y^3) + hx^4y^2 + jxy^5 = 0. \qquad (16_1)$$

For $r=9$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6 + zxy^4 + x^3y^3 = 0. \qquad ((15_1) \text{ if } b=c=d=e=f=h=k=0.) \qquad (17_1)$$

For $r=9$, $C \equiv \begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^3x^2y + x^6 + ax^3y^3 + y^6 = 0. \qquad (2_1). \qquad (18_1)$$

$$f_6 \equiv z^6 + az^3xy^2 + x^5y + bx^2y^4 = 0. \qquad (2_1). \qquad (19_1)$$

$$f_6 \equiv z^3(x^3+y^3) + ax^4y^2 + bxy^5 = 0. \qquad (2_1). \qquad (20_1)$$

For $r=9$, $C \equiv \begin{pmatrix} \alpha x & \alpha^5 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^5y + zx^5 + x^4y^2 = 0. \qquad ((16_1) \text{ if } b=c=d=e=g=j=0.) \qquad (21_1)$$

Twelve types of sextics remain invariant under collineations whose period can be expressed as $r=2^m \cdot 3^n$, where $m \geq 1$, $n \geq 1$. For $r=6$, there are three types; for $r=12$, there are five; for $r=18$, there are two; and for $r=24$, there are two.

For $r=6$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+dz^3y^3+ez^2x^2y^2+fzx^4y+hx^6+ky^6=0. \quad ((15_1) \text{ if } b=c=g=j=0.) \quad (22_1)$$

$$f_6\equiv az^5y+bz^4x^2+ez^2y^4+gzx^2y^3+hx^4y^2=0. \quad ((16_1) \text{ if } c=d=f=j=0.) \quad (23_1)$$

For $r=6$, $C\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^3(bx^2y+cy^3)+dx^6+ex^4y^2+fx^2y^4+gy^6=0. \quad (2_1). \quad (24_1)$$

For $r=12$, $C\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+cz^3y^3+x^4y^2+y^6=0. \quad ((24_1) \text{ if } b=d=f=0.) \quad (25_1)$$

For $r=12$, $C\equiv\begin{pmatrix} \alpha x & \alpha^7 y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+x^5y+gx^3y^3+xy^5=0. \quad ((5_1) \text{ or } (7_1) \text{ if } b=c=d=e=0.) \quad (26_1)$$

This sextic is invariant under $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that its group ($n=24$) is generated by A and C. (Cf. (17_2).)

For $r=12$, $C\equiv\begin{pmatrix} \alpha x & \alpha^8 y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+z^3y^3+fzx^4y+y^6=0. \quad ((22_1) \text{ if } e=h=0.) \quad (27_1)$$

This sextic is invariant under $B\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$ also, so that it belongs to the group ($n=24$) generated by C and B. (Cf. (18_2).)

For $r=12$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{10} y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^3y^3+x^6+x^2y^4=0. \quad ((24_1) \text{ if } a=b=e=g=0.) \quad (28_1)$$

$$f_6\equiv z^6+bz^3x^2y+ex^4y^2+y^6=0. \quad ((24_1) \text{ if } c=d=f=0.) \quad (29_1)$$

For $r=18$, $C\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^3x^2y+x^6+y^6=0. \quad ((18_1) \text{ if } a=0;\ (24_1) \text{ if } a=c=e=f=0.) \quad (30_1)$$

For $r=18$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{13} y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+x^5y+x^2y^4=0. \quad ((5_1);\ (19_1) \text{ if } a=0.) \quad (31_1)$$

For $r=24$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{19} y & z \\ x & y & z \end{pmatrix}$,

$$f_6=z^6+x^5y+xy^5=0. \quad ((7_1) \text{ if } b=c=d=e=g=0.) \quad (32_1)$$

This equation is readily seen to be invariant under $\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$ also, so that the group belonging to 32_1 is the non-cyclic G_{48}. (Cf. (24_2).)

For $r=24$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{20}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+zx^4y+y^6=0. \qquad ((22_1) \text{ if } d=e=h=0.) \qquad (33_1)$$

This equation is evidently invariant under $\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$ also, so that the group belonging to (33_1) is a non-cyclic G_{48}. (Cf. (25_2).)

If we consider, in the next place, collineations whose period is 5 or a power of 5, we discover three types invariant for $r=5$, and one for $r=25$.

For $r=5$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{3}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^5x+z^4y^2+bz^2x^3y+czx^2y^3+dx^6+exy^5=0. \qquad (34_1)$$

$$f_6\equiv az^5y+z^3x^3+bz^2x^2y^2+czxy^4+dx^5y+ey^6=0. \qquad (35_1)$$

$$f_6\equiv z^4x^2+az^3xy^2+z^2y^4+bzx^4y+x^3y^3=0. \qquad (36_1)$$

For $r=25$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{6}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+x^6+xy^5=0. \qquad (4_1). \qquad (37_1)$$

Collineations whose period can be expressed as $r=5\cdot 2^m\cdot 3^n$ leave invariant ten types. Of these, five belong to $r=10$, two to $r=15$, two to $r=20$, and one to $r=30$.

For $r=10$, $C\equiv\begin{pmatrix} \alpha^2 x & \alpha y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5x+z^4y^2+x^6=0. \qquad ((34_1) \text{ if } b=c=e=0.) \qquad (38_1)$$

For $r=10$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{3}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4x^2+z^2y^4+x^3y^3=0. \qquad ((36_1) \text{ if } a=b=0.) \qquad (39_1)$$

$$f_6\equiv z^4y^2+bz^2x^3y+x^6+xy^5=0. \qquad ((34_1) \text{ if } a=c=0.) \qquad (40_1)$$

For $r=10$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{6}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+x^6+ax^4y^2+bx^2y^4+y^6=0. \qquad (4_1). \qquad (41_1)$$

For $r=10$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{8}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4y^2+zx^2y^3+x^6=0. \qquad ((34_1) \text{ if } a=b=e=0.) \qquad (42_1)$$

For $r=15$, $C\equiv\begin{pmatrix} ax & a^3y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+z^3x^3+y^6=0. \qquad ((35_1) \text{ if } b=c=d=0.) \qquad (43_1)$$

For $r=15$, $C\equiv\begin{pmatrix} ax & a^6y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+x^6+ax^3y^3+by^6=0. \qquad (4_1). \qquad (44_1)$$

For $r=20$, $C\equiv\begin{pmatrix} ax & a^6y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+x^6+x^2y^4=0. \qquad (41_1) \text{ if } a=c=0.) \qquad (45_1)$$

For $r=20$, $C\equiv\begin{pmatrix} ax & a^{13}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4y^2+x^6+xy^5=0. \qquad ((34_1) \text{ if } a=b=c=0.) \qquad (46_1)$$

For $r=30$, $C\equiv\begin{pmatrix} ax & a^6y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+x^6+y^6=0. \qquad ((41_1) \text{ if } a=b=0.) \qquad (47_1)$$

If the period of the collineation is 7 or a multiple of 7, we have ten types which remain invariant. To $r=7$ belong six types; to $r=14$, three; to $r=21$, one.

For $r=7$, $C\equiv\begin{pmatrix} ax & a^2y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4y^2+az^3x^2y+z^2x^4+xy^5=0. \qquad (48_1)$$

$$f_6\equiv z^5y+z^4x^2+azxy^4+x^3y^3=0. \qquad (49_1)$$

For $r=7$, $C\equiv\begin{pmatrix} ax & a^3y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+az^3xy^2+zx^4y+x^2y^4=0. \qquad (50_1)$$

$$f_6\equiv z^4x^2+z^3y^3+azx^3y^2+xy^5=0. \qquad (51_1)$$

$$f_6\equiv z^5x+az^2x^2y^2+zy^5+x^5y=0. \qquad (52_1)$$

The last sextic is evidently invariant under $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$ also. Consequently the group belonging to (52_1) is a non-cyclic G_{21}. (Cf. (31_2).)

For $r=7$, $C\equiv\begin{pmatrix} ax & a^4y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+az^2x^3y+bzx^2y^3+xy^5=0. \qquad (53_1)$$

For $r=14$, $C\equiv\begin{pmatrix} ax & a^9y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4y^2+z^2x^4+xy^5=0. \qquad ((48_1) \text{ if } a=0.) \qquad (54_1)$$

For $r=14$, $C\equiv\begin{pmatrix} ax & a^{10}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+zx^4y+x^2y^4=0. \qquad ((50_1) \text{ if } a=0.) \tag{55_1}$$

For $r=14$, $C\equiv\begin{pmatrix} ax & a^{11}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^6+z^2x^3y+xy^5=0. \qquad ((53_1) \text{ if } b=0.) \tag{56_1}$$

For $r=21$, $C\equiv\begin{pmatrix} ax & a^{17}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5x+zy^5+x^5y=0. \qquad ((52_1) \text{ if } a=0.) \tag{57_1}$$

The last sextic is obviously invariant under $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$ also. Consequently the group belonging to (57_1) is a non-cyclic G_{63}. (Cf. (32_2).)

We have still to consider those collineations whose period is 11, 13, 17 or 19. For $r=11$, there are two invariant types; for $r=13$, there are two types; for $r=17$, there is one type; and for $r=19$, there is one type.

For $r=11$, $C\equiv\begin{pmatrix} ax & a^{3}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5x+z^2y^4+x^3y^3=0. \tag{58_1}$$

$$f_6\equiv z^5y+z^3x^3+x^2y^4=0. \tag{59_1}$$

For $r=13$, $C\equiv\begin{pmatrix} ax & a^{3}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+z^3x^3+xy^5=0. \tag{60_1}$$

$$f_6\equiv z^3x^2y+zx^5+y^6=0. \tag{61_1}$$

For $r=17$, $C\equiv\begin{pmatrix} ax & a^{4}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^5y+z^2x^4+xy^5=0. \tag{62_1}$$

For $r=19$, $C\equiv\begin{pmatrix} ax & a^{4}y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4xy+zx^5+y^6=0. \tag{63_1}$$

So far we have obtained sixty-three types which remain invariant under cyclic groups belonging to class (a); the highest cycle we obtained was of order 30. There remain a few sextics which are invariant under non-cyclic groups of this class, groups generated by two or more transformations of class (a). For these types, which are given below, n stands for the order of the group, and r for the period of the transformation.

We have for $n=4$, $A\equiv\begin{pmatrix}-x & y & z\\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix}x & y & -z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^6+z^4(bx^2+cy^2)+z^2(dx^4+ex^2y^2+fy^4)$$
$$+gx^6+hx^4y^2+jx^2y^4+ky^6=0. \qquad (\text{Cf. } (1_2).) \qquad (64_1)$$

For $n=8$, $A\equiv\begin{pmatrix}-x & y & z\\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix}ix & -iy & z\\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+ex^6+fx^4y^2+gx^2y^4+hy^6=0. \quad ((6_1) \text{ if } c=d=0.) \quad (65_1)$$

For $n=9$, $A\equiv\begin{pmatrix}x & y & \alpha z\\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix}\alpha x & \alpha^2 y & z\\ x & y & z\end{pmatrix}$, $r=3$.

$$f_6\equiv az^6+z^3(cx^3+dy^3)+hx^6+jx^3y^3+ky^6=0. \quad ((15_1) \text{ if } b=e=f=g=0.) \quad (66_1)$$

For $n=12$, $A\equiv\begin{pmatrix}x & -y & z\\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix}\alpha x & \alpha^2 y & z\\ x & y & z\end{pmatrix}$, $r=6$,

$$f_6\equiv z^4x^2+z^2y^4+x^4y^2=0. \quad ((23_1) \text{ is } a=g=0.) \quad (67_1)$$

This sextic is evidently invariant under $\begin{pmatrix}y & z & x\\ x & y & z\end{pmatrix}$ also, so that the group belonging to (67_1) is a G_{36}. (Cf. (21_2).)

For $n=12$, $A\equiv\begin{pmatrix}x & y & -z\\ x & y & x\end{pmatrix}$, $B\equiv\begin{pmatrix}\alpha x & \alpha^4 y & z\\ x & y & z\end{pmatrix}$, $r=6$,

$$f_6\equiv z^6+x^6+ex^4y^2+fx^2y^4+y^6=0. \quad ((24_1) \text{ if } b=c=0.) \quad (68_1)$$

For $n=16$, $A\equiv\begin{pmatrix}-x & y & z\\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix}\alpha x & \alpha^3 y & z\\ x & y & z\end{pmatrix}$, $r=8$,

$$f_6\equiv z^4x^2+x^4y^2+y^6=0. \quad ((9_1) \text{ if } d=0;\ (13_1) \text{ if } c=0.) \quad (69_1)$$

For $n=18$, $A\equiv\begin{pmatrix}x & y & \alpha z\\ x & y & z\end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix}\alpha x & \alpha^2 y & z\\ x & y & z\end{pmatrix}$, $r=3$,

$$f_6\equiv z^6+x^6+ax^3y^3+y^6=0. \quad (5_1). \quad (70_1)$$

This sextic is evidently invariant under $\begin{pmatrix}y & x & z\\ x & y & z\end{pmatrix}$ also, so that the group belonging to (70_1) is a G_{36}. (Cf. (22_2).)

For $n=24$, $A\equiv\begin{pmatrix}x & y & -z\\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix}\alpha x & \alpha^4 y & z\\ x & y & z\end{pmatrix}$, $r=12$,

$$f_6\equiv z^6+x^4y^2+y^6=0. \quad ((25_1) \text{ if } c=0.) \quad (71_1)$$

(B) *Collineations of Class* (b).

We shall now discuss sextics which are invariant under groups belonging to class (b). Considering first groups whose orders are powers of 2, we find that the simplest one is the non-cyclic G_4 generated by $\begin{pmatrix}x & y & -z\\ x & y & z\end{pmatrix}$ and $\begin{pmatrix}y & x & z\\ x & y & z\end{pmatrix}$. The

equation belonging to this group is

$$f_6 \equiv az^6 + z^4(bx^2+cxy+by^2) + z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4)$$
$$+kx^6+lx^5y+mx^4y^2+nx^3y^3+mx^2y^4+lxy^5+ky^6=0. \qquad (1_2)$$

This equation is equivalent to (64_1), which remains invariant under the non-cyclic G_4, generated by $\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$ and $\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$.

For $n=8$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $B \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^6 + bz^4(x^2+y^2) + z^2(ex^4+gx^2y^2+ey^4) + kx^6+mx^4y^2+mx^2y^4+ky^6=0. \qquad (2_2)$$

For $n=8$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & iz \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^4(x^2+bxy+y^2) + cx^6+dx^5y+ex^4y^2+fx^3y^3+ex^2y^4+dxy^5+cy^6=0. \quad (3_1). \qquad (3_2)$$

For $n=8$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$, $r=4$,

$$f_6 \equiv az^6 + bz^4xy + cz^2(x^4+dx^2y^2+y^4) + fxy(x^4+gx^2y^2+y^4)=0. \quad (7_1). \qquad (4_2)$$

For $n=16$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$, $r=8$,

$$f_6 \equiv z^6 + dz^2x^2y^2 + x^5y + xy^5 = 0. \quad (8_1). \qquad (5_2)$$
$$f_6 \equiv bz^4xy + z^2(x^4+y^4) + x^3y^3 = 0. \quad (10_1). \qquad (6_2)$$

For $n=16$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^5 y & z \\ x & y & z \end{pmatrix}$, $r=8$,

$$f_6 \equiv z^4xy + x^6 + ax^4y^2 + ax^2y^4 + y^6 = 0. \quad (11_1). \qquad (7_2)$$
$$f_6 \equiv z^4(x^2+y^2) + ax^5y + bx^3y^3 + axy^5 = 0. \quad (12_1). \qquad (8_2)$$

If we consider groups whose orders are powers of 3, we have

For $n=9$, $A \equiv \begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv az^5y + bz^4x^2 + cz^3xy^2 + z^2(cx^3y+by^4) + z(ax^5+cx^2y^3)$$
$$+bx^4y^2+axy^5=0. \quad (16_1). \qquad (9_2)$$

For groups whose orders can be expressed as $n=2^m \cdot 3^h$, where $m \geqq 1$, $h \geqq 1$, we have:

For $n=6$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv az^6 + (bx^3+cx^2y+cxy^2+bx^3)z^3 + dx^6+ex^5y+fx^4y^2$$
$$+gx^3y^3+fx^2y^4+exy^5+dy^6=0. \quad (2_1). \qquad (10_2)$$

For $n=6$, $A \equiv \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C \equiv \begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv az^6+bz^4xy+cz^3(x^3+y^3)+ez^2x^2y^2+fz(x^4y+xy^4) +hx^6+jx^3y^3+hy^6=0. \qquad (15_1). \qquad (11_2)$$

For $n=12$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6 \equiv az^6+dx^6+ex^5y+fx^4y^2+gx^3y^3+fx^2y^4+exy^5+dy^6=0. \qquad (5_1). \qquad (12_2)$$

For $n=12$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6 \equiv az^6+dz^3y^3+ez^2x^2y^2+fzx^4y+hx^6+ay^6=0. \qquad (22_1). \qquad (13_2)$$

For $n=12$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^6+z^4(bx^2+cy^2)+z^2(cz^4+ex^2y^2+by^4)+ax^6 +bx^4y^2+cx^2y^4+ay^6=0. \qquad (64_1). \qquad (14_2)$$

For $n=18$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv z^6+bz^3(x^3+y^3)+x^6+gx^3y^3+y^6=0. \qquad (10_2) \text{ or } (66_1). \qquad (15_2)$$

For $n=18$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6+bz^4xy+cz^3(x^3+y^3)+ez^2x^2y^2+bz(x^4y+xy^4) +x^6+cx^3y^3+y^6=0. \qquad (11_2). \qquad (16_2)$$

For $n=24$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^7 y & z \\ x & y & z \end{pmatrix}$, $r=12$,

$$f_6 \equiv z^6+x^5y+x^3y^3+xy^5=0. \qquad (26_1). \qquad (17_2)$$

For $n=24$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^8 y & z \\ x & y & z \end{pmatrix}$, $r=12$,

$$f_6 \equiv z^6+z^3y^3+zx^4y+y^6=0. \qquad (27_1). \qquad (18_2)$$

For $n=24$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6+z^2x^2y^2+x^6+y^6=0. \qquad (13_2). \qquad (19_2)$$

(19_2) permits necessarily, in addition, $\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, giving us for its group $n=72$, instead of $n=24$.

For $n=24$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $D\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^6+bz^4(x^2+y^2)+bz^2(x^4+ex^2y^2+y^4) +x^6+bx^4y^2+bx^2y^4+y^6=0. \qquad (14_2). \qquad (20_2)$$

For $n=36$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z\end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} y & z & x \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^4x^2+z^2y^4+x^4y^2=0. \qquad (67_1). \qquad (21_2)$$

For $n=36$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z\end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z\end{pmatrix}$, $r=3$,

$$f_6\equiv z^6+x^6+ax^3y^3+y^6=0. \qquad (70_1). \qquad (22_2)$$

For $n=24$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z\end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+x^6+ex^4y^2+ex^2y^4+y^6=0. \qquad (68_1). \qquad (23_2)$$

For $n=48$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{19} y & z \\ x & y & z\end{pmatrix}$, $r=24$,

$$f_6\equiv z^6+x^5y+xy^5=0. \qquad (32_1). \qquad (24_2)$$

For $n=48$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{20} y & z \\ x & y & z\end{pmatrix}$, $r=24$,

$$f_6\equiv z^6+zx^4y+y^6=0. \qquad (33_1). \qquad (25_2)$$

For $n=54$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z\end{pmatrix}$, $r=3$, $B\equiv\begin{pmatrix} y & z & x \\ x & y & z\end{pmatrix}$,

$D\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z\end{pmatrix}$, $r=3$,

$$f_6\equiv z^6+bz^3(x^3+y^3)+x^6+bx^3y^3+y^6=0. \qquad (15_2). \qquad (26_2)$$

For $n=72$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z\end{pmatrix}$, $r=6$, $B\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$, $D\equiv\begin{pmatrix} y & z & x \\ x & y & z\end{pmatrix}$,

$$f_6\equiv z^6+az^2x^2y^2+x^6+y^6=0. \qquad (22_2). \qquad (27_2)$$

For $n=216$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix} y & z & x \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & y & z \\ x & y & z\end{pmatrix}$, $r=6$,

$D\equiv\begin{pmatrix} x & \alpha y & z \\ x & y & z\end{pmatrix}$, $r=6$,

$$f_6\equiv x^6+y^6+z^6=0. \qquad (27_2). \qquad (28_2)$$

For groups whose orders are multiples of 5, we have:

For $n=10$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z\end{pmatrix}$, $r=5$,

$$f_6\equiv az^5(x+y)+bx^6+cx^5y+dx^4y^2+ex^3y^3+dx^2y^4+cxy^5+by^6=0. \qquad (4_1). \qquad (29_2)$$

For $n=10$, $A\equiv\begin{pmatrix} z & y & x \\ x & y & z\end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z\end{pmatrix}$, $r=5$,

$$f_6\equiv az^5y+z^3x^3+bz^2x^2y^2+czxy^4+ax^5y+by^6=0. \qquad (35_1). \qquad (30_2)$$

Finally, for groups whose orders are multiples of 7, we have:

For $n=21$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^3 y & z \\ x & y & z \end{pmatrix}$, $r=7$,

$$f_6\equiv z^5x+az^2x^2y^2+zy^5+x^5y=0. \qquad (52_1). \qquad (31_2)$$

For $n=63$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^{17} y & z \\ x & y & z \end{pmatrix}$, $r=21$,

$$f_6\equiv z^5x+zy^5+x^5y=0. \qquad (57_1). \qquad (32_2)$$

(c) *Collineations of Class* (c).

The well-known groups which belong to class (c), namely the G_{168} and the G_{360}, are both simple groups. The G_{168} was derived by Klein* through consideration of transformations of the seventh order of elliptic functions. The sextic which is invariant under the G_{168} is the Hessian of the Klein quartic,

$$C_4\equiv x^3y+y^3z+z^3x=0,$$

viz.,

$$f_6\equiv z^5y-5z^2x^2y^2+zx^5+xy^5=0. \qquad (1_3)$$

It is thus seen to be a particular case of (31_2).

The simple group G_{360} was first exhibited by Valentiner,† and it has been studied in some detail by him and also by A. Wiman.‡ The simplest curve belonging to the G_{360} is

$$f_6\equiv 9z^5y+10z^3x^3-45z^2x^2y^2-135zxy^4+9x^5y+27y^6=0. \qquad (2_3)$$

This is seen to be a particular case of (35_1).

The G_{360} is made up of forty-five collineations of period 2, eighty of period 3, ninety of period 4, and one hundred and forty-four of period 5.

The remaining groups of class (c) are the regular body groups not belonging to class (a) or class (b).

From the tetrahedral G_{12}, which leaves the two binary forms $xy(x^4-y^4)$, $x^4+2i\sqrt{3}x^2y^2+y^4$ invariant, we obtain the sextic

$$f_6\equiv az^6+bz^2(x^4+2i\sqrt{3}x^2y^2+y^4)+cxy(x^4-y^4)=0, \qquad (3_3)$$

invariant under G_{24}, defined by

$$\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}, \begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}, \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}, \text{ and } \begin{pmatrix} x+iy & x-iy & \alpha z \\ x & y & z \end{pmatrix}, \qquad \alpha=\sqrt{i+\sqrt{3}}.$$

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* F. Klein, "Ueber die Transformation siebenter Ordnung der elliptischen Funktionen," *Mathematische Annalen*, Vol. XIV, pp. 428–471.

† Valentiner, "De endelige Transformations Gruppers Theori."

‡ A. Wiman, "Ueber eine einfache Gruppe von 360 ebenen Collineationen," *Mathematische Annalen*, Vol. XLVII, pp. 531–556.

If $b=0$, we have the sextic

$$f_6 \equiv az^6+cxy(x^4-y^4)=0 \tag{4_3}$$

invariant under G_{72}, defined by the above transformations and also

$$\begin{pmatrix} x & y & \beta z \\ x & y & z \end{pmatrix}, \qquad \beta^3=1.$$

The form (24_2) has also the transformation

$$\begin{pmatrix} x+\alpha^3 y & \alpha^3(x-\alpha^3 y) & \sqrt{2}\alpha z \\ x & y & z \end{pmatrix}, \qquad \alpha^{24}=1,$$

making a G_{144}. Consequently we have invariant under a G_{144}

$$f_6 \equiv xy(x^4+y^4)+z^6=0. \tag{5_3}$$

Similarly (26_2), if $b=-10$, has also the transformation

$$\begin{pmatrix} x+y+z & x+\alpha y+\alpha^2 z & x+\alpha^2 y+\alpha z \\ x & y & z \end{pmatrix}, \qquad \alpha^3=1,$$

so that

$$f_6 \equiv x^6+y^6+z^6-10(x^3y^3+y^3z^3+z^3x^3)=0 \tag{6_3}$$

is invariant under a G_{216}.

The same substitution may be applied to

$$f_6 \equiv x^6+y^6+z^6+a(x^3y^3+y^3z^3+z^3x^3) - \frac{a+10}{2}[xyz(x^3+y^3+z^3)+3x^2y^2z^2]=0, \qquad (16_2), \tag{7_3}$$

a curve invariant, therefore, under a G_{36}.

The icosahedron group leaves invariant the sextics already found by Klein,*

$$f_6 \equiv (x^5+y^5)z+(a-1)x^3y^3+(2+3a)x^2y^2z^2+(3a-8)xyz^4+az^6=0. \tag{8_3}$$

The G_{60} is defined by

$$\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}, \begin{pmatrix} \alpha x & \alpha^4 y & z \\ x & y & z \end{pmatrix}, \quad \alpha^5=1,$$

and

$$\begin{pmatrix} (\alpha^2+\alpha^3)x+(\alpha+\alpha^4)y+2z & (\alpha+\alpha^4)x+(\alpha^2+\alpha^3)y+2z & x+y+z \\ x & y & z \end{pmatrix}.$$

CHAPTER II.

NON-LINEAR TRANSFORMATIONS.

In any attempt to determine the types of sextics which are invariant under non-linear birational transformations, we need not consider any curve whose genus is higher than 7. For any such sextic remains invariant only under

* Klein, "Ikosaeder," p. 215, Eq. (9), and p. 218, Eq. (14).

linear transformations.* On the other hand, any curve of genus zero may be reduced by birational transformations to a straight line; a curve of genus 1 to a non-singular cubic; a curve of genus 2 to a quartic with one double point; a non-hyperelliptic curve of genus 3 to a non-singular quartic; and a non-hyperelliptic curve of genus 4 to a quintic with two double points. The groups belonging to the canonical form of each of these genera have been determined. In this paper only one particular case of sextics of genus less than 5 will be discussed, namely that of the sextic of genus 3, whose seven double points are distinct and independent. We shall consider first, however, sextics whose genus is 7, 6, or 5.

The only non-linear Cremona transformations under which sextics of genus 7 remain invariant, are quadratic transformations; and of these sextics there are various types depending upon the configuration of the double points. The following cases will be considered in order:

(a) An f_6 with three non-collinear P_2's.
1. All three P_2's distinct.
2. Tacnode (or node cusp) and a P_2 not on the tacnodal tangent.
3. Oscnode (or tacnode cusp).
4. Triple point.

(b) An f_6 with three collinear P_2's.
1. All three distinct.
2. Tacnode and a P_2 on the tacnodal tangent.
3. All three coincident.

(a) 1. If we take double points as the vertices of the triangle of reference, the most general sextic having three distinct non-collinear double points is

$$f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3)+z^2(hx^4+jx^3y+kx^2y^2 \\ +lxy^3+my^4)+z(nx^4y+px^3y^2+qx^2y^3+rxy^4)+sx^4y^2+tx^3y^3+ux^2y^4=0.$$

If f_6 is to be invariant under a quadratic transformation of the first kind,† certain relations must be true among the coefficients of f_6, so that we have:

For $r=2$, $T \equiv \begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(gx^4y+fx^3y^2+ex^2y^3+dxy^4) \\ +cx^4y^2+bx^3y^3+ax^2y^4=0. \quad (1)$$

* C. Küpper, "Ueber das Vorkommen von linearen Schaaren g^2_n auf Kurven n^{ter} Ordnung," *Sitzungsberichte der Böhmischen Gesellschaft* (Prag, 1892), pp. 264–272; V. Snyder, "On Birational Transformations of Curves of High Genus," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908), p. 10.

† C. A. Scott, "Modern Analytical Geometry" (1894), § 233.

If $a=c$, $d=g$, $e=f$, we have for $n=4$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6\equiv z^4(ax^2+bxy+ay^2)+z^3(dx^3+ex^2y+exy^2+dy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(dx^4y+ex^3y^2+ex^2y^3+dxy^4) \\ +ax^4y^2+bx^3y^3+ax^2y^4=0. \end{aligned} \tag{2}$$

If, in (2), $b=d$, $a=h$, $e=j$, we have for $n=12$,

$$T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}, \quad A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}, \quad B\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix},$$

$$\begin{aligned} f_6\equiv z^4(ax^2+bxy+ay^2)+z^3(bx^3+ex^2y+exy^2+by^3) \\ +z^2(ax^4+ex^3y+kx^2y^2+exy^3+ay^4)+z(bx^4y+ex^3y^2+ex^2y^3+bxy^4) \\ +ax^4y^2+bx^3y^3+ax^2y^4=0. \end{aligned} \tag{3}$$

If in (3), $b=e=0$, we have for $n=72$,

$$T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}, \quad A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}, \quad B\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}, \quad C\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}, \quad D\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix},$$

$$f_6\equiv z^4(x^2+y^2)+z^2(x^4+kx^2y^2+y^4)+x^4y^2+x^2y^4=0. \tag{4}$$

(a) 2. The most general equation of a sextic of this type is

$$\begin{aligned} f_6\equiv az^4y^2+z^3y(bx^2+cxy+dy^2)+z^2(ex^4+fx^3y+gx^2y^2+hxy^3+jy^4) \\ +zx(kx^4+lx^3y+mx^2y^2+nxy^3+py^4)+x^2(qx^4+rx^3y+sx^2y^2+txy^3+vy^4)=0, \end{aligned}$$

which has a tacnode at $(0, 0, 1)$ with $y=0$ as tacnodal tangent, and also a double point at $(0, 1, 0)$.

If f_6 is to be invariant under a quadratic transformation, the transformation must be of the second kind* and certain relations must hold between the coefficients of the sextic. When these conditions are satisfied, we have for $r=2$,

$$T\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix},$$

$$\begin{aligned} f_6\equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ +zx(kx^4+lx^3y+mx^2y^2+lxy^3+ky^4) \\ +x^2(qx^4+rx^3y+sx^2y^2+rxy^3+qy^4)=0. \end{aligned} \tag{5}$$

If in (5), $c=f=k=m=r=0$, we have:

For $n=4$, $T\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6\equiv az^4y^2+byz^3(x^2+y^2)+z^2(ex^4+gx^2y^2+ey^4) \\ +lzx^2y(x^2+y^2)+x^2(qx^4+sx^2y^2+qy^4)=0. \end{aligned} \tag{6}$$

(a) 3. The most general equation of a sextic which has an oscnode at $(0, 0, 1)$ is

* C. A. Scott, "Modern Analytical Geometry" (1894), § 234, (2).

$$f_6 \equiv az^2(mx^2-yz)^2+(mx^2-yz)(bxyz^2+cx^3z)$$
$$+dy^3z^3+z^2y^2\phi_2(x,y)+zy\phi_4(x,y)+\phi_6(x,y)=0.$$

The quadratic inversion which will leave this sextic invariant must be of the third kind.* Consequently we have:

For $r=2$, $T\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^2(mx^2-yz)^2+cx^3z(mx^2-yz)+yz(mx^2-yz)\phi_2(x,y)+\phi_6(x,y)=0. \quad (7)$$

If in (7) $c=0$, and $\phi_2(x,y)=\phi_1(x^2,y^2)$, $\phi_6(x,y)=\phi_3(x^2,y^2)$:

For $n=4$, $T\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv az^2(mx^2-yz)^2+yz(mx^2-yz)\phi_1(x^2,y^2)+\phi_3(x^2,y^2)=0. \quad (8)$$

(a) 4. If we take the triple point as one vertex (0, 0, 1) of the triangle of reference, we have, as the most general equation of a sextic with one triple point,

$$f_6 \equiv z^3\phi_3(x,y)+z^2\phi_4(x,y)+z\phi_5(x,y)+\phi_6(x,y)=0.$$

It is readily seen that this sextic is invariant under no non-linear Cremona transformation. The linear ones under which it remains invariant will be seen to belong to classes (a) and (b) as listed in Chapter I.

For $r=2$, $C\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^3(ax^3+bx^2y+bxy^2+ay^3)+z^2(cx^4+dx^3y+ex^2y^2+dxy^3+cy^4)$$
$$+z(fx^5+gx^4y+hx^3y^2+hx^2y^3+gxy^4+fy^5)+jx^6+kx^5y$$
$$+mx^4y^2+nx^3y^3+mx^2y^4+kxy^5+jy^6=0. \quad \text{(Equivalent to } (1_1).)$$

For $r=3$, $C\equiv\begin{pmatrix} \alpha x & y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^3(ax^3+dy^3)+yz^2(fx^3+jy^3)+y^2z(mx^3+qy^3)+rx^6+sx^3y^3+ty^6=0. \quad (2_1).$$

For $r=3$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^3(ax^3+dy^3)+ez^2x^2y^2+z(fx^4y+gxy^4)+hx^6+jx^3y^3+ky^6=0. \quad (15_1).$$

For $n=6$, $C\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv z^3(ax^3+bx^2y+bxy^2+ay^3)+jx^6$$
$$+kx^5y+mx^4y^2+nx^3y^3+mx^2y^4+kxy^5+jy^6=0. \quad (10_2).$$

For $n=6$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6 \equiv cz^3(x^3+y^3)+ez^2x^2y^2+fz(x^4y+xy^4)+hx^6+jx^3y^3+hy^6=0. \quad (11_2).$$

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* C. A. Scott, "Modern Analytical Geometry" (1894), § 234, (3).

For $n=9$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv z^3(cx^3+dy^3)+hx^6+jx^3y^3+ky^6=0. \qquad (66_1).$$

For $n=18$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv bz^3(x^3+y^3)+x^6+gx^3y^3+y^6=0. \qquad (15_2).$$

For $n=18$, $A\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv cz^3(x^3+y^3)+ez^2x^2y^2+x^6+cx^3y^3+y^6=0. \qquad (16_2).$$

For $n=54$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=3$,

$B\equiv\begin{pmatrix} y & z & x \\ x & y & z \end{pmatrix}$, $D\equiv\begin{pmatrix} \alpha x & \alpha^2 y & z \\ x & y & z \end{pmatrix}$, $r=3$,

$$f_6\equiv bz^3(x^3+y^3)+x^6+bx^3y^3+y^6=0. \qquad (26_2).$$

As can be seen immediately, no sextic of genus 7, which possesses three collinear double points, coincident or distinct, remains invariant under non-linear Cremona transformations.

(b) 1. The most general equation of a sextic with three distinct double points (at $(0, 1, 0)$, $(m, 1, 0)$, $(1, 0, 0)$) is

$$f_6\equiv az^6+z^5\phi_1(x, y)+z^4\phi_2(x, y)+z^3\phi_3(x, y)+z^2\phi_4(x, y) \\ +xyz(x-my)f_2(x, y)+x^2y^2(x-my)^2=0.$$

If $\phi_1=\phi_3=f_2=0$, we have for $r=2$, $\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4\phi_2(x, y)+z^2\phi_4(x, y)+x^2y^2(x-my)^2=0.$$

For $n=4$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2 \\ +fxy^3+ey^4)+x^2y^2(x+y)^2. \qquad (1_2).$$

(b) 2. If two of the double points are coincident and the other distinct, we have, as the equation of the sextic,

$$f_6\equiv z^4\phi_2(x, y)+z^3yf_2(x, y)+z^2y\phi_3(x, y)+zy^2f_3(x, y)+y^2\phi_4(x, y)=0.$$

This sextic has a tacnode at $(1, 0, 0)$ and a double point at $(0, 0, 1)$.

If $f_2=f_3=0$, the sextic is seen to belong to type (1_1), which is invariant under $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$.

For $r=4$, $A\equiv\begin{pmatrix} x & y & iz \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4\phi_2(x, y)+y^2\phi_4(x, y)=0. \qquad (3_1).$$

For $r=4$, $A\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+z^2xy(cx^2+ey^2)+gx^4y^2+hx^2y^4+jy^6=0. \qquad (6_1).$$

For $n=4$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+z^2y^2(dx^2+fy^2)+gx^4y^2+hx^2y^4+jy^6=0. \qquad (64_1).$$

For $n=8$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(ax^2+by^2)+gx^4y^2+hx^2y^4+jy^6=0. \qquad (65_1).$$

(b) 3. The most general equation of a sextic which has three coincident collinear double points (at (1, 0, 0)) is

$$f_6\equiv az^6+z^5f_1(x, y)+z^4f_2(x, y)+z^3f_3(x, y)+z^2f_4(x, y)+zy^2\,\phi_3(x, y)+ty^6=0.$$

For $r=2$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, we have

$$\begin{aligned} f_6\equiv az^6+cz^5y+z^4(dx^2+ey^2)+z^3y(gx^2+hy^2)& \\ +z^2(jx^4+lx^2y^2+ny^4)+zy^3(px^2+sy^2)+ty^6=0.& \qquad (1_1). \end{aligned}$$

For $r=3$, $A\equiv\begin{pmatrix} x & \alpha y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+bz^5x+dz^4x^2+z^3(fx^3+hy^3)+z^2x(jx^3+my^3)+pzx^2y^3+ty^6=0. \qquad (2_1).$$

For $r=4$, $A\equiv\begin{pmatrix} ix & y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+cz^5y+ez^4y^2+hz^3y^3+z^2(jx^4+ny^4)+szy^5+ty^6=0. \qquad (3_1).$$

For $r=4$, $A\equiv\begin{pmatrix} ix & -iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4(dx^2+ey^2)+z^2xy(kx^2+my^2)+ty^6=0. \qquad (6_1).$$

For $r=6$, $A\equiv\begin{pmatrix} x & \alpha y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+bz^5x+dz^4x^2+fz^3x^3+jz^2x^4+ty^6=0. \qquad (5_1).$$

For $n=4$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^6+z^4(dx^2+ey^2)+z^2(jx^4+lx^2y^2+ny^4)+ty^6=0. \qquad (64_1).$$

For $n=12$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} \alpha^4x & y & \alpha z \\ x & y & z \end{pmatrix}$, $r=6$,

$$f_6\equiv z^6+dz^4x^2+jz^2x^4+ty^6=0. \qquad (68_1).$$

The only non-linear Cremona transformations under which sextics of genus 6* remain invariant are quadratic transformations, and of these sextics there are various types depending upon the configuration of the double points. The following cases will be considered:

(α) No three collinear.
  1. All distinct.
  2. One tacnode and two other P_2's.
    α) 2 P_2's distinct.
    β) 2 P_2's forming a second tacnode.
  3. Oscnode and one other P_2.
  4. Triple point and one other P_2.
  5. Four coincident.

(β) Three collinear.
  1. All distinct.
  2. Two tacnodes (one on tangent of the other).
  3. Three consecutive.
  4. Four consecutive.

(α) 1. The most general equation of a sextic with four double points (at (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)) which is invariant under a quadratic transformation of the first kind, $\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, is

$$\begin{aligned} f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(gx^4y+fx^3y^2+ex^2y^3+dxy^4) \\ +cx^4y^2+bx^3y^3+ax^2y^4=0, \end{aligned} \tag{1}$$

where $2(a+b+c+d+e+f+g+h+j)+k=0$.

If it is invariant under a harmonic homology also, we have:

For $n=4$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A=\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6 \equiv z^4(ax^2+bxy+ay^2)+z^3(dx^3+ex^2y+exy^2+dy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+zxy(dx^3+ex^2y+exy^2+dy^3) \\ +ax^4y^2+bx^3y^3+ax^2y^4=0, \end{aligned} \tag{2}$$

where $2(2a+b+2d+2e+h+j)+k=0$.

For $n=12$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A=\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} z & y & x \\ x & y & z \end{pmatrix}$,

* Cf. Snyder, "Normal Curves of Genus 6, and Their Groups of Birational Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX (1908).

$$f_6 \equiv z^4(ax^2+dxy+ay^2)+z^3(dx^3+ex^2y+exy^2+dy^3) \\ +z^2(ax^4+ex^3y+kx^2y^2+exy^3+ay^4)+zxy(dx^3+ex^2y+exy^2+dy^3) \\ +ax^4y^2+dx^3y^3+ax^2y^4=0. \qquad (3)$$

where $2(3a+3d+3e)+k=0$.

For $n=120$, $T\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} y & x & z \\ x & y & z \end{pmatrix}$,

$B\equiv\begin{pmatrix} z & y & x \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & (x-y) & (x-z) \\ x & y & z \end{pmatrix}$,

$a=2$, $d=-2$, $e=-1$, $k=6$, and we have

$$f_6\equiv 2z^4(x^2-xy+y^2)-z^3(2x^3+x^2y+xy^2+2y^3) \\ +z^2(2x^4-x^3y+6x^2y^2-xy^3+2y^4)-zxy(2x^3+x^2y+xy^2+2y^3) \\ +2x^2y^2(x^2-xy+y^2)=0. \qquad (4)$$

(α) 2, α). This case appears as a subcase of (a) 2. The sextic in (a) 2 with a tacnode at $(0,0,1)$ with $y=0$ as tangent, and a double point at $(0,1,0)$, invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, was

$$f_6\equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ +zx(kx^4+lx^3y+mx^2y^2+lxy^3+ky^4)+x^2(qx^4+rx^3y+sx^2y^2+rxy^3+qy^4)=0.$$

If f_6 is to have another double point, we may take it at $(1,1,0)$. Then the coefficients of f_6 must satisfy the conditions $2q+2r+s=0$, $2k+2l+m=0$. Imposing these conditions, we have as the equation of the sextic invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, and which has a tacnode and two double points,

$$f_6\equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4) \\ +x(x-y)^2[(kz+qx)(x+y)^2+xy(lz+rx)]=0.$$

For $n=4$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $c=f=k=l=r=0$, and we have

$$f_6\equiv az^4y^2+byz^3(x^2+y^2)+z^2(ex^4+gx^2y^2+ey^4)+qx^2(x-y)^2(x+y)^2=0.$$

For $n=8$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$, $b=0$, and we have

$$f_6\equiv az^4y^2+z^2(ex^4+yx^2y^2+ey^4)+qx^2(x-y)^2(x+y)^2=0.$$

(α) 2, β) The most general equation of a sextic of genus 6 having two tacnodes (neither one of which is on the tacnodal tangent of the other) is

$$f_6\equiv az^4y^2+z^3y\phi_2(x,y)+z^2\phi_4(x,y)+zx^2\phi_3(x,y)+x^4f_2(x,y)=0.$$

This sextic remains invariant under no non-linear Cremona transformations, but the linear ones are:

For $r=2$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+z^3y(bx^2+cxy+dy^2)+z^2(ex^4+fx^3y+gx^2y^2+cxy^3+ay^4) \\ +zx^2(kx^3+lx^2y+fxy^2+by^3)+px^6+kx^5y+ex^4y^2=0. \quad (1)$$

For $n=4$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+z^3y(bx^2+dy^2)+z^2(ex^4+gx^2y^2+ay^4) \\ +zx^2(lx^2y+by^3)+px^6+ex^4y^2=0. \quad (2)$$

For $n=8$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $C\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+z^2(ex^4+gx^2y^2+ay^4)+px^6+ex^4y^2=0. \quad (3)$$

(α) 3. The most general equation of a sextic with oscnode at (0, 0, 1) and another P_2 at (0, 1, 0), invariant under a quadratic transformation $\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$ is,

$$f_6\equiv az^2(mx^2-yz)^2+cx^2z(mx^2-yz)+yz(mx^2-yz)\phi_2(x,y)+x^2\phi_4(x,y)=0. \quad (1)$$

For $n=4$, $T\equiv\begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^2(mx^2-yz)^2+yz(mx^2-yz)(ax^2+by^2)+x^2\phi_2(x^2,y^2)=0. \quad (2)$$

(α) 4. The most general equation of a sextic with a triple point (at (0, 0, 1)) and a P_2 at (0, 1, 0) is

$$f_6\equiv z^3\phi_3(x,y)+z^2\phi_4(x,y)+zxf_4(x,y)+x^2yf_3(x,y)=0,$$

which has a simple point at (1, 0, 0).

If $f_3(x,y)=\phi_3(y,x)$, $f_4(x,y)=\phi_4(y,x)$, we have:

For $r=2$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^3\phi_3(x,y)+z^2\phi_4(x,y)+zx\phi_4(y,x)+x^2y\phi_3(y,x)=0. \quad (1)$$

For $n=6$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & az \\ x & y & z \end{pmatrix}$, $r=3$.

$$f_6\equiv z^3\phi_3(x,y)+x^2y\phi_3(y,x)=0. \quad (2)$$

For $n=8$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & iy & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^3x^3+z^2(ex^4+jy^4)+zx(jx^4+ey^4)+x^2y^4=0. \quad (3)$$

For $n=18$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z\end{pmatrix}$, $A\equiv\begin{pmatrix} \alpha^4 x & \alpha y & z \\ x & y & z\end{pmatrix}$, $r=9$,

$$f_6\equiv z^3(ax^3+by^3)+x^2y(bx^3+ay^3)=0. \tag{4}$$

(α) 5. The only sextic having four coincident P_2's which is invariant under any transformation is

$$f_6\equiv az^3xy^2+z^2y^2(bx^2+cy^2)+zx(dx^4+dx^2y^2+ey^4)+y^2(fx^4+gx^2y^2+hy^4)=0.$$

This is invariant under the single harmonic homology, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$.

(β) 1. If a sextic with three collinear P_2's and one other one, is inverted as to a triangle of the double points, the third double point on one of the sides of the triangle becomes a tacnode; and the latter case, which is as general as the given one, has been discussed in (α) 2, α).

(β) 2. The most general equation of a sextic with two tacnodes, one of which is on the tacnodal tangent of the other, is

$$f_6\equiv az^4y^2+yz^3\phi_2(x, y)+z^2\phi_4(x, y)+y^2z\phi_3(x, y)+y^4f_2(x, y)=0,$$

where the tacnode (1, 0, 0) is on the tacnodal tangent, $y=0$, of the other tacnode (0, 0, 1).

This sextic is invariant only under the linear transformations of the non-cyclic G_4.

For $r=2$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+bz^3xy^2+z^2(cx^4+dx^2y^2+ey^4)+zxy^2(fx^2+gy^2)+(hx^2+jy^2)y^4=0.$$

For $n=4$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+z^2(cx^4+dx^2y^2+ey^4)+y^4(hx^2+jy^2)=0.$$

(β) 3. The most general equation of a sextic with three consecutive collinear double points (at (0, 0, 1)) and one other double point (at (0, 1, 0)), is

$$f_6\equiv z^4y\phi_1(x, y)+z^3y\phi_2(x, y)+z^2y\phi_3(x, y)+zxyf_3(x, y)+x^2\phi_4(x, y)=0.$$

This sextic is invariant under linear transformations of the non-cyclic G_4.

For $r=2$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+bz^3xy^2+z^2y^2(cx^2+dy^2)+zxy^2(fx^2+gy^2)+x^2(hx^4+jx^2y^2+ky^4)=0.$$

For $n=4$, $A\equiv\begin{pmatrix} x & -y & z \\ x & y & z\end{pmatrix}$, $B\equiv\begin{pmatrix} x & y & -z \\ x & y & z\end{pmatrix}$,

$$f_6\equiv az^4y^2+z^2y^2(cx^2+dy^2)+x^2(hx^4+jx^2y^2+ky^4)=0.$$

(β) 4. If all four double points are consecutive, they must lie on a conic, and the sextic is invariant only under a harmonic homology with center on the

tangent and axis through the double point. (Cf. Snyder, "Normal Curves of Genus 6," p. 331.)

The only non-linear Cremona transformations under which sextics of genus 5 remain invariant are quadratic transformations; of these sextics there are various types, depending upon the configuration of the double points. These various types appear as particular cases of sextics of genus either 7 or 6, and they will be discussed in the following order:

(1) An f_6 with five distinct P_2's.
(2) An f_6 with one tacnode and
  a) Three distinct P_2's (general).
  b) One other tacnode with a P_2 at point of intersection of the tacnodal tangents.
  c) Three distinct collinear P_2's, one of which is on the tacnodal tangent.
  d) Three coincident collinear P_2's.
(3) An f_6 with an oscnode and two P_2's.

(1) From (a) 1 we have as the equation of the sextic of genus 7 which is invariant under $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6 \equiv z^4(ax^2+bxy+cy^2)+z^3(dx^3+ex^2y+fxy^2+gy^3) \\ +z^2(hx^4+jx^3y+kx^2y^2+jxy^3+hy^4)+z(gx^4y+fx^3y^2+ex^2y^3+dxy^4) \\ +cx^4y^2+bx^3y^3+ax^2y^4=0. \end{aligned}$$

If this sextic is to have two more double points which are interchanged by C, we need impose merely the conditions that it have, as a double point, $(\alpha, \beta, 1)$, not one of the invariant points; and then our sextic, since it remains invariant under the involution C, must have, as a fifth double point, $\left(\frac{1}{\alpha}, \frac{1}{\beta}, 1\right)$, the image of $(\alpha, \beta, 1)$ under C.

If the sextic (1) of (α) 1 is to have a fifth double point and yet remain invariant under $\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, the fifth double point must be $(1, 1, -1)$. If $(1, 1, -1)$ is to be a double point of (α) 1, (1), we must have $d+e+f+g=0$. Imposing this condition, we have:

For $r=2$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$,

$$\begin{aligned} f_6 \equiv ax^2(y^2-z^2)^2+bxy(z^2-xy)^2+cy^2(z^2-x^2)^2+dz(x^3-y^3)(z^2-xy) \\ +eyz(x^2-y^2)(z^2-x^2)+fyz(yz^2-x^3)(x-y)+hz^2(x^2-y^2)^2 \\ +jz^2xy(x-y)^2=0. \qquad (1) \end{aligned}$$

If $d=e=f=0$, (1) is invariant for $n=4$, $C\equiv\begin{pmatrix} yz & xz & xy \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$.

(2), a) If the sextic of (a) 2 which is invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$ is to have two more double points, which are interchanged by C, we need impose merely the conditions that $f_6=0$ has as one double point $(\alpha, \beta, 1)$ not on the fundamental system of the transformation, and then it will necessarily have as another double point $\left(\alpha, \frac{\alpha^2}{\beta}, 1\right)$, the image of $(\alpha, \beta, 1)$ under C, since f_6 remains invariant under C.

If the sextic of (α) 2, α), which has $(0, 0, 1)$ as a tacnode with $y=0$ as the tangent, and $(0, 1, 0)$ and $(1, 1, 0)$ as two other double points, is to have a fifth double point and yet remain invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, this fifth double point must be on the invariant conic $(x+y)(x-y)=0$, and certain further restrictions must be imposed on the coefficients of $f_6=0$.

(2), b) If, on the sextic of (a) 2 which is invariant under $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, we impose the condition that it shall have a double point at $(1, 0, 0)$, the double point at $(0, 1, 0)$ becomes a tacnode with tangent $z=0$, and we have:

For $r=2$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$,

$$f_6\equiv az^4y^2+yz^3(bx^2+cxy+by^2)+z^2(ex^4+fx^3y+gx^2y^2+fxy^3+ey^4)$$
$$+x^2yz(nx^2+mxy+ny^2)+rx^4y^2=0.$$

This is also a particular case of (α) 2, β). If $a=e=r$, $b=n$, $f=c=m$, our sextic is invariant under $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$ also, therefore under a group of order $n=8$.

If, in addition, $c=0$, $f_6=0$ is invariant under $\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$ also, therefore under a group of order $n=16$.

If, furthermore, $b=0$, we have:

For $n=32$, $C\equiv\begin{pmatrix} xy & x^2 & yz \\ x & y & z \end{pmatrix}$, $A\equiv\begin{pmatrix} x & z & y \\ x & y & z \end{pmatrix}$, $B\equiv\begin{pmatrix} -x & y & z \\ x & y & z \end{pmatrix}$, $D\equiv\begin{pmatrix} x & -y & z \\ x & y & z \end{pmatrix}$,

$$f_6\equiv z^4y^2+z^2(x^4+gx^2y^2+y^4)+x^4y^2=0.$$

(2), c) If we impose on the sextic of (b) 1, which has three collinear double points (at $(0, 1, 0)$, $(m, 1, 0)$, $(1, 0, 0)$), the conditions that it shall have a tacnode at $(0, 0, 1)$ with $x=my$ as the tacnodal tangent, we have as the equation of the sextic (when $m=1$),

$$f_6\equiv(x-y)^2(z^4+x^2y^2)+z(x-y)f_2(x, y)(z^2+xy)+z^2f_4(x, y)=0.$$

It is readily seen that this sextic is invariant under $C \equiv \begin{pmatrix} xz & yz & xy \\ x & y & z \end{pmatrix}$.

If $f_2=0$, our sextic is invariant under a group, whose order is $n=4$, generated by C and $A \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$.

(2), d) If we impose on the sextic of (b) 3, which has three coincident collinear double points at (1, 0, 0), the conditions that it shall have (0, 0, 1) as a tacnode with $y=0$ as the tacnodal tangent, we have:

$$f_6 \equiv z^4y^2 + yz^3\phi_2(x, y) + z^2\phi_4(x, y) + y^3z\phi_2(x, y) + ky^6 = 0,$$

which is invariant under $C \equiv \begin{pmatrix} xz & yz & y^2 \\ x & y & z \end{pmatrix}$. f_6 is seen to be a particular case of the sextic in (β) 2.

If $\phi_2=0$, we have:

For $n=4$, $C \equiv \begin{pmatrix} xz & yz & y^2 \\ x & y & z \end{pmatrix}$, $A \equiv \begin{pmatrix} x & y & -z \\ x & y & z \end{pmatrix}$,

$$f_6 \equiv z^4y^2 + z^2\phi_4(x, y) + ky^6 = 0.$$

(3) If the sextic of (a) 3, which is invariant under $C \equiv \begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$, is to have two more double points, which are interchanged by C, we need impose merely the condition that $f_6=0$ shall have as one double point (α, β, 1) not on the fundamental system of the inversion; it will then necessarily have as another double point the image of (α, β, 1) under C, since f_6 remains invariant under C.

From (α) 3 we find that the sextic with oscnode at (0, 0, 1) with $y=0$ as tangent, and a double point at (0, 1, 0), invariant under $C \equiv \begin{pmatrix} xy & y^2 & mx^2-yz \\ x & y & z \end{pmatrix}$, is

$$f_6 \equiv az^2(mx^2-yz)^2 + cx^3z(mx^2-yz) + yz(mx^2-yz)\phi_2(x, y) + x^2\phi_4(x, y) = 0.$$

If this sextic is to remain invariant under C and yet have an additional double point, this double point must be on the invariant conic $mx^2-2yz=0$, and certain further conditions must be imposed upon the coefficients of $f_6=0$.

We shall now consider, in conclusion, those sextics of genus 3 which have seven distinct double points.* If we let

$$f_6(x) = 0 \qquad 1)$$

be the equation of a sextic with seven distinct double points and

$$\phi_i = 0 \qquad (i=1, 2, 3) \qquad 2)$$

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* See p. 323 above.

the equations of three linearly independent adjoint cubics, then, if we eliminate x between 1) and 2), we shall obtain

$$F_4(y)=0 \qquad 3)$$

as the equation of the non-singular quartic into which our sextic is transformed by means of the adjoint cubics. If in 3) we make the substitution 2), we obtain $X(x)\equiv f_6(x)\cdot M_6(x)=0$, where $M_6=0$ is a sextic which has double points at the same seven points as $f_6=0$. If we consider $f_6=0$ as the locus of the eighth basis point ξ of a pencil of cubics of the net, then $M=0$ is traced by the ninth basis point ξ' of the pencil. Therefore $M=0$ is the image of $f_6=0$ under the Geiser transformation which has the seven double points as fundamental points. $M=0$ may be: 1), distinct from $f=0$; 2), coincident with $f=0$ (not pointwise); or 3), pointwise coincident. Case 1) does not concern us, as we are interested only in invariant sextics. However, we shall need to consider both 2) and 3).

Case 2) If $f_6=0$ is to remain invariant (not pointwise) under the Geiser transformation, then, if we choose any point on $f_6=0$, not on the invariant sextic of the transformation, as an eighth basis point ξ of a pencil of adjoint cubics, the ninth point ξ' must be another point on $f=0$. This pencil of cubics then cuts out a g_2' and the sextic is hyperelliptic and reducible not to a quartic, since the non-singular quartic $(p=3)$ does not possess a g_2', but to a quintic with a triple point. (The $F_4(y)=0$ is a double conic.) To show that a hyperelliptic sextic with seven distinct double points does exist, we shall proceed in the following way:

On the hyperboloid $x_1x_2-x_3x_4=0$, where $\dfrac{x_1}{x_3}=\dfrac{x_4}{x_2}=\lambda$ and $\dfrac{x_1}{x_4}=\dfrac{x_3}{x_2}=\mu$, consider the space curve of order 6 which is defined by the general $F(\lambda^2, \mu^4)=0$. If we consider

$$F\equiv\lambda^2\phi_4(\mu, \nu)+\lambda\nu\psi_4(\mu, \nu)+\nu^2\theta_4(\mu, \nu)=0 \qquad 1)$$

as the equation of the curve—in fact there is a (1, 1) correspondence between the values of (λ, μ, ν) satisfying 1) and the points of the space sextic—the curve is easily seen to be of genus 3, for (1, 0, 0) is a fourfold point and (0, 1, 0) is a double point. Then any projection of it will be of genus 3. If we connect every point of the curve with a fixed point P not on the hyperboloid, we have a projecting cone of order 6. Any plane section of this cone is a sextic of genus 3. Moreover, the seven double points of this sextic must be distinct, for no line from P can cut the hyperboloid (and consequently the curve) in more than two points. But $F(\lambda^2, \mu^4)$ possessed a g_2' which must be retained by projection. Therefore we have obtained a plane sextic of genus 3

with seven distinct double points, which is hyperelliptic. The canonical form for the hyperelliptic curve of genus 3 is the quintic with a triple point which may be obtained by projecting the space curve from a point on it. The groups of transformations belonging to our hyperelliptic sextic are those belonging to this canonical quintic; these have been discussed by Wiman.*

Case 3) If $f=0$ is to remain pointwise invariant under the Geiser transformation, *i. e.*, if it is the invariant sextic of the transformation, then $\xi=\xi'$ and there is a (1, 1) correspondence between the points of this sextic and the non-singular quartic $F(y)=0$ mentioned above. Then the groups of transformations belonging to the non-hyperelliptic $f_6=0$ are the groups which belong to the non-singular quartic $F(y)=0$, into which the $f_6=0$ is transformed. These groups have also been discussed by Wiman in the paper mentioned above. Corresponding to the linear transformations of points of $F(y)=0$ are the transforms of these collineations, by which pencils of adjoint cubics are sent into pencils of adjoint cubics.

Consequently we have seen that the transformations belonging to any sextic of genus 3 are the conjugates of the transformations belonging to the canonical form of the genus, with respect to the birational transformation which reduces the sextic to the canonical form.

* A. Wiman, "Ueber die Hyperelliptischen Curven," *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI.

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*Geometrical and Invariantive Theory of Quartic Curves Modulo 2.**

BY L. E. DICKSON.

§ 1. The bitangents and singular points of a quartic curve Q modulo 2 depend only upon the terms of Q in which at least one exponent is odd. We shall write $Q=E+O$, where

$$E=ax^4+by^4+cz^4+jx^2y^2+kx^2z^2+ly^2z^2,$$
$$O=dx^3y+exy^3+fx^3z+gxz^3+hy^3z+iyz^3+mx^2yz+nxy^2z+pxyz^2.$$

The *derived*† points are the points for which the three first partial derivatives of Q all vanish; they depend upon O, but not upon E.

Again, the bitangents depend upon O alone. For example, $z=0$ is a bitangent if and only if $d\equiv e\equiv 0 \pmod 2$, so that the terms free of z in Q form a perfect square modulo 2. The line $x=ry+sz$ is a bitangent if the quartic in y and z obtained by eliminating x has no terms in y^3z and yz^3. An equivalent definition is that $l=0$ is a bitangent if and only if Q can be expressed in the form $lC+q^2 \pmod 2$.

Under a linear transformation, Q becomes E_1+O_1 modulo 2, in which the terms O_1 involving at least one odd exponent are derived solely from the similar part O of Q.

Thus Q is dominated by its O in several respects, just as is the case with the terms of the second degree in the equation of a conic in non-homogeneous coördinates.

§ 2. THEOREM. *Any two bitangents intersect in a derived point.*

After applying a linear transformation, we may assume that $x=0$ and $y=0$ are bitangents. Then $h=i=0$, $f=g=0$, so that O is quadratic in z. Thus no partial derivative of Q has a term z^3, so that each partial derivative is zero for $x=y=0$. Hence the intersection of the two bitangents is a derived point.

* Presented to the American Mathematical Society, April 2, 1915.

† There is at least one, since Euler's theorem shows that the derivatives are dependent, the sum of their products by x, y, z being identically zero modulo 2.

§ 3. THEOREM. *Unless all of the bitangents concur, every derived point is the intersection of two bitangents.*

By hypothesis there are three bitangents forming a triangle. They may be transformed linearly into $x=0$, $y=0$, $z=0$. Then O becomes

$$O'=mx^2yz+nxy^2z+pxyz^2.$$

For the case $m=n=p=0$, $Q=E$ is a double conic and every point of the plane is a derived point and every line a bitangent.

The derived points of $E+O'$ are those for which

$$ny^2z+pyz^2=0, \quad mx^2z+pxz^2=0, \quad mx^2y+nxy^2=0.$$

The derived points having $z=0$ are evidently the intersections of the bitangent $z=0$ with one of the bitangents $x=0$, $y=0$, $mx+ny+pz=0$ (these being factors of O'). Next, a derived point with $z=1$ is $(s\,r\,1)$, where

$$nr^2+pr=0, \quad ms^2+ps=0.$$

Then $y=rz$ and $x=sz$ are bitangents meeting at $(s\,r\,1)$.

In view of the last two theorems, the derived points coincide completely with the intersections of the bitangents, provided the last are not all concurrent. But if there are two or more bitangents to a real quartic and all concur at P, there is a derived point besides P. For (§ 4) O can then be transformed into O_4 or O_6. The bitangents to $E+O_4$ are the infinitude of lines through $P=(001)$, while any point with $x=0$ is a derived point. The bitangents to O_6 are $x=0$ and $y=0$, while P and (010) are the derived points. Finally, a real quartic with a single bitangent can be transformed into $E+O_{11}$ (§ 4), whose single derived point is P and single bitangent is $y=0$.

Non-equivalent Systems of Real Quartics, §§ 4–8.

§ 4. Henceforth we consider only quartics Q with integral coefficients and subject them to linear transformations with integral coefficients only. The 2^6 quartics $E+O$ in which the six coefficients of E take independently the values 0 and 1, while O is fixed, shall be said to form a *system*.

We shall prove that every such system is equivalent to one and but one of the fourteen systems $E+O_i$, in which O_0 is identically zero, while $O_1, \ldots,$ O_{13} are given in the table together with all of their bitangents (where x denotes $x=0$), in which

$$\rho^2+\rho+1=0, \quad A^3+A^2+1=0, \quad b^2+\rho^2b+1=0.$$

O_1	$x^2yz+xy^2z+xyz^2$	all 7 real lines
O_2	xy^2z+xyz^2	$x,\ y,\ z,\ y=z$
O_3	x^2yz	$x,\ y,\ z=\alpha y$ (α any)
O_4	x^3y	$x,\ y=\alpha x$ (α any)
O_5	x^3y+O_1	$x,\ y,\ y=x,\ z=\rho x,\ z=\rho x+y$
O_6	x^3y+xy^2z	$x,\ y$
O_7	$x^3y+x^2yz+xyz^2$	$x,\ y,\ z=\rho x$
O_8	$x^3y+yz^3+xyz^2$	$y,\ z=Ax$
O_9	xy^2z+O_8	$y,\ z=Ax,\ z=Ax+A^2y$
O_{10}	$x^3y+y^3z+yz^3+xy^2z$	$y,\ z=x+\rho y,\ z=\rho x+by$
O_{11}	x^3y+y^3z	y
O_{12}	$x^3y+xz^3+y^3z+yz^3$	$x=(s^3+1)y+sz,\ s^7+s+1=0$
O_{13}	$x^3y+xz^3+y^3z+xy^2z$	$x=\sigma^3y+\sigma z,\ \sigma^7+\sigma^3+1=0$

§ 5. First, let the quartic have three real bitangents forming a triangle. Applying a real transformation, we obtain $O=O'$ of § 3. We obtain at once O_0, O_1, O_2 or O_3.

Second, let there be three intersecting real bitangents $x=0$, $y=0$, $x=y$, but no real ones forming a triangle. Then

$$O=dx^3y+exy^3+O',\quad m=n.$$

If $m+p=1$, $z=dx+ey$ is a bitangent, contrary to hypothesis. Hence $m=p$. Since $O\neq O'$, we may interchange x and y, if necessary, and set $d=1$. Replacing x by $x+ey$, we have $d=1$, $e=0$, and thus obtain O_4 or O_5.

Third, let there be exactly two real bitangents $x=0$, $y=0$. Then

$$O=dx^3y+exy^3+O',\quad m\neq n.$$

Replace z by $z+\alpha x+\beta y$, where α and β are integers. We get

$$(d+m\alpha+p\alpha)x^3y+(e+n\beta+p\beta)xy^3+O'+E.$$

Since $m+p$ and $n+p$ are not both zero, we may take $e=0$ in O. Then $d=1$, since $z=0$ is not a bitangent. If $p=n$, then $m+p+1=0$ and $x=z$ is a bitangent. Hence $p=n+1=m$. According as $m=0$ or 1, we have O_6 or O_7.

§ 6. Fourth, let $y=0$ be the only real bitangent. Then $f=g=0$ in O. Since no one of the six remaining real lines is a bitangent, the two numbers of each pair

$$d,\ e;\quad h,\ i;\quad m+n+h,\ i;\quad d+i+m+p,\ h+e;$$
$$d,\ e+n+p;\quad d+i+m+p,\ e+h+i+n+p$$

are not both zero. Suppose that $d=0$, so that $e=1$, $n=p$, by the first and fifth pairs. If $i=0$, then $h=1$, $m=n$, by the second and third pairs, contrary to the

fourth. Thus $i=1$ and by interchanging x and z we obtain an O with $d=1$, $f=g=0$. Hence we may set $d=1$. Replacing x by $x+ey$, we have $e=0$. Then replacing x by $x+mz$, we have also $m=0$.

Thus $d=1$, $e=m=0$, while the two numbers of each pair

$$h,\, i; \quad n+h,\, i; \quad 1+i+p,\, h; \quad 1+i+p,\, h+i+n+p$$

are not both zero. For $h=0$, we have $i=p=1$; according as $n=0$ or 1, we get O_8 or O_9. For $h=n=1$, we have $i=1$; according as $p=0$ or 1, we get O_{10} or $O_{10}+xyz^2$, the latter being derived, apart from terms E, from O_9 by replacing z by $z+y$. For $h=1$, $n=0$, we have $i=p$ and hence O_{11} or $O''=O_{11}+yz^3+xyz^2$. The only bitangents to O'' are $y=0$, $z=Ax+(A+1)y$, which are derived from those to O_8 by replacing x by $x+y$ and z by $z+y$; the same replacement carries O_8 to $O''+E$.

§ 7. Finally, consider quartics without real bitangents. We may take $d=1$, make $f=0$ by replacing y by $y+fz$, and make $e=m=0$ as in § 6. Hence

$$O=x^3y+Gxz^3+Hy^3z+Iyz^3+Nxy^2z+Pxyz^2.$$

Then $G=1$, since $y=0$ is not a bitangent. The only* bitangents are $x=ry+sz$, where

$$r=s^3+Ps+I, \quad r^2s+Nr+H=0.$$

These have the solutions $r,\, s=0,\, 0;\ 1,\, 0;\ 0,\, 1;\ 1,\, 1$ if

$$I=H=0; \quad I=1,\, H=N; \quad I=P+1,\, H=0; \quad I=P,\, H=N+1,$$

respectively. No one of the latter pairs of equations is to hold, since there is to be no real bitangent. Hence $H=1$ and either $N=1$, $I=0$ or $N=0$, $I=P+1$. If $N=P=0$, then $I=1$ and $O=O_{12}$. If $N=1$, $I=P=0$, $O=O_{13}$. If $N=I=0$, $P=1$, we apply (xzy) and get O_{13}. If $N=P=1$, $I=0$, we apply (xyz) and then replace x by $x+z$, obtaining $O_{12}+z^4$.

§ 8. The only case in which two of our O_j have the same number of real bitangents and the same number of imaginary bitangents involving the roots of irreducible congruences modulo 2 of the same degree is that of O_{12} and O_{13}. Hence it remains only to prove that O_{12} is not equivalent to any $E+O_{13}$ under a real linear transformation.

Let B_s and β_σ denote the bitangents to O_{12} and O_{13} given in the table of § 4. Since the congruence for s is irreducible, the square of any root is a root. Now B_s becomes B_{s^2} under the transformation

$$T: \qquad x=Y+Z, \quad y=Z, \quad z=X,$$

which therefore permutes the seven bitangents B_{s^i} $(i=1, 2, 4, 8, 16, 32, 64)$

* Dickson, *Trans. Amer. Math. Soc.*, April, 1915.

cyclically; it replaces O_{12} by $O_{12}+y^2z^2+z^4$. Hence, if O_{12} and O_{13} are equivalent, there is a real linear transformation which replaces any chosen bitangent to O_{12} by any chosen one to O_{13}. We take $\sigma=s^4+s^5$, which is a root* of $\sigma^7+\sigma^3+1\equiv 0 \pmod 2$. The transformation (c_{ij}) replaces B_s by $lX+mY+nZ=0$, where

$$l=c_{11}+(s^3+1)c_{21}+sc_{31}, \quad n=c_{13}+(s^3+1)c_{23}+sc_{33}.$$

If the resulting line is β_σ, we have $n=l\sigma$. Now

$$l\sigma=l(s^4+s^5)=c_{31}s^6+(c_{31}+c_{21}+c_{11})s^5+(c_{21}+c_{11})s^4+c_{21}s^2+c_{21}.$$

If this equals the above cubic function n of s we have, in view of the linear independence of $s^6, \ldots, s, 1$ modulo 2, $c_{31}=c_{21}=c_{11}=0$, whereas the elements of the first column of (c_{ij}) are not all zero.

We have now completed the proof that *every system is equivalent to one and but one of fourteen systems* $E+O_i$ $(i=0, 1, \ldots, 13)$.

Invariants Distinguishing between the Systems.

§ 9. We readily construct a rational integral function c_j of the coefficients of Q which is congruent to zero modulo 2 if and only if the real line $l_j=0$ is a bitangent to Q. For example, if $l_j=x$, we may set $c_j=hi+h+i$, whose vanishing modulo 2 implies $h=i=0$, and hence that $x=0$ is a bitangent.

Let $s_1, \ldots, s_7$ denote the elementary symmetric functions of $c_1, \ldots, c_7$. Since any real linear transformation permutes the lines $l_j=0$ and hence the c's, each s_i is an invariant of Q. Now s_i is unity for a quartic with exactly $7-i$ real bitangents, since exactly i of the c's are unity for such a quartic, while the remaining c's are zero. Moreover, s_i is zero for a quartic with more than $7-i$ real bitangents. Hence the values of $s_1, \ldots, s_7$ determine the number of real bitangents.

We can construct in a similar way invariants which determine the number of bitangents whose coefficients are in the Galois field of the 2^n polynomials in a root of a congruence of degree n irreducible modulo 2. We have only to employ in place of the above seven real lines $l_i=0$ the lines with coefficients in the present field.

The values of all these invariants with $n\leq 4$ distinguish between two systems of quartics not having the same number of bitangents with coefficients in each of the fields of order 2^n $(n=1, 2, 3, 4)$, and hence differentiate all the systems $E+O_j$ $(j=0, \ldots, 11)$ from each other and from the two with $j=12, 13$. A new invariant is needed to differentiate the last two systems from each other.

* Found by use of the table by Bussey, *Bull. Amer. Math. Soc.*, Vol. XII (1905–6), p. 33. The headings i-λ are misprints for $i\lambda$.

We proceed to construct an invariant J whose value is 1 for any quartic equivalent to an $E+O_{12}$ and 0 for the remaining quartics, so that J is a characteristic invariant for the system $E+O_{12}$. By applying the transformations specified at the beginning of § 7 to a quartic with $d=1$, we obtain $E+O$, where O is of the form in § 7, with coefficients

$$G=g+f(e+p+n), \qquad H=h+e(m+n+f),$$
$$I=i+fh+Ge+pm+efm, \quad N=n+ef, \quad P=p+m+ef.$$

By §§ 7, 8, $E+O$ is equivalent to $E+O_{12}$ if and only if $G=H=1, N=P=I+1$, and hence if and only if $GH(N+P+1)(N+I)=1$. The case $d=0, e=1$ is reduced to the preceding by interchanging x and y. Finally, if $d=e=0$, $z=0$ is a real bitangent, whence $J=0$. Thus

$$J=dGH\{(n+p+m+1)(i+fh+e+pm+efm)+NP\}$$
$$+(d+1)ef(i+ph+mh)\{(n+p+m+1)(g+fh+pn)+m(n+p)\}.$$

In view of its construction, J is an invariant of the general quartic modulo 2. As a check this was verified for the substitution (xy).

Real Linear Automorphs of Each System.

§10. To test the equivalence of two quartics of the same system $E+O_j$, we require the real linear transformations which leave the system unaltered, *i. e.*, its automorphs.

For $j=0$ or 1, every real linear transformation is an automorph.

For $j=2$ or 3, the real bitangents are $x=0$ and the three lines through (100). Since that point and the line $x=0$ are invariant, the transformation is a binary one on y, z. Every such real binary transformation is an automorph.

An automorph of $E+O_4$ must leave fixed the intersection (001) of all the bitangents and must permute the derived points $(0kl)$, k, l arbitrary. The resulting real transformations

$$x=X, \quad y=rX+Y, \quad z=dX+eY+Z \tag{1}$$

are evidently automorphs. The same result holds for $O_5=O_4+O_1$, since $E+O_1$ is invariant under all real linear transformations.

For $j=6$ or 7, the only real bitangents are $x=0$, $y=0$; the only real derived points are (001) and (010). Hence each of the latter must be fixed and the transformation is the identity or $z=X+Z$.

For $j=8$ or 9, the only real bitangent is $y=0$; the only real point on an imaginary bitangent (or the only real derived point) is $P=(010)$. Hence an

automorph is binary on x, z. It must permute the three bitangents $z=Ax$ through P. We get

$$t: \quad x=X+Z, \quad y=Y, \quad z=X, \tag{2}$$

and t^2, $t^3=I$. Conversely, these are automorphs.

An automorph of $E+O_{10}$ leaves fixed the only real bitangent $y=0$ and the only real point (101) on an imaginary bitangent. Of the resulting transformations, the only automorphs are

$$x=rX+sY+(r+1)Z, \quad y=Y, \quad z=(r+1)X+(r+s+1)Y+rZ.$$

These are the four powers of

$$x=Z, \quad y=Y, \quad z=X+Y. \tag{3}$$

An automorph of $E+O_{11}$ leaves fixed the only bitangent $y=0$ and the only derived point (001). The actual automorphs are

$$x=X+sY, \quad y=Y, \quad z=sX+mY+Z. \tag{4}$$

Among the automorphs of $E=O_{12}$ are the seven powers of T in § 8, which permute the seven bitangents cyclically. There are no other automorphs, since the identity transformation alone leaves B_s unaltered. In fact, one condition is $n=ls$, in the notations of § 8, whence c_{21}, c_{23}, c_{31}, c_{13} are zero. Thus the transformation is $x=X+cY$, $y=Y$, $z=kY+Z$. It leaves B_s fixed if and only if $c=sk$, whence $c=k=0$.

The only real linear transformation replacing the bitangent β_σ to O_{13} by β_{σ^2} is seen similarly to be

$$x=Y, \quad y=Y+Z, \quad z=X, \tag{5}$$

which replaces O_{13} by $O_{13}+Y^4$. Hence the only automorphs of the system $E+O_{13}$ are the seven powers of (5).

§ 11. The number N of systems equivalent to a given one $E+O_i$ is found by dividing the order 168 of the group of all real linear transformations modulo 2 by the number of automorphs of the system. Using the results in § 10, we get $N=1$ for O_0 or O_1; $N=28$ for O_2, O_3; $N=21$ for O_4, O_5; $N=84$ for O_6, O_7; $N=56$ for O_8, O_9; $N=42$ for O_{10}, O_{11}; $N=24$ for O_{12}, O_{13}. The sum of these N's, each in duplicate, is $512=2^9$. This count of the total number of systems is in accord with the fact that O has 9 coefficients each with 2 values. We thus have a complete check upon §§ 4–10.

Invariantive Classification of the Quartics in a System.

§ 12. Let a real linear automorph T of the system $E+O_j$ replace $E+O_j$ by $E'+O_j$. A polynomial $I(a, \ldots, l)$ in the arbitrary integral coefficients of E shall be called an invariant of the system $E+O_j$ if, when $a', \ldots, l'$ are

replaced by their expressions in $a, \ldots, l$, $I(a', \ldots, l')$ becomes congruent modulo 2 to $I(a, \ldots, l)$ for all integral values of $a, \ldots, l$ and for every linear automorph T of the system.

We shall obtain a fundamental set of invariants of each system and apply them to effect a complete classification of the quartics in each system. These invariants of the systems, together with the invariants characterizing the systems (§ 9), enable us to find (§28) a fundamental set of invariants of Q.

For the system having O identically zero modulo 2, E is identically congruent to the square of a quadratic form q. The invariants (and covariants) of q modulo 2 have been determined.* If q has no real linear factor, it is equivalent to x^2+yz or y^2+yz+z^2.

§ 13. For the system $E+O_1$ every real linear transformation is an automorph. All such transformations are generated by

$$(xy): \quad (ab)(kl), \qquad (xz): \quad (ac)(jl),$$
$$x'=x+z: \quad c'=c+a+k, \quad l'=l+j+1. \tag{α}$$

Evident invariants are therefore jkl and $P=jklabc$.

First, let both of these invariants be unity. We get E_1+O_1,

$$E_1=x^4+y^4+z^4+x^2y^2+x^2z^2+y^2z^2.$$

It follows that E_1+O_1 is invariant under all real linear transformations. Thus it has no real linear factor (true also of E_2+O_1).

Second, let $jkl=1$, $P=0$. After an interchange of two variables, we have $a=0$, $j=k=1$. In view of α and

$$x'=x+y: \quad b'=b+a+j, \quad l'=l+k+1, \tag{β}$$

we may set also $b=c=0$; we get E_2+O_1,

$$E_2=x^2y^2+x^2z^2+y^2z^2.$$

Third, let $jkl=0$. We may set $j=0$. By use of α and

$$y'=y+z: \quad c'=c+b+l, \quad k'=k+j+1, \tag{γ}$$

we may set also $l=k=0$. If a, b, c are all zero, E is identically zero. In the contrary case we may set $a=1$ and make $c=0$ by use of the product $\alpha\beta$. We get E_i+O_1, where

$$E_3\equiv0, \quad E_4=x^4, \quad E_5=x^4+y^4.$$

No two of these quartics E_i+O_1 are equivalent, since the first, O_1, is a product of four real linear factors, the second has the single real linear factor x, while the third has none (or since they contain 6, 4, 2 real points respectively).

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* Dickson, "Madison Colloquium," 1914, p. 76 (references, p. 70).

We readily construct invariants of the system $E+O_1$ which differentiate the last three types. We need the coefficients of the equivalent form having $j=k=l=0$.

If $j=0$, we apply $\alpha^l\gamma^k$ and get

$$ax^4+by^4+[c+(a+k)l+bk]z^4+O_1.$$

If $j=1$, $k=0$, we apply $\beta^l(yz)\gamma$ and get

$$ax^4+cy^4+[b+c+l(a+1)]z^4+O_1.$$

If $j=k=1$, $l=0$, we apply $(xz)\alpha\gamma$ and get

$$cx^4+by^4+(a+b+c+1)z^4+O_1.$$

A function having the value unity when the quartic is equivalent to one with $E_3\equiv 0$ and the value zero in the contrary case, is seen by an inspection of the preceding results to be

$$(1+j)\{(a+1)(b+1)(c+1+kl)\}+j(k+1)(a+1)(c+1)(b+1+l)$$
$$+jk(l+1)(c+1)(b+1)a.$$

It may be written in the symmetrical form

$$B=(1+jkl)(a+1)(b+1)(c+1)+(j+1)kl(a+1)(b+1)$$
$$+(k+1)jl(a+1)(c+1)+(l+1)jk(b+1)(c+1).$$

To find an invariant C of the system having the value 1 or 0 according as $E+O_1$ is or is not equivalent to x^4+O_1, we note that it follows from the third case above that $C=a+b+c$ when $j=k=l=0$. Hence,

$$C=(j+1)[a+b+c+(a+k)l+bk]+j(k+1)[a+b+l(a+1)]$$
$$+jk(l+1)(a+1)$$
$$=a(l+1)+b(k+1)+c(j+1)+jkl+jk+jl+kl.$$

Although the invariance of B and C follows from their construction and the non-equivalence of the E_i+O_1 $(i=3, 4, 5)$, this was verified by use of (xy), (xz), α.

§ 14. Each system $E+O_i$ $(i=2, 3)$ is invariant only under the group G of the real binary linear transformations on y, z. We have

$$E+O_2\equiv ax^4+x^2\lambda^2+x\pi+q^2 \pmod 2,$$
$$\lambda=jy+kz,\quad \pi=yz(y+z),\quad q=by^2+lyz+cz^2.$$

Evidently π is invariant under G. Hence the invariants of $E+O_2$ under G are the polynomials in a and the invariants of λ and q under G. Among the latter are evidently

$$P=(j+1)(k+1),\quad l,\quad bcl,\quad I=(l+1)(b+1)(c+1),\quad R=ljk+bk+cj,$$

R being the resultant of λ and q.

First, let $P=0$. We may set $j=1, k=0$, since

$$(yz): \qquad (bc)(jk), \tag{6}$$

$$y=Y+Z: \qquad c'=c+b+l, \quad k'=k+j. \tag{7}$$

A linear combination of our last four invariants gives $b(l+c+1)$. Hence b is fixed if $l=c$, a relation between our invariants l and $R=c$. But if $l=c+1$, $z=Y+Z$ adds unity to b without otherwise changing $E+O_2$, so that we may set $b=0$. The respective types are

$$ax^4+by^4+cz^4+x^2y^2+cy^2z^2+O_2, \quad ax^4+cz^4+x^2y^2+(c+1)y^2z^2+O_2.$$

Second, let $P=1$, whence $j=k=0$. If $l=I=0$, b and c are not both zero, and we may take $b=1$, $c=0$, in view of (6), (7). If $l=0$, $I=1$, then $b=c=0$. If $l=1$, either $bcl=1$ or we may set $b=1$, $c=0$. The respective types are

$$ax^4+y^4+O_2, \quad ax^4+O_2, \quad ax^4+y^4+cz^4+y^2z^2+O_2.$$

Hence our invariants form a fundamental set. Aside from a possible factor x or y (which may be determined by inspection), there is no real linear factor except* z and $y+z$ of ax^4+O_2 for $a=0$ and $y+z$ of the final type for $a=c=0$. Hence there are 11 types without real linear factors.

§ 15. The automorphs of the system $E+O_3$ form the binary group G on y, z. Using q of § 14, we have

$$E+O_3 \equiv ax^4+x^2\beta+q^2 \pmod 2, \quad \beta=jy^2+yz+kz^2.$$

Hence the invariants of $E+O_3$ are the polynomials in a and the invariants of β and q under G, the latter† being polynomials in

$$l, \quad jk, \quad bcl, \quad I, \quad \sigma=l(j+k)+bk+cj+b+c,$$

I as in § 14. Since we desire the types, it is about as simple to proceed independently as to make use of the result cited that $l, \ldots, \sigma$ form a fundamental set of invariants of β and q.

If $l=0$, $jk=1$, either $I=1$, whence $b=c=0$ and $E+O_3$ has the factor x^2, or $I=0$, when we may set $b=1$, $c=0$ by use of (6) and

$$y=Y+Z: \qquad c'=c+b+l, \quad k'=k+j+1. \tag{8}$$

If $l=jk=1$, either $b=c=1$ or $bcl=0$; in the latter case, we may set $b=0$, $c=0$ by use of (6), (8).

If $jk=0$, we may set $j=0$, $k=0$ by (6), (8). Then $\sigma=b+c$. First, let $bcl=0$, $l=1$. If $\sigma=0$, then $b=c=0$. If $\sigma=1$, we may set $b=1$, $c=0$ by (6). Second, let $l=0$. Then I and σ give bc and $b+c$, so that b and c can be at most interchanged and this is done by (6). We may drop the case $b=0$, $c=1$.

* The four possibilities are the four real bitangents.

† Dickson, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI, pp. 126, 127.

Hence our invariants characterize the resulting types and thus form a fundamental set. The thirteen types without real linear factors are

$$\begin{array}{lll} ax^4+y^4+x^2y^2+x^2z^2+x^2yz, & \phi=ax^4+x^2y^2+x^2z^2+y^2z^2+x^2yz, & \phi+y^4+z^4, \\ ax^4+y^4+z^4+y^2z^2+x^2yz, & \psi=x^4+y^2z^2+x^2yz, & \psi+y^4, \\ x^4+y^4+x^2yz, & ax^4+y^4+z^4+x^2yz. & \end{array}$$

§ 16. The further systems are treated by new methods, illustrated in this section and § 18. For the system $E+O_{13}$ the seven automorphs are (5) and its powers. Under (5), O_{13} receives the increment Y^4 and $E+O_{13}$ becomes $E'+O_{13}$, in which

$$a'=c, \quad b'=b+a+j+1, \quad c'=b, \quad j'=k+l, \quad k'=l, \quad l'=j. \tag{5'}$$

Evident invariants of the system are

$$I=(j+1)(k+1)(l+1), \quad P=abcI.$$

If $I=1$, whence $j=k=l=0$, either $P=1$, whence $a=b=c=1$, or $P=0$, giving seven sets a, b, c, not all unity. These sets are permuted cyclically by (5′), so that we may take $a=b=c=0$.

If $I=0$, the seven sets j, k, l, not all zero, are permuted cyclically by (5′), so that we may take $j=1$, $k=l=0$. The identity is the only automorph preserving these values. Hence we need invariants to fix a, b, c. The sum of aj and its six conjugates under the powers of (5′) is the invariant

$$a(j+k+l)+b(j+l)+c(k+j)+(k+1)(j+l).$$

Similarly from ak and al we obtain two invariants which, when combined linearly with the preceding one, give

$$\alpha=aj+bl+c(k+l), \quad \beta=ak+b(j+k)+cl+jl+j+l,$$
$$\gamma=al+bk+cj+jk+lk+jl.$$

For our form, $j=1$, $k=l=0$, whence $\alpha=a$, $\beta=b+1$, $\gamma=c$ fix a, b, c. Thus *$I, P, \alpha, \beta, \gamma$ form a fundamental set of invariants of the system $E+O_{13}$, and the ten types of quartics are*

$$x^4+y^4+z^4+O_{13}, \quad O_{13}, \quad ax^4+by^4+cz^4+x^2y^2+O_{13}.$$

No one of these and no quartic $E+O_{12}$ has a real linear factor, having no real bitangent.

§ 17. The treatment of $E+O_{12}$ is similar to the last. Employing the automorph T of § 8, we have

$$T': \quad a'=c, \quad b'=a, \quad c'=a+b+j+1, \quad j'=k, \quad k'=k+l, \quad l'=j+1.$$

Evident invariants are

$$M=jk(l+1), \quad \pi=(a+1)(b+1)(c+1)M.$$

If $\pi=1$, then $j=k=1$, $l=a=b=c=0$. If $\pi=0$, $M=1$, then $j=k=1$, $l=0$, and the seven sets a, b, c, not all zero, are permuted cyclically by T', so that we may set $a=1$, $b=c=0$. If $M=0$, the seven sets j, k, l are permuted by T' and we may set $j=k=l=0$. Then invariants fixing a, b, c are

$$al+b(k+1)+c(j+1)+l(j+1)(k+1),\quad a(j+k)+b(j+1)+cl+k+l,$$
$$a(j+l+1)+bl+c(j+k)+kl(j+1).$$

Hence the last three and M, π give a fundamental set of invariants of the system. The 10 types are

$$x^2y^2+x^2z^2+y^2z^2+O_{12},\quad x^4+x^2y^2+x^2z^2+O_{12},\quad ax^4+by^4+cz^4+O_{12}.$$

§ 18. For $E+O_{10}$ the automorphs are the powers of (3). Under it,

$$T:\qquad a'=c,\quad b'=b+c+l,\quad c'=a,\quad j'=l+1,\quad k'=k,\quad l'=k+j.$$

We readily obtain the invariants

$$\begin{array}{llll} k, & k(l+j), & kl(j+1), & \sigma=a(j+l+1)+c(k+j+l),\\ ac, & a+c, & & \tau=(k+1)\{aj+b+cl+l(j+1)\}. \end{array}$$

If $k=0$, T permutes $(j, l)=(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ cyclically. Hence we may set $j=l=0$. Then a, b, c are fixed by $\sigma=a$, $\tau=b$ and $a+c$.

Let $k=1$, $k(l+j)=0$. By use of T, we may set $l=j=0$. Now T^2 changes only b, adding $a+c+1$ to it; hence if $a+c=0$, we may set $b=0$; the invariant ac then reduces to a. For $a+c=1$, invariants reducing to a and b when $k=1$, $l=j=0$, are

$$\lambda=k(j+1)(l+1)a+kljc,$$
$$\mu=k(j+1)(l+1)(a+c)b+klj(a+c)(b+c+1),$$

which were formed by starting with the first term (the desired one) and adding the term derived from it by T. Since T carries the second term to the first, we have an invariant.

Finally, let $k=1$, $k(l+j)=1$. The invariant $kl(j+1)$ then becomes l. Thus l and j are invariant. If $a+c=1$, we apply T and set $a=1$, $c=0$; then make $b=0$ by T^2. Next, let $a+c=0$. If $a=c=l+1$, we make $b=0$ by T. If $a=c=l$, b is fixed by the new invariant

$$\pi=k(j+l)(a+c+1)(a+l+1)b.$$

The one real bitangent $y=0$ is a factor only when $k=a=c=0$.

Our ten invariants form a fundamental set. The nineteen types of quartics without real linear factors are

$$\begin{array}{ll} ax^4+by^4+cz^4+O_{10}\ (a,\ c \text{ not both } 0), & ax^4+az^4+x^2z^2+O_{10},\\ ax^4+by^4+(a+1)z^4+x^2z^2+O_{10}, & x^4+\phi,\\ (l+1)(x^4+z^4)+\phi, & l(x^4+z^4)+by^4+\phi, \end{array}$$

where

$$\phi=(l+1)x^2y^2+x^2z^2+ly^2z^2+O_{10}.$$

§ 19. The only automorphs of the system $E+O_6$ are $z=X+Z$ and its square. It replaces $E+O_6$ by $E'+O_6$, in which the altered coefficients are

$$a'=a+c+k, \quad j'=j+l+1.$$

Evident invariants are b, c, k, l, lj and

$$\alpha=(l+1)(a+cj+jk), \quad \beta=la(c+k+1).$$

If $l=0$, we may make $j=0$, whence $a=\alpha$. If $l=c+k=1$, we may make $a=0$, whence $j=lj$. If $l=1$, $c=k$, we have $j=lj$, $a=\beta$. Hence the invariants form a fundamental set. The twenty-one types without real linear factors are

$$ax^4+by^4+z^4+kx^2y^2+O_6, \quad ax^4+y^4+kx^2z^2+O_6\ (a=1 \text{ or } k=1),$$
$$by^4+cz^4+jx^2y^2+(c+1)x^2z^2+y^2z^2+O_6,$$
$$ax^4+by^4+cz^4+jx^2y^2+cx^2z^2+y^2z^2+O_6\ (a=1 \text{ or } c=1).$$

§ 20. For $E+O_7$, $a'=a+c+k$, $j'=j+l$. The invariants are

$$b, \quad c, \quad k, \quad l, \quad j(l+1), \quad al+clj+jk, \quad a(l+1)(c+k+1).$$

If $l=1$, we make $j=0$. If $l=0$, $c+k=1$, we make $a=0$. The thirty types without real linear factors are

$$ax^4+by^4+cz^4+kx^2z^2+y^2z^2+O_7\ (\text{except } a=c=k=0),$$
$$by^4+cz^4+jx^2y^2+(c+1)x^2z^2+O_7\ (\text{except } b=c=0),$$
$$ax^4+by^4+z^4+jx^2y^2+x^2z^2+O_7, \quad x^4+y^4+jx^2y^2+O_7.$$

§ 21. For $E+O_8$ the automorphs are the three powers of t given by (2). Now t replaces $E+O_8$ by $E'+O_8$, in which

$$t: \qquad a'=a+c+k, \quad c'=a, \quad j'=j+l, \quad l'=j.$$

A fundamental set of invariants is given by

$$b, \quad k, \quad kac, \quad P=(k+1)(a+1)(c+1), \quad \pi=(l+1)(j+1),$$
$$\rho=al+cj+jkl, \quad \sigma=aj+c(j+l)+k(l+1).$$

If $\pi=1$, then $l=j=0$. If also $P=1$, then $k=a=c=0$ and y is a factor. But if $P=0$, $k=0$, then a, c are not both zero and by use of t and its powers we may set $a=c=1$. If $k=1$, either $ac=1$ or we may make $a=c=0$ by use of t and its powers.

If $\pi=0$, we may set $l=1$, $j=0$ by use of t and its powers. Then $a=\rho$, $c=\sigma$, b and k are fixed by invariants.

The twenty types without a real linear factor are

$$ax^4+by^4+az^4+kx^2z^2+O_8, \quad ax^4+by^4+cz^4+kx^2z^2+y^2z^2+O_8,$$

with a, c, k not all zero.

§ 22. The automorphs of $E+O_9$ are the powers of t given by (2), while

$$t: \qquad a'=a+c+k, \quad c'=a, \quad j'=j+l+1, \quad l'=j.$$

A fundamental set of invariants is given by

$$b,\quad k,\quad lj,\quad \lambda=kac,\quad \mu=(k+1)(a+1)(c+1),$$
$$\sigma=a(j+1)+c(j+l)+kl(j+1),\quad \tau=a(j+l)+c(l+1)+jk(l+1).$$

Let $lj=1$, whence $l=j=1$. According as $k=1$ or $k=0$ the powers of t permute the three sets a, c not all 1, or not all 0, respectively, so that we may set $a=c$. Then λ, μ give the value of a.

If $lj=0$ we may take $l=j=0$ by use of powers of t. Then $\sigma=a$, $\tau=c$.

The twenty types without a real linear factor are

$$ax^4+by^4+az^4+x^2y^2+kx^2z^2+y^2z^2+O_9 \text{ (except } a=k=0),$$
$$ax^4+by^4+cz^4+kx^2z^2+O_9 \text{ (except } a=c=k=0).$$

§ 23. The four automorphs of $E+O_{11}$ are given by (4). We get

$$a'=a+s(c+k),\quad b'=b+sa+mc+sj+msk+ml+m+s,$$
$$l'=l+sk,\quad j'=j+(m+s)k+sl+s.$$

A fundamental set of invariants is given by

$$c,\quad k,\quad l(k+1),\quad lj(k+1),\quad A=a(c+k+1),\quad B=(k+1)(l+1)(a+cj),$$
$$C=c(k+1)(l+1)(b+ja),\quad D=l(k+1)(c+1)(a+j)b,$$
$$E=k(c+1)(l+a),\quad F=kla+kb+k(l+j)c+kj(l+1).$$

First, let $k=0$, $l(k+1)=0$, whence $l=0$. We make $j=0$ by choice of s. Then $a\doteq B$, $cb=C$. If $c=1$, b is fixed. If $c=0$, we make $b=0$ by taking $s=0$, $m=b$.

Second, let $k=0$, $l(k+1)=1$, whence $l=1$. Then j is fixed by $lj(k+1)$. If $c=1$, we make $a=0$, $b=0$. If $c=0$, $a=A$, $(a+j)b=D$, so that b is fixed if $a+j=1$. But if $a=j$, we can make $b=0$.

Third, let $k=1$. We can make $l=j=0$. Then $A=ac$, $E=a(c+1)$ fix a, while $F=b$.

The eighteen types with no real linear factor are

$$ax^4+by^4+z^4+O_{11},\quad x^4+O_{11},\quad z^4+jx^2y^2+y^2z^2+O_{11},\quad x^4+by^4+y^2z^2+O_{11},$$
$$x^4+x^2y^2+y^2z^2+O_{11},\quad ax^4+by^4+cz^4+x^2z^2+O_{11}.$$

§ 24. The automorphs of $E+O_4$ are given by (1). We have

$$a'=a+br+cd+jr+kd+lrd+r,\quad b'=b+(c+l)e,$$
$$j'=j+ke+lre+ld,\quad k'=k+lr.$$

A fundamental set of invariants is given by

$$c,\quad l,\quad P=(k+1)(l+1),\quad \beta=(b+1)(c+l+1),\ \pi=P(j+1),$$
$$\lambda=P(c+1)(b+j)a,\quad \mu=l(c+1)[a+k(b+j+1)],$$
$$\tau=ck(l+1)(b+j),\quad \tau a,\quad \sigma=lc(a+bk+jk+j+k).$$

First, let $l=1$, $c=0$. We may make $b=k=j=0$. Then $\mu=a$.

Second, let $l=c=1$. We make $k=j=0$. Then $\beta=b+1$, $\sigma=a$.

Third, let $l=0$, $c=1$. We make $b=0$ and take $e=0$ henceforth. Then k is fixed by P and j is fixed by π if $k=0$, while $j=\tau$ if $k=1$. If $j=k=1, \tau a=a$. But if j, k are not both 1, we may set $a=0$, since

$$a'=a+d(1+k)+r(1+j).$$

Fourth, let $l=c=0$. Then k and b are fixed by P and β. If $P=1$, $k=0$ and $j=\pi+1$. Then $a=\lambda$ if $b+j=1$; while if $b+j=0$, we make $a'=a+r=0$. Finally, let $P=0$, whence $k=1$. We make $j'=j+e=0$. Then, for $e=r=0$, $a'=a+d$ can be made zero.

The twelve types with no real linear factor are

$$x^4+y^2z^2+x^3y, \quad ax^4+by^4+z^4+y^2z^2+x^3y,$$
$$ax^4+z^4+x^2y^2+x^2z^2+x^3y, \quad z^4+jx^2y^2+kx^2z^2+x^3y \ (jk=0),$$
$$x^4+y^4+x^3y,^* \quad y^4+x^2z^2+x^3y.$$

§ 25. For $E+O_5$, we must add to the expressions for a', j', k' in § 24 rd, $d+e+re$, r, respectively. A fundamental set of invariants is

$$c, \quad l, \quad lk, \quad lckj, \quad \xi=b(c+l+1), \quad \eta=lck(b+j)a,$$
$$\zeta=c(l+1)[a+b(k+1)+jk+j+k], \quad \lambda=(l+1)(c+1)[a+k(b+j+1)],$$
$$\mu=l(c+1)[j+b(k+1)], \quad \nu=l(c+1)(k+1)(b+j)a.$$

If $l=0$, $c=1$, we make $b=k=j=0$. Then $\zeta=a$.

If $l=c=0$, we make $k=j=0$. Then $\xi=b$, $\lambda=a$.

If $l=1$, $c=0$, we make $b=0$ and take $e=0$ henceforth. Then $lk=k$, $\mu=j$. Since $a'=a+r(j+1)+kd$, we may make $a'=0$ unless $j=1$, $k=0$. In the latter case, $\nu=a$.

If $l=c=1$, $\xi=b$, $lk=k$, while

$$j'=j+(k+1)e, \quad a'=a+r(b+j+1)+(k+1)d.$$

For $k=0$, we can make $j=a=0$. Finally, let $k=1$. Then $lckj=j$. If $b=j$, we can make $a'=0$. But if $b+j=1$, $\eta=a$.

The fourteen types with no real linear factor are

$$ax^4+z^4+O_5, \quad x^4+y^4+O_5, \quad x^4+x^2y^2+y^2z^2+O_5,$$
$$jx^2y^2+x^2z^2+y^2z^2+O_5, \quad by^4+z^4+y^2z^2+O_5,$$
$$by^4+z^4+bx^2y^2+x^2z^2+y^2z^2+O_5, \quad ax^4+by^4+z^4+(b+1)x^2y^2+x^2z^2+y^2z^2+O_5.$$

Rejection of Reducible Types.

§ 26. By §§ 12–25, there are 203 non-equivalent types of quartics with no real linear factor. Of these, eight have no real point, one has seven real points, and six have six real points.†

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* To this binary form can be reduced any quartic equivalent to a binary one and having no real linear factor and not a perfect square.

† *Trans. Amer. Math. Soc.*, April, 1915.

In order to obtain a list of types of irreducible quartics, it remains to delete those types which are products BC of quadratic forms without real linear factors. As noted at the end of § 12, such a quadratic form is equivalent to x^2+yz or $(y+\rho z)(y+\rho^2 z)$, where $\rho^2+\rho+1=0$.

Consider quartics Q with an imaginary linear factor f whose coefficients involve the single imaginary ρ. Since f is a bitangent to Q, the table in § 4 shows that, in $Q=E+O_i$, i is 3, 4, 5, 7, or 10, and gives the possible forms of f. For example, if $i=3$, then $f=z+\rho y$, which is a factor of $E+O_3$ if and only if $a=0$, $b=l=c$, $j=k=1$. If $b=0$, the factor x occurs. If $b=1$, we have

$$y^4+z^4+x^2y^2+x^2z^2+y^2z^2+x^2yz;$$

viz., $\phi+y^4+z^4(a=0)$, at the end of § 15. An $E+O_4$ with the factor $y+\rho x$ has the factor x or y. An $E+O_5$ with either of the factors $z+\rho x$, $z+\rho x+y$ has both (if without the factor y) and is

$$x^4+z^4+x^2y^2+x^2z^2+y^2z^2+O_5=(x^2+xz+z^2)(x^2+y^2+z^2+xy+xz);$$

it is the final quartic in § 25 for $a=1$, $b=0$.

An $E+O_7$ with the factor $z+\rho x$, but not x, is

$$x^4+z^4+x^2z^2+O_7;$$

it is the first of the two at the end of § 20 with $a=1$, $b=j=0$.

An $E+O_{10}$ with the factor $z+x+\rho y$ is

$$Q_a=ax^4+(a+1)(y^4+z^4)+ax^2y^2+x^2z^2+ay^2z^2+O_{10}.$$

Now Q_1 becomes Q_0 under the transformation T of § 18; while Q_0 is the third one of the list at the end of § 18 for $a=0$, $b=1$.

§ 27. Finally, let Q be the product BC of two quadratic forms without real or imaginary linear factors. As noted, we may take

$$B=x^2+yz,\quad C=a_1yz+a_2xz+a_3xy+b_1x^2+b_2y^2+b_3z^2,\quad \Delta=a_1a_2a_3+\Sigma a_ib_i=1.$$

First, let a_2 and a_3 be not both zero. Interchanging y and z, if necessary, we may set $a_3=1$. The automorph

$$x=X+Z,\quad y=Y+Z,\quad z=Z \tag{9}$$

of B replaces C by C', in which $a_3'=1$, $a_2'=a_2+1$. Hence we may set $a_3=1$, $a_2=0$. Using $\Delta=1$, we get

$$C=ayz+xy+bx^2+b_2y^2+(ab+1)z^2.$$

Let T be an automorph of B which replaces C by a form C' differing from C only by terms in x^2, y^2, z^2. The single derived point (apex) of B is $P=(100)$;

the apex of C is $A=(a01)$. The three real points not on B and distinct from P lie on the covariant line $x+y+z=0$. The only real transformation, leaving fixed this line as well as P and A, is seen at once to be the identity or

$$x=X+Y,\ y=Y,\ z=Y+Z.$$

It replaces C by a form in which only b_2 is changed, the increment to b_2 being $a+b+ab$. The latter can be made unity unless $a=b=0$. Hence in C we may set $b_2=0$ unless $b_2=1$, $a=b=0$.

Second, let $a_2=a_3=0$. By $\Delta=1$, $a_1=b_1=1$. Hence $C=B+l^2$, where $l=b_2y+b_3z$. Unless $l\equiv0$, we may set $b_2=1$ and make $b_3=0$ by use of (9).

Hence C can be transformed by an automorph of B into one and but one of the seven forms

$$B,\quad B+y^2,\quad e=xy+y^2+z^2,\quad f=ayz+xy+bx^2+(ab+1)z^2.$$

The desired quartics are therefore equivalent to products of the latter by B. The product B^2 occurs at the end of § 12. To $B(B+y^2)$ we apply (xzy) and get the third type (for $a=0$) at the end of § 24. Next, $Be=x^2y^2+x^2z^2+O_{10}$; it is the last type in § 18 for $l=b=0$. For $a=b=1$, $Bf=x^4+y^2z^2+O_6$; it is the last type in § 19 for $a=1$, $b=c=j=0$. For $a=b=0$, we replace x by $x+z$ in Bf and get $z^4+x^2z^2+y^2z^2+O_5$; it is next to the last type at the end of § 25 for $b=0$. In Bf with $a=b+1$ we replace x by $x+z$ and get

$$bx^4+(b+1)z^4+by^2z^2+x^2z^2+O_9.$$

If $b=1$, we apply transformation (2) and get $z^4+x^2z^2+O_9$, which is the preceding for $b=0$. This type is a case of the last one in § 22.

Our lists in §§ 12–25 therefore contain $5+6$ quartics (including the two in § 12) which are reducible. *There remain* 192 *non-equivalent types* of irreducible quartics modulo* 2.

Fundamental Set of Invariants of the Quartic Q.

§ 28. Evident linear combinations of the invariants of Q constructed in § 9 give characteristic invariants of the various systems; C_j is characteristic for the system $E+O_j$ provided $C_j=1$ if and only if Q is equivalent to a quartic of this system. For example, invariant J of § 9 is characteristic for the system $E+O_{12}$.

For each j we constructed in §§ 12–25 a fundamental set S_j of invariants I_j of $E+O_j$, in which the coefficients of E are arbitrary integers, and determined type forms E_1+O_j, E_2+O_j, ... to which all such forms are equivalent.

* The only one equivalent to a binary form is that in the last line of § 24.

Let the coefficients of $Q=E+O$ be arbitrary integers subject only to the condition $C_j=1$. Compute by §§ 4–7 the coefficients $a', \ldots, l'$ of E' in the form $E'+O_j$ into which Q can be transformed. Let K_j be the function of the coefficients of Q which is obtained from $I_j(a', \ldots, l')$ by giving to $a', \ldots, l'$ their values in terms of the coefficients of Q. Then if I_j ranges over the set S_j and j ranges over the values $0, 1, \ldots, 13$, the C_j and the products C_jK_j are invariants of Q forming a fundamental set.

To prove this, let $I_{t,j}$ be the linear combination of the I_j which gives that invariant of $E+O_j$ which is characteristic of the class of quartics $E+O_j$ which are equivalent to E_t+O_j; thus $I_{t,j}$ has the value 1 if and only if $E+O_j$ is equivalent to E_t+O_j. Let $K_{t,j}$ be the corresponding K_j. Then $C_jK_{t,j}$ has the value 1 if and only if Q is equivalent to E_t+O_j (the value 0 if not equivalent to it); hence it is an invariant of Q and is characteristic of the class of quartics equivalent to E_t+O_j. Since the various CK completely characterize the classes of quartics, they form a fundamental set of invariants of Q.

*Invariantive Classification of Pairs of Conics Modulo 2.**

By L. E. Dickson.

§ 1. *Relation to a Generalization of Recent Problems on Point Sets.*

The projective classification of sets of points in a modular space has been utilized in several recent papers by the writer to effect a complete projective classification of modular curves or surfaces of given orders for low moduli. Finite sets of points in an ordinary space have been investigated quite recently by Coble.† For either modular or ordinary space, we therefore have information also concerning sets of elements (lines or planes) dual to points. But a new situation arises if we consider systems composed of points and lines (or planes). To such a problem we can at once reduce the problem of the classification of one or more conics modulo 2. In fact with such a conic

$$F=a_1yz+a_2xz+a_3xy+b_1x^2+b_2y^2+b_3z^2$$

is associated covariantly‡ an apex $A=(a_1, a_2, a_3)$ and a line

$$L=(\beta_1+1)x+(\beta_2+1)y+(\beta_3+1)z,$$

where

$$\beta_1=b_1+\alpha_2\alpha_3, \quad \beta_2=b_2+\alpha_1\alpha_3, \quad \beta_3=b_3+\alpha_1\alpha_2 \qquad (\alpha_i=a_i+1).$$

Conversely, A and L evidently uniquely determine a conic F.

When F is a double line, the a's are congruent to zero and there is no apex. When F is a pair of imaginary lines, the coefficients of L are all congruent to zero ("Colloq.," p. 76, table). The converse of each statement is true.

Hence *the projective classification of pairs of conics F, F' modulo 2 is equivalent to the projective classification of the systems A, L, A', L' of two points and two lines modulo 2, and the degenerate systems with fewer points or fewer lines.*

§ 2. *Geometrical Classification of Pairs of Conics.*

For the present, let the conics have apices A and A' which are distinct and lines L and L' which are distinct.

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* Presented at the California meeting of the American Mathematical Society, August 3, 1915.

† *Trans. Amer. Math. Soc.*, Vol. XVI (1915).

‡ Dickson, "Madison Colloquium Lectures," 1914, p. 69, cited as "Colloq."

First, let neither A nor A' be on L or L'. Since there are only three real points on $l=AA'$, the intersection P of L and L' is on l. After a real transformation, we have $A=(001)$, $P=(011)$, $L=y+z$. Then A' is the third real point (010) on $l(x=0)$, and L' is the third real line $x+y+z=0$ through P. Thus $F=xy+x^2+z^2$, $F'=xz+y^2$.

Second,* let neither A nor A' be on L', while A', but not A, is on L. After a transformation, we have $l=x$, $L=x+z$, $L'=x+y+z$, whence $A'=(010)$. Since the intersection of l and L' is (011), A is the remaining real point (001) on l. Thus $F=xy+y^2+z^2$, $F'=xz+y^2$.

Third, let A', but not A, be on L; A, but not A', on L'. We may take $L=x+z$, $L'=x+y$, $l=x$. Then $F=xy+y^2+z^2$, $F'=xz+y^2+z^2$.

Fourth, let A be on L, A' on L'. If A is on L', we may take $L=x+y$, $L'=l=x$, $A'=(010)$. Then $F=xy$, $F'=xz+z^2$. But, fifth, if A is not on L' and A' is not on L, we may take $l=x$, $L=x+y$, $L'=x+z$, whence $F=xy$, $F'=xz$.

Sixth, let A be on L, A' not on L'. If A is not on L' and A' not on L, we may take $l=x$, $L=x+y$, $L'=x+y+z$. Then $A'=(010)$, $F=xy$, $F'=xz+y^2$. But, seventh, if A is on L', and A' not on L, we may take $L=x+y$, $A=(001)$, $A'=(010)$, whence $L'=y$, $F=xy$, $F'=xz+x^2+y^2+z^2$. Again, eighth, if A is not on L', but A' on L, we may take $A=(001)$, $L=x$, $L'=x+y+z$, whence $A'=(010)$, $F=xy+y^2$, $F'=xz+y^2$. Ninth, if A is on L', and A' on L, we may set $L=x$, $L'=y$, $A'=(010)$, whence $F=xy+y^2$, $F'=xz+x^2+y^2+z^2$.

Next, if there are not two distinct apices (including the cases of an absence of one or both apices) and not two distinct lines L and L', two configurations of the same type are evidently projectively equivalent. The types of conics will not be listed. The same result was seen to hold in the above nine cases.

Two pairs of conics modulo 2 *are projectively equivalent if and only if they have the same properties as regards existence of apices and covariant lines, distinctness of apices and lines, and incidence of apices and lines.*

§ 3. *Modular Invariants of a Pair of Conics.*

Since $P=\alpha_1\alpha_2\alpha_3$ is 0 or 1 according as F has an apex or not, it is an invariant of F modulo 2. Similarly, we have the following invariants of F and F': $P'=\alpha'_1\alpha'_2\alpha'_3$; $J=\beta_1\beta_2\beta_3$, with the value 0 or 1 according as L is not or is iden-

* The companion case given by interchanging accented and unaccented letters will not be treated separately. Similarly throughout.

tically zero; $J'=\beta_1'\beta_2'\beta_3'$; $\Delta_1=\Sigma(\beta_i+1)a_i$, which is 0 or 1 according as* A is on L or not on L; Δ_1' for F'; $\delta_1=\Sigma(\beta_i+1)a_i'$, which is 0 or 1 according as A' is on or not on L; δ_1' for A on L';

$$v=\prod_{i=1}^{3}(1+a_i+a_i'),\quad \beta=\prod_{i=1}^{3}(1+\beta_i+\beta_i'),$$

where $v=0$ or 1 according as A and A' are distinct or coincident, while $\beta=0$ or 1 according as L and L' are distinct or identical.

In view of § 2 and a general theorem ("Colloq.," p. 14), *these invariants form a fundamental system of invariants of F and F' modulo* 2.

§ 4. *Certain Formal Invariants of F and F' Modulo* 2.

When the coefficients of F and F' are independent variables, instead of integers taken modulo 2 as hitherto, the invariants under all linear transformations with integral coefficients taken modulo 2 are called formal invariants.

Since a_1, a_2, a_3 are cogredient with x, y, z,

$$\Delta=F(a_1,\,a_2,\,a_3)=a_1a_2a_3+a_1^2b_1+a_2^2b_2+a_3^2b_3$$

is a formal invariant of F. If we set

$$B_1=b_1^2+a_2a_3+a_2^2+a_3^2,\,\ldots,\,B_3=b_3^2+a_1a_2+a_1^2+a_2^2,$$

we find that ("Colloq.," p. 70)

$$L=B_1x+B_2y+B_3z$$

is a formal covariant of F, which reduces to L of § 1 for integral coefficients taken modulo 2. Hence

$$\Delta_1=\Sigma a_iB_i,\quad \Delta_1'=\Sigma a_i'B_i',\quad \delta_1=\Sigma a_i'B_i,\quad \delta_1'=\Sigma a_iB_i'$$

are formal invariants of F and F'. From the universal covariant

$$Q_1=\Sigma x^4+\Sigma x^2y^2+xyz\Sigma x,$$

we obtain the formal invariants of F

$$C=\Sigma a_1^4+\Sigma a_1^2a_2^2+a_1a_2a_3\Sigma a_1,\quad E=\Sigma B_1^4+\Sigma B_1^2B_2^2+B_1B_2B_3\Sigma B_1,$$

which reduce, for integral values of a_i, b_i, to $1+P$ and $1+J$ of § 2. The remaining two universal covariants, which form with Q_1 a fundamental system ("Colloq.," p. 76), vanish for integral values. Every formal invariant of F of degree <4 is a linear function of Δ, Δ_1, and

$$\alpha=\Sigma a_ib_i+\Sigma a_i^2+a_1a_2+a_1a_3+a_2a_3.$$

To form intermediate formal invariants, we construct the same function of $F+kF'$ that a given invariant is of F. Thus Δ_1 and Δ become

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* In this sentence, A denotes the set a_1, a_2, a_3, whether or not all the a's are zero.

$$\Delta_1+k\delta_1+k^2\delta_1'+k^3\Delta_1', \quad \Delta+k\delta+k^2\delta'+k^3\Delta',$$

where

$$\delta=a_1a_2a_3'+a_1a_3a_2'+a_2a_3a_1'+a_1^2b_1'+a_2^2b_2'+a_3^2b_3'$$

vanishes if and only if vertex A of F is on F', while δ' vanishes if and only if A' is on F. From C we get

$$C+k\Sigma a_1^2(a_2a_3'+a_2'a_3)+k^2C_2+k^3\Sigma a_1'^2(a_2a_3'+a_2'a_3)+k^4C',$$
$$C_2=\sum_3(a_1^2a_2'^2+a_1'^2a_2^2+a_1^2a_2'a_3'+a_1'^2a_2a_3).$$

For integral a_i, b_i, the coefficients of k and k^3 vanish, while

$$C_2\equiv v+P+P'+1 \pmod 2,$$

so that, if the conics have apices, the apices coincide if and only if $C_2\equiv 0$.

The formal invariant intermediate between α and α' is

$$\rho=\Sigma(a_ib_i'+a_i'b_i+a_i'a_j).$$

For integers, $\rho\equiv\delta+\delta_1\equiv\delta'+\delta_1' \pmod 2$. If in C_2 we replace the a's by B's, we get the formal invariant

$$\sigma=\Sigma B_1^2B_2'^2+\Sigma(B_1^2B_2'B_3'+B_1'^2B_2B_3).$$

For integers, $\sigma\equiv\beta+J+J'+1 \pmod 2$. From E we get

$$E+kE_1+k^2E_2+k^3E_3+k^4E_4+k^5E_3'+k^6E_2'+k^7E_1'+k^8E',$$

where, if $c_{ij}=a_ia_j'+a_i'a_j$,

$$\begin{aligned}
E_1&=\Sigma c_{12}B_1^2B_2,\\
E_2&=\Sigma(c_{12}^2B_1^2+c_{12}c_{13}B_1^2+c_{23}^2B_2B_3+B_1^2B_2B_3'),\\
E_3&=\Sigma(c_{12}B_1^2B_2'+c_{12}c_{23}^2B_2),\\
E_4&=\sigma+\Sigma c_{23}^2(c_{23}^2+c_{13}^2+c_{12}c_{13}+B_2B_3'),
\end{aligned}$$

where the sums are in the sense of symmetric functions. Evidently E_i' is derived from E_i by interchanging accented and unaccented letters.

On the Solutions of Linear Non-homogeneous Partial Differential Equations.

By L. L. Steimley.

The integrals of a partial differential equation of the first order were first classified by Lagrange, who separated them into three groups; namely, the general, the complete, and the singular integrals. For a long time this classification was thought to be complete. In fact, Forsyth, in his "Differential Equations," published first in 1885, gives a supposed proof of a theorem stating that every solution of such a differential equation is included in one or other of the three classes named. This error is also carried through the second and third English editions and the two German editions, the last one being published in 1912.

In 1891 Goursat pointed out, in his "Équations aux dérivées partielles du premier ordre," that solutions exist which do not belong to any of these three classes, and showed, indeed, that the existing theory is not complete even for the simplest forms.

In November, 1906, Forsyth, in his presidential address to the London Mathematical Society, emphasized the fact that the theory is incomplete, and in his closing remark said: "It appears to me that there is a very definite need for a re-examination and a revision of the accepted classification of integrals of equations even of the first order: in the usual establishment of the familiar results, too much attention is paid to unspecified form, and too little attention is paid to organic character, alike of the equations and of the integrals. Also, it appears to me possible that, at least for some classes of equations, these special integrals may emerge as degenerate forms of some semi-general kinds of integrals; but it is even more important that methods should be devised for the discovery of these elusive special integrals." Forsyth also, in an address delivered at the Fourth International Congress of Mathematicians, takes advantage of the opportunity offered, to again emphasize the incompleteness of the existing theory of partial differential equations of the first order.

Some indication of the nature of the incompleteness can be seen from the equation

$$xp+yq=z.$$

A general solution by Lagrange's method is afforded by

$$f(\phi_1, \phi_2)=0,$$

where f is an arbitrary function and where $\phi_1=\dfrac{z}{x}$ and $\phi_2=\dfrac{z}{y}$. We can easily verify that

$$\psi=z-\frac{x^2}{y}=0$$

affords a solution of the original equation; but this solution cannot be put in the form of the Lagrange general integral, since the Jacobian

$$\frac{\partial(\phi_1, \phi_2, \psi)}{\partial(z, x, y)}$$

is not identically zero. Consequently the so-called general solution is not a general solution in the true sense of the term, since not every solution of the differential equation can be put in that form.

In the present paper we shall deal with the linear non-homogeneous equation

$$\sum_{i=1}^{n} X_i p_i = Z, \qquad p_i=\frac{\partial z}{\partial x_i}, \tag{1}$$

where X_i and Z are functions of the variables $z, x_1, x_2, \ldots, x_n$. We shall assume, as we may without loss of generality, that all common factors of Z, $X_1, X_2, \ldots, X_n$ have been removed; and consequently we need not take into account the values of z which satisfy simultaneously

$$Z=0, \quad X_i=0, \qquad i=1, 2, \ldots, n.$$

We also assume that there is a set of values of the variables $z, x_1, x_2, \ldots, x_n$, in the vicinity of which the functions X_i and Z are single-valued and analytic. Throughout the paper we shall confine our attention to such a domain.

In this paper a new and complete classification is given of all the integrals of equation (1). In the new classification, the so-called special integrals (except for trivial ones) are contained in the general solution. A means is developed by which all these elusive special integrals can be readily determined as soon as the Lagrange general integral is known.

In order to obtain the solution of (1) we take the set of equations

$$\frac{dx_i}{X_i}=\frac{dz}{Z}, \qquad i=1, 2, \ldots, n. \tag{2}$$

It is well known that (2) possesses n functionally independent integrals, one or more of which contain z explicitly. Let these integrals be

$$\phi_i(z, x_1, x_2, \ldots, x_n)=c_i, \qquad i=1, 2, \ldots, n, \tag{3}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. Not all the above n equations are free of z.

Any $\phi_i(z, x_1, x_2, \ldots, x_n)=c_i$ affords a solution of the original equation (1) if it involves z explicitly. Since it is an integral of the equations (2), the relations

$$\sum_{r=1}^{n} \frac{\partial \phi_i}{\partial x_r} dx_r + \frac{\partial \phi_i}{\partial z} dz=0, \qquad i=1, 2, \ldots, n, \tag{4}$$

are consistent with (2). From (2) and (4) follows immediately

$$\sum_{r=1}^{n} \frac{\partial \phi_i}{\partial x_r} X_r + \frac{\partial \phi_i}{\partial z} Z=0, \qquad i=1, 2, \ldots, n. \tag{5}$$

If the equation

$$f(\phi_1, \phi_2, \ldots, \phi_n)=0,$$

in which f is an arbitrary function of the quantities $\phi_1, \phi_2, \ldots, \phi_n$, determines one or more values of z, then it affords solutions of (1), and is what may be called the Lagrange general integral.

We have already seen that not all the solutions of every equation (1) can be put in this form, and since they cannot, we wish to see just how they differ from it. We also wish to get a solution which will be general in the true sense of the term.

Let

$$\psi(z, x_1, x_2, \ldots, x_n)=0 \tag{6}$$

be any equation whatever with the restrictions that it determines a z which satisfies equation (1) and that there is a set of values of the variables $z, x_1, x_2, \ldots, x_n$, in the domain which we are considering, such that in their vicinity ψ and z are single-valued and analytic functions.

Since $\phi_1, \phi_2, \ldots, \phi_n$ are functionally independent, not all the n-th order determinants of the matrix

$$\left\| \begin{array}{ccccc} \frac{\partial \phi_1}{\partial z} & \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial z} & \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi_n}{\partial z} & \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{array} \right\|$$

vanish identically.

Consequently one of the Jacobians

$$J=\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

or

$$J_s=\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)}, \qquad s=1, 2, \ldots, n,$$

does not vanish identically.

If $J \not\equiv 0$, then we can solve the system $\phi_i = \phi_i(z, x_1, x_2, \ldots, x_n)$, $i=1, 2, \ldots, n$, for $x_1, x_2, \ldots, x_n$, in terms of $z, \phi_1, \phi_2, \ldots, \phi_n$. Substituting these values in equation (6), we get

$$\psi(z, x_1, x_2, \ldots, x_n) = g(z, \phi_1, \phi_2, \ldots, \phi_n) = 0. \tag{6'}$$

Two cases now arise. Either z is expressible in terms of $\phi_1, \phi_2, \ldots, \phi_n$, or equation (6′) may be written in the form

$$\theta(z) g_2(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

where $\theta(z)$ is independent of $\phi_1, \phi_2, \ldots, \phi_n$. $\theta(z)$ may be a constant.

If z is expressible in terms of $\phi_1, \phi_2, \ldots, \phi_n$, let us write

$$z = g_1(\phi_1, \phi_2, \ldots, \phi_n). \tag{7}$$

Differentiating z with respect to x_i $(i=1, 2, \ldots, n)$ we get

$$\frac{\partial z}{\partial x_i} = p_i = \sum_{r=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_i} + \frac{\partial \phi_r}{\partial z} p_i\right), \qquad i=1, 2, \ldots, n. \tag{8}$$

Multiplying the i-th equation in (8) by X_i and adding the equations thus obtained, we get

$$\sum_{i=1}^{n} X_i p_i = \sum_{i=1}^{n} X_i \sum_{r=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_i} + \frac{\partial \phi_r}{\partial z} p_i\right) = \sum_{r=1}^{n} \sum_{i=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(\frac{\partial \phi_r}{\partial x_i} X_i + \frac{\partial \phi_r}{\partial z} X_i p_i\right).$$

Hence, employing (5) and (1), we have

$$\sum_{i=1}^{n} X_i p_i = \sum_{r=1}^{n} \frac{\partial g_1}{\partial \phi_r}\left(-\frac{\partial \phi_r}{\partial z} Z + \frac{\partial \phi_r}{\partial z} Z\right),$$

or

$$Z = 0. \tag{9}$$

That is, if

$$z = g_1(\phi_1, \phi_2, \ldots, \phi_n)$$

is a solution of equation (1), then this value of z satisfies $Z=0$.

Consequently, if we wish to get solutions of form (7), we need to examine only those values of z which are determined by equation (9) and see whether they satisfy equation (1). Since no knowledge of differential equations or integration is involved in getting these solutions, it follows that they are trivial.

If z is not a function of $\phi_1, \phi_2, \ldots, \phi_n$, then equation (6′) can be written in the form

$$\theta(z) g_2(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

where $\theta(z)$ is a function of z alone and may be a constant.

Since the values of z determined by $\theta(z)=0$ afford solutions of equation (1), we can easily determine the character of θ. The values of z determined by $\theta(z)=0$ are of the form $z=$ a constant. (By exception this constant may

be infinity.) If $z=\alpha$, a finite constant, is a solution of equation (1), a direct substitution in that equation shows that for $z=\alpha$ we have

$$Z\equiv 0$$

for all values of $x_1, x_2, x_3, \ldots, x_n$.

Therefore $\theta(z)=0$ determines only those (non-infinite) values of z which satisfy $Z=0$.

If

$$J_s=\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)}\not\equiv 0,$$

where s is one of the integers $1, 2, \ldots, n$, then the system $\phi_i=\phi_i(z, x_1, x_2, \ldots, x_n)$, $i=1, 2, \ldots, n$, can be solved for $z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n$ in terms of $x_s, \phi_1, \phi_2, \ldots, \phi_n$.

On substituting these values of z and x_i (x_s excluded) in (6), we get

$$\psi(z, x_1, x_2, \ldots, x_n)=h(x_s, \phi_1, \phi_2, \ldots, \phi_n)=0. \tag{6''}$$

Again two cases arise. Either x_s is expressible in terms of $\phi_1, \phi_2, \ldots, \phi_n$, or

$$h(x_s, \phi_1, \phi_2, \ldots, \phi_n)\equiv\theta_s(x_s)h_2(\phi_1, \phi_2, \ldots, \phi_n)=0,$$

where $\theta_s(x_s)$ is a function of x_s alone and may be a constant.

If x_s is expressible as a function of $\phi_1, \phi_2, \ldots, \phi_n$, let it be written as

$$x_s=h_1(\phi_1, \phi_2, \ldots, \phi_n). \tag{10}$$

Differentiating x_s with respect to x_i ($i=1, 2, \ldots, n$), we get

$$\delta_{is}=\sum_{r=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_i}+\frac{\partial\phi_r}{\partial z}p_i\right), \qquad i=1, 2, \ldots, n, \tag{11}$$

where δ_{is} is equal to 1 or 0 according as i is or is not equal to s.

Multiplying equations (11) in order by $X_1, X_2, \ldots, X_n$ and adding, we get

$$X_s=\sum_{i=1}^{n}X_i\sum_{r=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_i}+\frac{\partial\phi_r}{\partial z}p_i\right)=\sum_{r=1}^{n}\sum_{i=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(\frac{\partial\phi_r}{\partial x_i}X_i+\frac{\partial\phi_r}{\partial z}X_ip_i\right).$$

Hence, employing (5) and (1), we get

$$X_s=\sum_{r=1}^{n}\frac{\partial h_1}{\partial\phi_r}\left(-\frac{\partial\phi_r}{\partial z}Z+\frac{\partial\phi_r}{\partial z}Z\right),$$

or

$$X_s=0. \tag{12}$$

That is, if

$$x_s=h_1(\phi_1, \phi_2, \ldots, \phi_n)$$

affords a solution of (1), then this value of x_s also satisfies (12).

Consequently we have the trivial case again. Such solutions can be determined by processes independent of integration and involve no knowledge of

partial differential equations. To get these solutions we take those values of z which cause X_s to vanish, and determine whether they satisfy equation (1).

If x_s is not a function of $\phi_1, \phi_2, \ldots, \phi_n$, then equation (6″) assumes the form

$$\theta_s(x_s) h_2(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

where $\theta_s(x_s)$ is a function of x_s alone and may be a constant.

Summing up our results we have the following theorem:

Let the general linear non-homogeneous partial differential equation be written in the form

$$\sum_{i=1}^{n} X_i(z, x_1, x_2, \ldots, x_n) p_i = Z(z, x_1, x_2, \ldots, x_n), \qquad p_i = \frac{\partial z}{\partial x_i}. \qquad \text{(A)}$$

We assume that there is a set of values of the variables $z, x_1, x_2, \ldots, x_n$, in the vicinity of which the functions $Z, X_1, X_2, \ldots, X_n$ are single-valued and analytic. We confine our attention to such a domain. We assume $Z, X_1, X_2, \ldots, X_n$ to have no common factor.

Solutions may be obtained by examining the values of z determined by $Z=0$, by $X_1=0$, by $X_2=0, \ldots,$ by $X_n=0$, and by seeing whether these satisfy the differential equation (A). *These solutions are, however, trivial.*

All other solutions are determined as follows:

Let n functionally independent solutions of the equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = \frac{dz}{Z}$$

be

$$\phi_i(z, x_1, x_2, \ldots, x_n) = \phi_i, \qquad i = 1, 2, \ldots, n.$$

One of the Jacobians

$$J = \frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

or

$$J_s = \frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(z, x_1, x_2, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)}, \qquad s = 1, 2, \ldots, n,$$

does not vanish identically, since $\phi_1, \phi_2, \ldots, \phi_n$ are functionally independent.

Two cases arise.

CASE I. *If $J \not\equiv 0$, the general single-valued analytic solution, except for the trivial solutions afforded by*

$$Z = 0,$$

may be written in the form

$$\theta(z) F(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

F being an arbitrary function of $\phi_1, \phi_2, \ldots, \phi_n$ involving z, and $\theta(z) = 0$ being

an equation in z alone, determining only those (non-infinite) values of z which satisfy $Z=0$.

Case II. *If* $J_s \not\equiv 0$ (*s being an integer such that* $1 \leq s \leq n$), *the general single-valued analytic solution, except for the trivial solutions afforded by*

$$X_s = 0,$$

may be written in the form

$$\theta_s(x_s) F(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

F being an arbitrary function of $\phi_1, \phi_2, \ldots \phi_n$) *involving z, and* $\theta_s(x_s)$ *being a function of* x_s *alone.*

In any case the general single-valued analytic solution of (A), *exclusive of the trivial solutions mentioned above, is*

$$\theta(z)\theta_1(x_1)\theta_2(x_2)\ldots\theta_n(x_n)F(\phi_1, \phi_2, \ldots, \phi_n) = 0,$$

F being an arbitrary function of $\phi_1, \phi_2, \ldots, \phi_n$ *which involves z explicitly; and* $\theta(z)$, $\theta_1(x_1)$, $\theta_2(x_2)$, $\ldots$, $\theta_n(x_n)$ *being functions restricted as before.*

As an illustration of the theory let us take the example

$$p_1 \tan x_1 + p_2 \tan x_2 = \tan z.$$

Here we get for our auxiliary solutions

$$\phi_1 = \frac{\sin x_1}{\sin z}, \quad \phi_2 = \frac{\sin x_2}{\sin z}.$$

The Lagrange general solution is then

$$f(\phi_1, \phi_2) = 0.$$

where f is an arbitrary function of ϕ_1 and ϕ_2 involving z explicitly.

We can easily verify that

$$\psi = \sin z - \frac{\sin^{n+1} x_1}{\sin^n x_2} = 0$$

affords a solution of the original equation. But, since

$$\frac{\partial(\phi_1, \phi_2, \psi)}{\partial(z, x_1, x_2)} \not\equiv 0,$$

ψ cannot be put in the form of the Lagrange general integral.

Since $J = \dfrac{\partial(\phi_1, \phi_2)}{\partial(x_1, x_2)} \not\equiv 0$, we know that ψ can be put in the form

$$\theta(z)F(\phi_1, \phi_2);$$

thus we have

$$\psi \equiv \sin z\left(\frac{\phi_2^n - \phi_1^{n+1}}{\phi_2^n}\right).$$

We see that in this example $\theta(z) = \sin z = 0$ determines only those values of z which cause $Z, \equiv \tan z$, to vanish.

Likewise, since

$$J_1 = \frac{\partial(\phi_1, \phi_2)}{\partial(z, x_2)} \not\equiv 0 \quad \text{and} \quad J_2 = \frac{\partial(\phi_1, \phi_2)}{\partial(z, x_1)} \not\equiv 0,$$

ψ can be put in the forms $\theta_1(x_1)F_1(\phi_1, \phi_2)$ and $\theta_2(x_2)F_2(\phi_1, \phi_2)$; thus we have

$$\psi \equiv \sin x_1\left(\frac{1}{\phi_1} - \frac{\phi_1^n}{\phi_2^n}\right) \equiv \sin x_2\left(\frac{1}{\phi_2}\,\frac{\phi_1^{n+1}}{\phi_2^{n+1}}\right).$$

An example in which $\theta = 0$ or $\theta_i = 0$ may determine an infinite value of the variable is afforded by the equation

$$x_1^2 p_1 - x_1 x_2 p_2 = -x_2^2.$$

Here the auxiliary solutions are

$$\phi_1 = x_1 x_2, \quad \phi_2 = z - \frac{x_2^2}{3x_1}.$$

A so-called special solution is

$$\psi = 3zx_1 - x_2^2 = 0,$$
$$= \frac{3}{x_2}\phi_1\phi_2.$$

Here $\theta_2(x_2)$ is $\frac{3}{x_2}$; equating θ_2 to zero, we have $x_2 = \infty$.

An example of the trivial solutions is afforded by the equation

$$(z - e^{x_2})p_1 + p_2 = z.$$

Here the auxiliary solutions are

$$\phi_1 = \frac{e^{x_2}}{z}, \quad \phi_2 = x_1 - z + e^{x_2}.$$

Trivial solutions are

$$\psi_1 = z - e^{x_2} = 0, \quad \text{from } X_1 = 0,$$

and

$$\psi_2 = z = 0, \quad \text{from } Z = 0.$$

Neither ψ_1 nor ψ_2 is a function of ϕ_1 and ϕ_2.

INDIANA UNIVERSITY.

A Method in the Calculus of Variations.

By Rainard B. Robbins.

Introduction.

The Calculus of Variations originated in the discussion of the problem of finding the curve of quickest descent, the brachistochrone. This problem was proposed to other mathematicians by John Bernoulli* in 1696, and modifications of it have been discussed ever since. There was essentially but one method used by various mathematicians before Lagrange in dealing with this problem, a method developed most fully by Euler, who applied it to the general problem of minimizing a definite integral. The essentials of this method consist in

1) Considering a definite integral as the limit of a sum and a differential equation as the limit of an equation in differences;

2) Using the fact that the first derivative of a function vanishes for values of the variable minimizing the function.

Although Euler put his work in geometric form, his argument may be stated more tersely in analytic form as follows:

The problem is to find a function $y(x)$ such that $y=y(x)$ minimizes $\int_{x_0}^{x_n} F(x, y, dy/dx)\,dx$. Divide the interval x_0x_n into n equal parts of length Δ and use the notation $x_i=x_0+i\Delta$. Then consider the problem of determining $(y_0, y_1, \ldots, y_n)$ such as to minimize

$$\phi(y_0, y_1, \ldots, y_n) \equiv F(x_1, y_1, p_1)\Delta + F(x_2, y_2, p_2)\Delta + \ldots + F(x_n, y_n, p_n)\Delta,$$

where $p_i=(y_i-y_{i-1})/\Delta$. A set of necessary conditions is that

$$\frac{\partial\phi}{\partial y_i} \equiv \Delta F_y(x_i, y_i, p_i) - \{F_p(x_{i+1}, y_{i+1}, p_{i+1}) - F_p(x_i, y_i, p_i)\} = 0, \quad i=1, 2, \ldots, n.$$

Then Euler reasoned that since an integral is the limit of a sum and a differential equation is the limit of a difference equation, the function $y(x)$ minimizing the integral must satisfy

$$F_y(x, y(x), y'(x)) - \frac{d}{dx}F_p(x, y(x), y'(x)) = 0.$$

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* See Ostwald's *Klassiker der Exakten Wissenschaften*, Nr. 46.

Other methods have shown the conclusion to be correct. Furthermore, Knezer* has proved this by a modification of Euler's method.

When Lagrange† introduced the term "variation" and the suggestive notation "δ" to represent it and showed that $d(\delta f)=\delta(df)$, analytic methods replaced the former geometric methods, with a gain in rigor and possibility of further extension of the theory. However, it would seem that this turn of affairs probably prevented the early recognition of as close connections between the problem of the Calculus of Variations and the ordinary problem of maxima and minima as actually exist.

The first part of this paper is a development of the well known tests for a minimum of a function of several variables. It is introduced here because the methods of obtaining these tests are to be used later. The second part may be thought of as a comparison of an algebraic problem of minimizing a sum with the transcendental problem of minimizing a definite integral, the sum and the integral corresponding as in Euler's investigations. The theorems of Part II may be stated roughly as follows:

I. If, as Δ decreases indefinitely, the solution $(y_1, \ldots, y_{n-1})$ of the algebraic problem approaches uniformly to corresponding ordinates of a curve $y=y(x)$, where $y(x)$ is continuous together with its first two derivatives in the interval under consideration, then this curve $y=y(x)$ satisfies the first three necessary conditions‡ for a solution of the transcendental problem.

II. If the absolute minimum of the algebraic problem approaches a curve with a continuously turning tangent, this curve gives the absolute minimum of the transcendental problem if such exists.

In Part III the converse problem is considered; it is shown that if $y(x)$ minimizes the transcendental problem, then for n sufficiently large a solution of the algebraic problem exists as close as we please to $y(x)$.

In Part IV the problem of the Calculus of Variations is considered following formal analogies with the algebraic problem. An important characteristic of the method used in the algebraic problem is that the variations $(\eta_1, \ldots, \eta_{n-1})$ are unrestricted. To get a similar state of affairs in the transcendental problem of minimizing $\int_{x_0}^{x_n} F(x, y, y')\,dx$, we let the second derivative play the rôle of independent variable. This leads to the entrance of multiple integrals and shows clearly certain formal analogies between the algebraic and transcendental problems.

* "Euler und die Variationsrechnung," *Abhandlungen zur Geschichte der Mathematischen Wissenschaften*, XXV.

† Ostwald's *Klassiker*, Nr. 47.

‡ See Bolza, "Lectures on the Calculus of Variations," pp. 101, 102.

I. THE MINIMUM OF A FUNCTION OF SEVERAL VARIABLES.

Consider $\phi(x_1, x_2, \ldots, x_n)$ a real function of the real variables $(x_1, \ldots, x_n)$ of class* C''' in a certain neighborhood R of the origin, $|x_i| < k, i=1,2,\ldots,n$. Then for points $(x_1, \ldots, x_n)$ within this region† R,

$$\text{I.} \quad \phi(x_1, \ldots, x_n) - \phi(0, \ldots, 0) = \sum_{i=1}^{n} x_i \phi_i(0, \ldots, 0) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \phi_{ij}(0, \ldots, 0) + l^2(l),$$

in which we use the notation $l^2 = x_1^2 + \ldots + x_n^2$, (l) represents a quantity infinitesimal with l, and

$$\phi_i(x_i, \ldots, x_n) \equiv \frac{\partial \phi}{\partial x_i}(x_1, \ldots, x_n); \quad \phi_{ij}(x_1, \ldots, x_n) \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_1, \ldots, x_n).$$

To find necessary conditions that $\phi(x_1, \ldots, x_n)$ have a relative minimum at $(0, \ldots, 0)$, we assume that $(0, \ldots, 0)$ gives a relative minimum. Then for $(x_1, \ldots, x_n)$ sufficiently near the origin,

$$\phi(0, \ldots, 0) \overline{\gtrless} \phi(x_1, \ldots, x_n)$$

and the left-hand side of I is positive or zero. In particular,

$$\text{I}'. \quad \phi(x_1, 0, \ldots, 0) - \phi(0, \ldots, 0) = x_1 \phi_1(0, \ldots, 0) + \frac{1}{2!} x_1^2 \phi_{11}(0, \ldots, 0) + x_1^2(x_1) \overline{\gtrless} 0.$$

In order that this should not change sign with x_1, we must have $\phi_1(0, \ldots, 0) = 0$. In this way we obtain

$$1) \quad \phi_1(0, \ldots, 0) = 0, \quad \phi_2(0, \ldots, 0) = 0, \quad \ldots, \quad \phi_n(0, \ldots, 0) = 0.$$

Then I becomes

$$\text{II.} \quad \phi(x_1, \ldots, x_n) - \phi(0, \ldots, 0) = \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \phi_{ij}(0, \ldots, 0) + l^2(l),$$

and I′ becomes

$$\text{II}'. \quad \phi(x_1, 0, \ldots, 0) - \phi(0, \ldots, 0) = \frac{1}{2!} x_1^2 \phi_{11}(0, \ldots, 0) + x_1^2(x_1).$$

For x_1 sufficiently small the sign on the right in II′ depends on the sign of $\phi_{11}(0, \ldots, 0)$, if $\phi_{11}(0, \ldots, 0) \neq 0$. Thus as a second set of necessary con-

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* $\phi(x_1, \ldots, x_n)$ is said to be of class $C^{(m)}$ in a given region if ϕ, together with all its partial derivatives of orders 1, 2, ..., m, is continuous in this region.

† See Goursat, "Cours d'analyse, t. I, p. 110.

ditions that $(0, \ldots, 0)$ give a relative minimum for $\phi(x_1, \ldots, x_n)$ we have

2) $$\phi_{ii}(0, \ldots, 0) \geqq 0; \quad i=1, 2, \ldots, n.$$

To examine II further, we introduce $v_1, \ldots, v_n$ defined by the equations

$$x_1 = lv_1, \quad x_2 = lv_2, \quad \ldots, \quad x_n = lv_n.$$

Then II becomes

$$\phi(x_1, \ldots, x_n) - \phi(0, \ldots, 0) = \frac{1}{2!} l^2 \left[\sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \phi_{ij}(0, \ldots, 0) + (l) \right].$$

For l sufficiently small the sign on the right is the sign of the quadratic form, unless this vanishes. Then as a third necessary condition that $(0, \ldots, 0)$ give a relative minimum for $\phi(x_1, \ldots, x_n)$ we have

3) The quadratic form $\sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \phi_{ij}(0, \ldots, 0)$ must be positive definite or semi-definite.

Since $\sum_{i=1}^{n} x_i^2 = l^2$, we have $\sum_{i=1}^{n} v_i^2 = 1$; consequently 3) can be stated thus:

$$\Omega(v_1, \ldots, v_n) \equiv \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \phi_{ij}(0, \ldots, 0)}{\sum_{i=1}^{n} v_i^2} \geqq 0.$$

We note that $\Omega(v_1, \ldots, v_n) = \Omega(cv_1, \ldots, cv_n)$. For this reason the function Ω takes all its values on the hypersphere in n-space, $\sum_{i=1}^{n} v_i^2 = 1$. On this locus Ω reaches its lower limit, *i. e.*, takes on a minimum value, which is positive or zero. Let

$$\sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j \phi_{ij}(0, \ldots, 0) \equiv L, \quad \sum_{i=1}^{n} v_i^2 \equiv M.$$

For values $(v_1, \ldots, v_n)$ for which Ω is a maximum or minimum, $\dfrac{\partial \Omega}{\partial v_i}$ vanishes for $i=1, 2, \ldots, n$. For such values we have

a) $$M \frac{\partial L}{\partial v_i} - L \frac{\partial M}{\partial v_i} = 0, \quad i=1, 2, \ldots, n.$$

Differentiating, $\dfrac{\partial L}{\partial v_i} = 2 \sum_{j=1}^{n} \phi_{ij} v_j$, $\dfrac{\partial M}{\partial v_i} = 2v_i$, and conditions a) become

b) $$\begin{cases} \phi_{11} v_1 + \phi_{12} v_2 + \cdots + \phi_{1n} v_n = \lambda v_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \phi_{n1} v_1 + \phi_{n2} v_2 + \cdots + \phi_{nn} v_n = \lambda v_n, \end{cases}$$

where λ is the value of L and therefore of Ω at $(v_1, \ldots, v_n)$, *i. e.*, either a maximum or a minimum of Ω. The necessary and sufficient condition that

this last set of equations have a solution $(v_1, \ldots, v_n)$ other than $(0, \ldots, 0)$ is that

$$\text{c)} \qquad \begin{vmatrix} (\phi_{11}-\lambda)\phi_{12}\ldots\ldots\phi_{1n} \\ \phi_{21}(\phi_{22}-\lambda)\ldots\ldots\phi_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_{n1}\phi_{n2}\ldots\ldots(\phi_{nn}-\lambda) \end{vmatrix} = 0.$$

The roots of this equation in λ are real* and condition 3) is equivalent to

3′) Every root of equation c) is positive or zero. For let $\lambda=\lambda'$ be a root of c). For $\lambda=\lambda'$ equations b) are consistent. Multiplying equations b) in succession by $v_1, v_2, \ldots, v_n$ and adding, we have for $\lambda=\lambda'$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_{ij}(0, \ldots, 0)v_iv_j=\lambda'\sum_{i=1}^{n}v_i^2,$$

or

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_{ij}(0, \ldots, 0)v_iv_j}{\sum_{i=1}^{n}v_i^2}=\lambda'.$$

Since the left-hand side is positive or zero, we have $\lambda' \geqq 0$. Conversely, if every root of c) is positive or zero, so also are all extremal values of $\Omega(v_1, \ldots, v_n)$.

To recapitulate: If $(0, \ldots, 0)$ is a point in the interior of a closed region R in which $\phi(x_1, \ldots, x_n)$ is of class C''', necessary conditions that $\phi(x_1, \ldots, x_n)$ have a relative minimum at $(0, \ldots, 0)$ are:

1) $$\frac{\partial\phi}{\partial x_i}(0, \ldots, 0)=0, \quad i=1, 2, \ldots, n.$$

2) $$\frac{\partial^2\phi}{\partial x_i\partial x_j}(0, \ldots, 0)\geqq 0, \quad i=1, 2, \ldots, n.$$

3) The quadratic form $\sum_{i=1}^{n}\sum_{j=1}^{n}\phi_{ij}(0, \ldots, 0)x_ix_j$ is positive definite or semi-definite; or

$\overline{3}$) The system b) has no solution other than $(0, \ldots, 0)$ for $\lambda<0$; or

$\overline{\overline{3}}$) The equation c) in λ has no negative roots.

Conversely, if 1), 2) are satisfied and if equations b) have no solution other than $(0, \ldots, 0)$ for $\lambda \leqq 0$, then $\phi(x_1, \ldots, x_n)$ has a relative minimum at the origin. For the least root of equation c) is the least value of Ω on the locus $\sum_{i=1}^{n}v_i^2=1$; if this is positive, so also is $\sum_{i=1}^{n}\sum_{j=1}^{n}v_iv_j\phi_{ij}(0, \ldots, 0)$, and therefore $\phi(x_1, \ldots, x_n)-\phi(0, \ldots, 0)>0$ for l sufficiently small and $(x_1, \ldots, x_n) \not\equiv (0, \ldots, 0)$.

* Proved by Cauchy, "Exercises de Mathématiques," IV, p. 145.

II. THE ALGEBRAIC PROBLEM AND ITS LIMIT.

Consider a real function of three independent real variables, $F(x, y, p)$ of class C''' in a certain simply connected region R of the xy plane and for all values of p. Let (x_0, y_0), (x_n, y_n) be two interior points of R, $x_0 < x_n$; let the equation $x_0 + n\Delta = x_n$, n being a positive integer, determine the quantity Δ, and let $x_i \equiv x_0 + i\Delta$, $i = 1, 2, \ldots, n-1$. Now consider the sum

$$\begin{aligned}\phi(y_1, \ldots, y_{n-1}) &\equiv F\left(x_1, y_1, \frac{y_1 - y_0}{\Delta}\right)\Delta + F\left(x_2, y_2, \frac{y_2 - y_1}{\Delta}\right)\Delta \\ &\quad + \ldots + F\left(x_n, y_n, \frac{y_n - y_{n-1}}{\Delta}\right)\Delta \\ &\equiv \sum_{i=1}^{n} F\left(x_i, y_i, \frac{y_i - y_{i-1}}{\Delta}\right)\Delta,\end{aligned}$$

$y_1, \ldots, y_{n-1}$ being such that the points (x_i, y_i) lie entirely in R. This is obviously a function of precisely the type considered in Part I. We propose to consider for this particular function the conditions for a minimum derived in Part I, and at the same time to recapitulate the essentials of the argument.

For n a fixed positive integer suppose that $(Y_1, \ldots, Y_{n-1})$, such that (x_i, Y_i) is an interior point of R, $i = 1, 2, \ldots, n-1$, gives a relative minimum for $\phi(y_1, \ldots, y_{n-1})$. Then if $\eta_1, \ldots, \eta_{n-1}$ are any $n-1$ real quantities, a positive number δ can be found such that when $|\epsilon| < \delta$

$$1)\qquad \phi(Y_1 + \epsilon\eta_1, \ldots, Y_{n-1} + \epsilon\eta_{n-1}) - \phi(Y_1, \ldots, Y_{n-1}) \geqq 0.$$

Expanding by Taylor's series with a remainder (this expansion is legitimate with our hypotheses on F) and using the abbreviated bracket notation,

$$\left[x_i, Y_i, \frac{Y_i - Y_{i-1}}{\Delta}\right] \equiv (i),$$

$$\begin{aligned}2)\qquad &\sum_{i=1}^{n}\left\{F\left(x_i, Y_i + \epsilon\eta_i, \frac{Y_i - Y_{i-1} + \epsilon(\eta_i - \eta_{i-1})}{\Delta}\right) - F\left(x_i, Y_i, \frac{Y_i - Y_{i-1}}{\Delta}\right)\right\}\Delta \\ &= \epsilon\sum_{i=1}^{n}\left\{\eta_i F_y(i) + \frac{\eta_i - \eta_{i-1}}{\Delta}F_p(i)\right\}\Delta \\ &+ \frac{\epsilon^2}{2!}\sum_{i=1}^{n}\left\{\eta_i^2 F_{yy}(i) + 2\eta_i\frac{\eta_i - \eta_{i-1}}{\Delta}F_{yp}(i) + \left(\frac{\eta_i - \eta_{i-1}}{\Delta}\right)^2 F_{pp}(i)\right\}\Delta + \epsilon^2(\epsilon) \geqq 0,\end{aligned}$$

in which $\eta_0 = \eta_n = 0$ and (ϵ) indicates a quantity which is infinitesimal with ϵ.

Since the right-hand side of equation 2) must not change sign with ϵ, for any real values $(\eta_1, \ldots, \eta_{n-1})$ we must have

$$3)\qquad \sum_{i=1}^{n}\left\{\eta_i F_y(i) + \frac{\eta_i - \eta_{i-1}}{\Delta}F_p(i)\right\}\Delta = 0.$$

Then the coefficient of each η in 3) must vanish. We can get the coefficient of η_i in 3) very readily by inspection, but to make the analogy to the method

of the Calculus of Variations more evident we use the formula for partial summation:

$$\sum_{i=v}^{w} b_i(a_i-a_{i-1}) \equiv \sum_{i=v}^{w} a_i(b_i-b_{i+1}) - a_{v-1}b_v + a_w b_{w+1}.$$

Since $\eta_0=\eta_n=0$, this identity gives us

$$\sum_{i=1}^{n}\left(\frac{\eta_i-\eta_{i-1}}{\Delta}\right)F_p(i)=\sum_{i=1}^{n}\eta_i\left(\frac{F_p(i)-F_p(i+1)}{\Delta}\right),$$

and 3) becomes

$$\sum_{i=1}^{n}\eta_i\left[F_y(i)-\frac{F_p(i+1)-F_p(i)}{\Delta}\right]\Delta=0.$$

Hence in this case conditions 1), Part I, are

$$F_y(i)-\frac{F_p(i+1)-F_p(i)}{\Delta}=0, \qquad i=1,2,\ldots\ldots,n-1.$$

Also in 2) the coefficient of ε^2 must be positive or zero; *i. e.*,

$$4)\qquad \sum_{i=1}^{n}\left\{\eta_i^2 P_i+2\eta_i\frac{\eta_i-\eta_{i-1}}{\Delta}Q_i+\left(\frac{\eta_i-\eta_{i-1}}{\Delta}\right)^2 R_i\right\}\Delta \geqq 0,$$

in which $P_i\equiv F_{yy}(i)$; $Q_i\equiv F_{yp}(i)$; $R_i\equiv F_{pp}(i)$.

Let j be a positive integer less than n, and choose $\eta_i=0$, $i\neq j$, $\eta_j\neq 0$. (This is analogous to a method often useful in the Calculus of Variations of giving the minimizing curve an increment for a very short distance). Then the coefficient of η_j^2 in 4) must be greater than or equal to zero, giving condition 2), Part I, for this particular problem,

$$\Delta P_j+2Q_j+\frac{R_{j+1}+R_j}{\Delta}\geqq 0, \qquad j=1,2,\ldots\ldots,n-1.$$

Treating the quadratic form 4) in a manner closely analogous to that used in dealing with the second variation in the general algebraic problem, Part I, we have immediately,

$$\text{A.}\qquad \frac{\sum_{i=1}^{n}\left\{\eta_i^2 P_i+2\eta_i\frac{\eta_i-\eta_{i-1}}{\Delta}Q_i+\left(\frac{\eta_i-\eta_{i-1}}{\Delta}\right)^2 R_i\right\}}{\sum_{i=1}^{n}\eta_i^2\Delta}\geqq 0.$$

Since the expression A is homogeneous of degree zero in the η's, it has the same value for the set $(\eta_1,\ldots\ldots,\eta_{n-1})$ as for $(\rho\eta_1,\ldots\ldots,\rho\eta_{n-1})$. Then it takes all its values on the locus $\eta_1^2+\ldots\ldots+\eta_{n-1}^2=1/\Delta$, a closed locus, and therefore it assumes its lower and its upper limits, and its lower limit is positive or zero. At points where extremal values are taken we have, from A,

$$\sum_{i=1}^{n}\eta_i^2\Delta\frac{\partial}{\partial\eta_i}(\text{numerator of } A)-\text{numerator of } A\cdot 2\eta_i\Delta=0, \quad i=1,2,\ldots\ldots,n-1.$$

For such points on $\sum_{i=1}^{n} \eta_i^2 \Delta = 1$ (or $\sum_{i=1}^{n-1} \eta_i^2 \Delta = 1$, since $\eta_n = 0$),

$$\frac{\partial}{\partial \eta_i} (\text{numerator of } A) = 2\eta_i \Delta \cdot \lambda,$$

where λ is the extremal value of the fraction at the point in question. Performing the indicated differentiation and using the notation $u_i - u_{i-1} = \delta u_i$,

$$\eta_i P_i - \eta_i \frac{\delta Q_{i+1}}{\Delta} - \frac{R_{i+1} \dfrac{\delta \eta_{i+1}}{\Delta} - R_i \dfrac{\delta \eta_i}{\Delta}}{\Delta} + Q_i \frac{\delta \eta_i}{\Delta} - Q_{i+1} \frac{\delta \eta_{i+1}}{\Delta} = \eta_i \lambda, \; i = 1, 2, \ldots, n-1;$$

or, re-arranging,

$$\text{B.} \quad R_{i+1} \frac{\dfrac{\delta \eta_{i+1}}{\Delta} - \dfrac{\delta \eta_i}{\Delta}}{\Delta} + \frac{\delta R_{i+1}}{\Delta} \frac{\delta \eta_i}{\Delta} + \left(\frac{\delta Q_{i+1}}{\Delta} - P_i + \lambda \right) \eta_i + Q_{i+1} \frac{\delta \eta_{i+1}}{\Delta} - Q_i \frac{\delta \eta_i}{\Delta} = 0,$$
$$i = 1, 2, \ldots, n-1.$$

If $(Y_1, \ldots, Y_{n-1})$ gives a relative minimum, system B can have no solution $(\eta_1, \ldots, \eta_{n-1}) \not\equiv 0$ for $\lambda < 0$. It should be remembered that $\eta_0 = \eta_n = 0$.

Summarizing: If $(Y_1, Y_2, \ldots, Y_{n-1})$ gives a minimum for $\phi(y_1, \ldots, y_{n-1})$, then

$$\text{I}'. \quad F_y(i) - \frac{F_p(i+1) - F_p(i)}{\Delta} = 0, \qquad i = 1, 2, \ldots, n-1.$$

$$\text{II}'. \quad \Delta^2 P_i + 2\Delta \cdot Q_i + R_{i+1} + R_i \geqq 0, \qquad i = 1, 2, \ldots, n-1.$$

III′. The set of equations B have no solution $(\eta_1, \ldots, \eta_{n-1}) \not\equiv 0$ for $\lambda < 0$ such that $\eta_0 = \eta_n = 0$.

Also, as in the general case, Part I, we have as sufficient conditions for a relative minimum, conditions I′, II′ and

III″. The set of equations B have no solution $(\eta_1, \ldots, \eta_{n-1}) \not\equiv 0$, such that $\eta_0 = \eta_n = 0$, for $\lambda \leqq 0$.

We wish now to see what becomes of these conditions I′, II′, III′ as n, the number of divisions of our interval, increases indefinitely, assuming that the set $(Y_1, \ldots, Y_{n-1})$ approach the corresponding ordinates of a curve of class C'', $y(x)$; we hope in this way to show that $y(x)$ satisfies certain necessary conditions for minimizing the definite integral, $\int_{x_0}^{x_n} F(x, y, dy/dx)\, dx$.

Let x' be a fixed abscissa of our interval, $x_0 < x' \leqq x_n$, and for any particular division of our interval let j be such that $x_{j-1} < x' \leqq x_j$. It is evident that j increases indefinitely with n, while $\underset{j=\infty}{\text{Lt}}\, x_j = x'$; this will be indicated thus:

$$\underset{n=\infty,\, j=\infty}{\text{Lt}} x_j = x'.$$

Assume that, as we increase the number of points of division x_i, the ordinates $(Y_1, \ldots, Y_{n-1})$ approach uniformly to the corresponding values of a function $y(x)$ of class C'' in the interval $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$; *i. e.*, given $\varepsilon > 0$, N can be found independent of i such that whenever the number of divisions n of the interval $(x_0 x_n)$ is greater than N,

$$|Y_i - y(x_i)| < \varepsilon \text{ for } i = 1, 2, \ldots, n-1.$$

Then because of the uniform continuity of $y(x)$ in the closed interval $(x_0 x_n)$, given $\varepsilon > 0$, N can be found independent of x' such that whenever $n > N$, $|Y_j - y(x')| < \varepsilon$ for any x' of our interval.

Assume furthermore that $\underset{n=\infty}{\mathrm{Lt}} (Y_i - Y_{i-1})/\Delta = y'(x_i)$, and that this limit is approached uniformly. Then because of the uniform continuity of $y'(x)$ in $(x_0 x_n)$, $\underset{n=\infty,\, j=\infty}{\mathrm{Lt}} (Y_j - Y_{j-1})/\Delta = y'(x')$, and this limit is approached uniformly for all values of x'.

These assumptions, together with the assumption that F_y is a continuous function of its three arguments for (x, y) in region R and for all values of p, gives us that

$$\underset{n=\infty,\, j=\infty}{\mathrm{Lt}} F_y\left[x_j,\, Y_j,\, \frac{Y_j - Y_{j-1}}{\Delta}\right] = F_y[x',\, y(x'),\, y'(x')],$$

and that this limit is approached uniformly for all values of x' involved. (When convenient, we shall denote the arguments $[x, y(x), y'(x)]$ by $[x]$.) Just as with F_y, it is evident that F_p, F_{yy}, F_{yp}, F_{pp}, F_{ypx}, F_{ypp}, F_{ppx}, F_{ppy}, F_{ppp}, with the arguments (j), approach uniformly to the corresponding functions with the arguments $[x']$, in the interval $x_0 < x' \overline{\gtrless} x_n$.

We shall now consider equations I′ as n increases indefinitely:

I′. $\quad \Delta F_y(i) + F_p(i) - F_p(i+1) = 0, \qquad i = 1, 2, \ldots, n-1.$

We wish to show that $y(x)$ satisfies Euler's differential equation; *i. e.*, that

I. $$F_y[x] - \frac{d}{dx} F_p[x] = 0.$$

From I′ we have

5) $$F_p(1) - F_p(j) + \sum_{i=1}^{j-1} F_y(i) \cdot \Delta = 0.$$

Since $\underset{n=\infty,\, j=\infty}{\mathrm{Lt}} x_j = x'$, and since $F_y(i)$ approaches $F_y[x_1]$ uniformly as n increases indefinitely,

$$\underset{n=\infty,\, j=\infty}{\mathrm{Lt}} \sum_{i=1}^{j-1} F_y(i) \cdot \Delta = \int_{x_0}^{x'} F_y[x]\, dx.$$

Also

$$\underset{n=\infty}{\mathrm{Lt}} F_p(1) = F_p[x_0]\,; \quad \underset{n=\infty,\, j=\infty}{\mathrm{Lt}} F_p(j) = F_p[x'].$$

Since the limit of a sum is the sum of the limits, it follows from 5) that

$$F_p[x_0]-F_p[x']+\int_{x_0}^{x'}F_y[x]\,dx=0.$$

Differentiating with respect to x', we obtain equation I for x in the interval $x_0<x\overline{\gtrless}x_n$. Since F_y and $\frac{d}{dx}F_p$ are continuous, it follows that equation I holds for all values of x in the closed interval $x_0\overline{\gtrless}x\overline{\gtrless}x_n$.

We may write I′ in the form

$$F_y(j)-F_{px}(j+\theta_1)-\frac{Y_{j+1}-Y_j}{\Delta}F_{py}(j+\theta_2)-\frac{\dfrac{Y_{j+1}-Y_j}{\Delta}-\dfrac{Y_j-Y_{j-1}}{\Delta}}{\Delta}F_{pp}(j+\theta_3)=0,$$

in which $(j+\theta_i)$ indicates that the arguments are taken between those indicated by (j) and by $(j+1)$. We know that $\underset{n=\infty,\,j=\infty}{\mathrm{Lt}}$ of each function involved in this last equation exists and is approached uniformly throughout the interval with the exception of the fraction in the last term. Then if we assume that $F_{pp}[x]\neq 0$ in the interval $x_0\overline{\gtrless}x\overline{\gtrless}x_n$, *i. e.*, that $F_{pp}(x,y,p)\neq 0$ along $y(x)$, we have

$$\underset{n=\infty,\,j=\infty}{\mathrm{Lt}}\frac{\delta Y_{j+1}-\delta Y_j}{\Delta^2}=\frac{F_y[x']-F_{px}[x']-y'(x')F_{py}[x']}{F_{pp}[x']}.$$

From equation I we obtain

$$y''(x')=\frac{F_y[x']-F_{px}[x']-y'(x')F_{py}[x']}{F_{pp}[x']}.$$

Hence,

$$\text{6)}\qquad \underset{n=\infty,\,j=\infty}{\mathrm{Lt}}\frac{\delta Y_{j+1}-\delta Y_j}{\Delta^2}=y''(x'),$$

and furthermore this limit is approached uniformly for $x_0<x'\overline{\gtrless}x_n$.

Consider condition II′, which we rewrite,

$$\text{II}'.\qquad \Delta^2P_j+2\Delta Q_j+R_{j+1}+R_j\overline{\gtrless}0.$$

It is evident that

$$\underset{n=\infty,\,j=\infty}{\mathrm{Lt}}\{\Delta^2P_j+2\Delta Q_j+R_{j+1}+R_j\}=2F_{pp}[x'].$$

Thus we get, as the limiting form of the second condition,

$$\text{II}.\qquad F_{pp}[x]\equiv R[x]\overline{\gtrless}0,\quad x_0\overline{\gtrless}x\overline{\gtrless}x_n.$$

Assuming as above that $F_{pp}[x]\neq 0$, we have

$$\text{II}''.\qquad F_{pp}[x]>0.$$

We shall now use condition III′ to show that if $u(x)$ is a solution of

$$\psi(u)\equiv\frac{d}{dx}(Ru')-(P-Q')u=0,$$

such that $u(x_0)=0$, $u(x)\not\equiv 0$ (the arguments of P, Q', R being $[x]$), then $u(x)$ has no root between $x=x_0$ and $x=x_n$; *i. e.*, $u(x)\neq 0$, in the interval $x_0<x<x_n$.

We shall make use here of the two following theorems:*

a) Given the system

$$\begin{cases} \dfrac{d}{dx}\left(k(x)\dfrac{du}{dx}\right)+[\lambda g(x)-l(x)]u=0, \\ u(a)=\alpha', \\ u'(a)=\alpha, \end{cases}$$

in which $k(x)>0$, $g(x)>0$, $k(x)$ is of class C', $a\overline{\gtrless}x\overline{\gtrless}b$ and $g(x)$, $l(x)$ are continuous in the same interval, and α, α' are constants not both zero; there is one and only one value of the parameter λ for which the solution of this system has a specified number (including zero) of roots between a and b and is such that $u(b)=0$.

b) Given the systems

$$\begin{cases} \dfrac{d}{dx}\left(\varkappa_1\dfrac{du}{dx}\right)-G_1u=0, \\ u(a)=0, \end{cases}$$

$$\begin{cases} \dfrac{d}{dx}\left(\varkappa_2\dfrac{du}{dx}\right)-G_2u=0, \\ u(a)=0, \end{cases}$$

in which $\varkappa_1$, $\varkappa_2$ are functions of x of class C', and G_1, G_2 are continuous in the interval $a\overline{\gtrless}x\overline{\gtrless}b$; furthermore, $G_2<G_1$ and $0<\varkappa_2\overline{\gtrless}\varkappa_1$. Let $u_1(x)$ be a non-identically vanishing solution of the first system and $u_2(x)$ likewise for the second. If $u_1(x)$ has k roots in the interval $a<x\overline{\gtrless}b$, then $u_2(x)$ has at least k roots in this interval and the i-th root of $u_2(x)$ is smaller than the i-th root of $u_1(x)$.

Suppose that $\psi(u)=0$ has a non-identically vanishing solution, $u(x)$, such that $u(x_0)=0$, $u(x')=0$ for some number x' such that $x_0<x'<x_n$. To prove our theorem we shall show that this supposition leads to a contradiction; in fact it leads to a solution of the difference equation system B, $(\eta_1,\ldots,\eta_{n-1})\not\equiv 0$, such that $\eta_0=\eta_n=0$, for λ negative.

Making use of the above theorems a), b), our present supposition gives us that there is one and only one value $\lambda_0<0$ such that the solution of the system

$$\text{7)}\qquad \begin{cases} \psi(u)+\lambda_0u=0, \\ u(x_0)=0, \\ u'(x_0)=\gamma, \end{cases}$$

* Given in this form by Professor Bôcher in lectures. See also in article by Bôcher, *Bulletin of Amer. Math. Soc.*, Vol. IV, p. 298, I, and p. 304, VII, and *Transactions of Amer. Math. Soc.*, Vol. I, p. 418.

in which γ is a positive constant, has $x=x_n$ as its first root to the right of $x=x_0$. If we replace λ_0 by $\lambda_0-\delta$, $\delta>0$, the solution of system 7), $u(x, \lambda_0-\delta)$ does not vanish in the interval $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$ and $u(x_n)$ is positive; for λ_0 replaced by $\lambda_0+\delta$ the solution $u(x, \lambda_0+\delta)$ vanishes within the interval $(x_0 x_n)$, and for δ sufficiently small, $u(x_n)$ is negative.

The plan is first to show that we can get one solution of the difference system B, with $\eta_0=0$, as near to $u(x, \lambda_0-\delta)$ as we please and therefore with $\eta_n>0$, and one as near to $u(x, \lambda_0+\delta)$ as we please and therefore with $\eta_n<0$. Then by holding n fast and varying λ between $\lambda_0+\delta$ and $\lambda_0-\delta$ we shall get a solution of B such that $\eta_0=\eta_n=0$, for some negative λ, which contradicts condition III′.

The solution of the linear differential system

$$8)\qquad \begin{cases} \psi(u)+\lambda u=0, \\ u(x_0)=0, \\ u'(x_0)=\gamma \end{cases}$$

can be obtained by the Cauchy–Lipschitz method as the limit approached by the solution of the linear difference equation system (using our previous notation, $\delta\eta_i=\eta_i-\eta_{i-1}$),

$$9)\qquad \begin{cases} R[x_i]\dfrac{\delta^2\eta_{i+2}}{\Delta^2}+R'[x_i]\dfrac{\delta\eta_{i+1}}{\Delta}+[Q'[x_i]-P[x_i]]\eta_i+\lambda\eta_i=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad i=0, 1, 2, \ldots, n-2, \\ \eta_0=0, \\ \eta_1=\Delta\gamma. \end{cases}$$

The theorem involved here may be stated as follows:

Given $f(x, y, v)$, $g(x, y, v)$ real continuous functions of the three real variables (x, y, v) in the region R: $|x-x_0|<a$, $|y-y_0|<b$, $|v-v_0|<b$, satisfying the inequalities

$$|f(x, y', v')-f(x, y, v)|<k_1|y'-y|+k_2|v'-v|,$$
$$|g(x, y', v')-g(x, y, v)|<k_1|y'-y|+k_2|v'-v|,$$

k_1, k_2 being constants and (x, y', v'), (x, y, v) being points in R; then there exist solutions $y(x)$, $v(x)$ of the differential equations

$$\frac{dy}{dx}=f(x, y, v), \quad \frac{dv}{dx}=g(x, y, v),$$

satisfying the initial conditions $y(x_0)=y_0$, $v(x_0)=v_0$. These functions are approached by the solutions of the difference equation system

$$\begin{cases} y_{i+1}-y_i=(x_{i+1}-x_i)f(x_i, y_i, v_i), \\ v_{i+1}-v_i=(x_{i+1}-x_i)g(x_i, y_i, v_i), \qquad i=1, 2, \ldots, n. \end{cases}$$

in which $x_{n+1} \equiv x$ is a point of the interval $|x-x_0| < k$ in which the solutions of the differential equations are continuous, $x_1, x_2, \ldots, x_n$ being points of the interval $(x_0 x)$; the approach mentioned occurs when the number of divisions of the interval $(x_0 x)$ is increased indefinitely and the length of the longest interval $(x_i x_{i+1})$ is decreased indefinitely.*

Furthermore, the solutions of the difference system approach $y(x)$, $v(x)$ uniformly in any interval which, with its end points, lies within the largest interval in which the solutions $y(x)$, $v(x)$ of the differential equations are continuous and f, g satisfy the conditions originally imposed.†

An examination of the proof of this theorem shows that it holds with no essential changes if in the difference equations we replace $f(x_i, y_i, v_i)$, $g(x_i, y_i, v_i)$ by functions which approach these values uniformly for all values of x' in the interval of convergence of the solution $y(x), v(x)$ when i increases with n in such a way that $\operatorname*{Lt}_{n=\infty} x_i = x'$. It is of course essential that the subscripts in the difference equation system be such that y_i, v_i can be determined in succession when y_0, v_0 are given.

It is this generalization of the theorem which is of use in the present problem.

The differential equation $\psi(u) + \lambda u = 0$ is equivalent to the system

$$10) \qquad \begin{cases} \dfrac{du}{dx} = v & \equiv f(x, y, v), \\ \dfrac{dv}{dx} = \dfrac{(P-Q'-\lambda)u - R'v}{R} & \equiv g(x, y, v), \end{cases}$$

and the difference equation system 9) is equivalent to the system

$$11) \qquad \begin{cases} \eta_{i+1} - \eta_i = (x_{i+1} - x_i) v_i, \\ v_{i+1} - v_i = (x_{i+1} - x_i) \dfrac{[P[x_i] - Q'[x_i] - \lambda]\eta_i - R'[x_i] v_i}{R[x_i]}, \end{cases}$$

which evidently corresponds to the above system of differential equations in applying the Cauchy–Lipschitz theorem. The functions f, g satisfy a Lipschitz condition since $R \neq 0$ and f, g have continuous first partial derivatives with regard to x, u, v in the neighborhood of the solutions $u(x), v(x)$ satisfying the initial conditions $u(x_0) = 0$, $v(x_0) = \gamma$.

But instead of the difference system 9) we should like to be able to substitute system B, Part II. More precisely, we wish to show that for η sufficiently large, solutions of B exist uniformly close to corresponding solutions of system 8) for certain values of λ.

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* Picard, "Traité d'analyse," t. 2, p. 322.

† Picard, "Traité d'analyse," t. 2, p. 332, and Lipschitz, "Lehrbuch der Analysis," Bd. 2, p. 504.

System B is equivalent to

$$12)\quad \begin{cases} \eta_{i+1}-\eta_i=(x_{i+1}-x_i)v_i, \\ v_{i+1}-v_i= \\ (x_{i+1}-x_i)\dfrac{\left(P_{i+1}-\dfrac{Q_{i+2}-Q_{i+1}}{\Delta}-\lambda\right)(\eta_i+\Delta v_i)-\left(\dfrac{R_{i+2}-R_{i+1}}{\Delta}+Q_{i+2}-Q_{i+1}\right)v_i}{R_{i+2}+\Delta Q_{i+2}}, \\ \qquad\qquad i=0,1,2,\ldots,n-2. \end{cases}$$

From the extension mentioned of the Cauchy–Lipschitz theorem we see that we can replace system 11) by system 12) if we can show that, as n increases, P_{i+1}, $(Q_{i+2}-Q_{i+1})/\Delta$, $(R_{i+2}-R_{i+1})/\Delta$ approach uniformly to $P[x_i]$, $Q'[x_i]$, $R'[x_i]$.

That P_{i+1} approaches uniformly to $P[x_{i+1}]$ has already been shown. Then P_{i+1} approaches uniformly to $P[x_i]$ because of the uniform continuity of $P[x]$ in the interval $x_0\overline{\gtrless}x\overline{\gtrless}x_n$.

To show that $(Q_{i+2}-Q_{i+1})/\Delta$ approaches uniformly to $Q'[x_i]$, we note that

$$Q_{i+2}-Q_{i+1}=(x_{i+2}-x_{i+1})Q_x(i+1+\theta_1)+(Y_{i+2}-Y_{i+1})Q_y(i+1+\theta_2)$$
$$+\left(\frac{Y_{i+2}-Y_{i+1}}{\Delta}-\frac{Y_{i+1}-Y_i}{\Delta}\right)Q_p(i+1+\theta_3),$$

in which $(i+1+\theta)$ means that the arguments are between $(i+2)$ and $(i+1)$. Dividing through by $x_{i+2}-x_{i+1}$, *i. e.* by Δ, and using the information regarding limits including equation 6), we see that $\underset{n=\infty,\, i+2=\infty}{\mathrm{Lt}}$ of the right-hand side exists.

$$\therefore \underset{n=\infty,\, i+2=8}{\mathrm{Lt}} \frac{Q_{i+2}-Q_{i+1}}{\Delta}=Q_x[x_{i+1}]+y'(x_{i+1})Q_y[x_{i+1}]+y''(x_{i+1})Q_p[x_{i+1}];$$

i. e.,

$$\underset{n=\infty,\, i+2=\infty}{\mathrm{Lt}}(Q_{i+2}-Q_{i+1})/\Delta=Q'[x_{i+1}].$$

Furthermore, the limits on the right are approached uniformly, and from this fact, together with the uniform continuity of $y(x)$, $y'(x)$ and $Q'(x)$ in the interval $x_0\overline{\gtrless}x\overline{\gtrless}x_n$, it follows that $(Q_{i+2}-Q_{i+1})/\Delta$ approaches uniformly to $Q'[x_i]$, $i=1,2,\ldots,n-2$, when n increases indefinitely. Exactly similar reasoning gives similar results for $(R_{i+2}-R_{i+1})/\Delta$.

Thus the solution of system 8) is approached by the solution of the difference equation system

$$13)\quad \begin{cases} R_{i+1}\dfrac{\dfrac{\delta r_{i+1}}{\Delta}-\dfrac{\delta\eta_i}{\Delta}}{\Delta}+\dfrac{\delta R_{i+1}}{\Delta}\cdot\dfrac{\delta\eta_i}{\Delta}+\left(\dfrac{\delta Q_{i+1}}{\Delta}-P_i+\lambda\right)\eta_i \\ \qquad +Q_{i+1}\dfrac{\delta\eta_{i+1}}{\Delta}-Q_i\dfrac{\delta\eta_i}{\Delta}=0, \qquad i=1,2,\ldots,n-1. \\ \eta_0=0,\ \eta_1=\gamma\cdot\Delta. \end{cases}$$

Furthermore, since the coefficients P, Q', R' in the differential system 10) are of class C' and $F_{pp} \neq 0$ in a region of the xy plane including the locus $y=y(x)$, $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$ and for all values of the third argument, a solution of system 11) or 12) approaches a corresponding solution of system 10) uniformly in the interval $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$, when n increases indefinitely; and similarly the solutions of system 13) approach uniformly to the solutions of system 8) as n increases indefinitely.

Then for $\lambda=\lambda_0-\delta$ and n sufficiently large the solutions of 13) are always positive, $i>0$ and $\eta_n(\lambda_0-\delta)>0$, while for $\lambda=\lambda_0+\delta$ and n sufficiently large, $\eta_n(\lambda_0+\delta)<0$. Now choose $n=N$ sufficiently large that $\eta_n(\lambda_0-\delta)>0$, $\eta_n(\lambda_0+\delta)<0$ and hold n fast. Then $\eta_n(\lambda)$ is a continuous function of λ, and for some λ' in the interval $\lambda_0-\delta<\lambda<\lambda_0+\delta$, and therefore negative, we have $\eta_n(\lambda')=0$. Thus, if there is a solution of $\psi(u)=0$, vanishing at $x=x_0$ and again at $x=x'$, where $x_0<x'<x_n$, and not vanishing identically, then there exists a λ' less than zero such that in the corresponding solution $(\eta_0\eta_1 \ldots \eta_{n-1}\eta_n)$, not identically zero, the variable η_n is zero. But this contradicts condition III′, and therefore a solution $u(x)$ of $\psi(u)=0$ such that $u(x_0)=0$ and $u(x)\not\equiv 0$ can not vanish in the interval $x_0<x<x_n$.

The results of this work can be summarized in the following

THEOREM I. *If the following conditions hold,*

1) $(Y_1 Y_2 \ldots Y_{n-1})$ *such that* $(x_1 Y_1), \ldots, (x_{n-1} Y_{n-1})$ *lie entirely in the interior of region* R *give a relative minimum for* $\phi(y_1 y_2 \ldots y_{n-1})$ *in* R;
2) *As* n *increases indefinitely* $(Y_1 Y_2 \ldots Y_{n-1})$ *tend uniformly toward the corresponding ordinates of a curve* $y(x)$ *of class* C'' *lying entirely in the interior of* R *and going through the fixed points* (x_0, y_0), (x_n, y_n);
3) *When* j *is allowed to increase with* n *in such a way that* x_j *approaches a definite value* x', $\underset{n=\infty,\, j=\infty}{Lt} (Y_j-Y_{j-1})/\Delta$ *exists and is equal to*
$$dy(x)/dx\,|_{x=x'};$$
4) $F_{pp}[x, y(x), y'(x)] \neq 0$, $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$ $(y'(x)=dy(x)/dx)$:

then it follows that

I. $y(x)$ *satisfies Euler's differential equation,*
$$F_y[x, y(x), y'(x)] - \frac{d}{dx} F_p[x, y(x), y'(x)] = 0.$$

II. $R>0$ *along** $y(x)$; *i. e.*, $F_{pp}[x, y(x), y'(x)]>0$, $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$.

* Notice that under hypotheses 1, 2, 3 we have proved that $F_{pp}[x, y(x), y'(x)] \geqq 0$.

III. *If $u(x)$ is a solution of Jacobi's equation,*

$$\frac{d}{dx}(Ru')-(P-Q')u=0,$$

vanishing for $x=x_0$ but not vanishing identically, then $u(x)$ does not vanish in the interval $x_0<x<x_n$.

Thus we see that $y(x)$ satisfies the first three necessary conditions* usually given for a function giving a relative minimum of $\int_{x_0}^{x_n} F(x, y, y')\,dx$.

It is well to notice that we can make this theorem somewhat stronger by replacing the first hypothesis by the part of it which we have really used; namely, instead of assuming that $(Y_1 \ldots . Y_{n-1})$ furnishes a relative minimum for ϕ in R, we need merely assume the three necessary conditions for a relative minimum in the algebraic problem.

We note also that our conclusions are not sufficient conditions for $y(x)$ to give even a weak minimum of $\int_{x_0}^{x_n} F(x, y, y')\,dx$. Furthermore, if we assume sufficient conditions for a proper minimum in the algebraic problem, and nothing more, we should not be surprised at being unable to show that the conjugate point† x' is beyond x_n, for the form A may be positive for any particular n and still approach zero as n increases indefinitely. However, if we assume sufficient conditions for a proper minimum in the algebraic problem and assume furthermore that as n increases indefinitely the least value λ of A approaches a positive limit, not zero, then our method shows that the conjugate point x' is beyond x_n if existent and our conclusions are sufficient conditions that $y(x)$ give a weak minimum for $\int_{x_0}^{x_n} F(x, y, y')\,dx$.

The above discussion leads to the consideration of the following

THEOREM II. *If we assume,*

1) *$(Y_1 \ldots . Y_{n-1})$ give the absolute minimum for $\sum_{i=1}^{n} F[x_i, y_i, (y_i - y_{i-1})/\Delta]\Delta$, (x_i, y_i) being in region R;*

2) *As n increases indefinitely, $(Y_1 \ldots . Y_{n-1})$ approach uniformly the values of the corresponding ordinates of a curve $y=y(x)$ lying in the interior of R, of class C';*

3) *As j increases with n in such a way that $\underset{n=\infty}{Lt}\, x_j=x'$,*

$$\underset{n=\infty,\, j=\infty}{Lt}(Y_j-Y_{j-1})/\Delta$$

exists and is equal to $y'(x')$, and this limit is approached uniformly in the interval $x_0<x'\overline{\leqq}x_n$:

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* See Bolza, "Lectures," pp. 101, 102.

† For definition of *conjugate point*, see Bolza, "Lectures," p. 60.

then it follows that

I. $\underset{n=\infty}{Lt} \sum_{i=1}^{n} F[x_i, Y_i, (Y_i - Y_{i-1})/\Delta]\Delta = \int_{x_0}^{x_n} F[x, y(x), y'(x)]\,dx.$

II. *If there is a curve of class C' lying entirely within R and giving the absolate minimum for $\int_{x_0}^{x_n} F(x, y, y')\,dx$, then $y(x)$ is such a curve.*

PROOF: From hypotheses 2), 3) and the uniform continuity of $F(x, y, p)$ for (x, y) in R and p varying between any two fixed limits, it follows that as n increases indefinitely $F[x_i, Y_i, (Y_i - Y_{i-1})/\Delta]$ approaches uniformly to $F[x_i, y(x_i), y'(x_i)]$, $i = 1, 2, \ldots, n$. From this in turn follows conclusion I.

Let $\bar{y}_1 \ldots \bar{y}_{n-1}$ be the ordinates of any other curve $y = \bar{y}(x)$ corresponding to the abscissas $x_1, \ldots, x_{n-1}$, $\bar{y}(x)$ being of class C' and $y = \bar{y}(x)$ being interior to region R. We wish to show that

$$\int_{x_0}^{x_n} F(x, \bar{y}(x), \bar{y}'(x))\,dx \geqq \int_{x_0}^{x_n} F(x, y, y')\,dx.$$

For n sufficiently large $\sum_{i=1}^{n} F[x_i, \bar{y}_i, (\bar{y}_i - \bar{y}_{i-1})/\Delta]\Delta$ is as near as we please to $\int_{x_0}^{x_n} F(x, \bar{y}, \bar{y}')\,dx$. But from the first hypothesis

$$\sum_{i=1}^{n} F\left(x_i, \bar{y}_i, \frac{\bar{y}_i - \bar{y}_{i-1}}{\Delta}\right)\Delta \geqq \sum_{i=1}^{n} F\left(x_i, Y_i, \frac{Y_i - Y_{i-1}}{\Delta}\right)\Delta.$$

Then using conclusion I, we read from this inequality conclusion II.

III. FROM THE TRANSCENDENTAL PROBLEM TO THE ALGEBRAIC PROBLEM.

Although sufficient conditions for a solution of the algebraic problem do not give a solution of the limiting transcendental problem, it is interesting to note that if, conversely, $y(x)$ satisfies sufficient conditions for a weak minimum of $\int_{x_0}^{x_n} F(x, y, y')\,dx$, then for n sufficiently large we can find a solution of the algebraic problem. To prove this we consider first the

THEOREM: *If the following conditions hold,*

1) *$y = y(x)$ is a solution of Euler's equation, of class C'' in the interval $x_0 \leqq x \leqq x_n$, going through the points (x_0, y_0), (x_n, y_n) and lying entirely in the interior of the region* R;*
2) *The conjugate point to x_0 is beyond x_n or non-existent;*
3) *F_{pp} is positive along $y(x)$;*
4) *$y(x)$ is of class C' in an interval $x_0 - \delta \leqq x \leqq x_n + \delta$, where δ is some positive constant:*

then, given $\epsilon > 0$, N exists, independent of i, such that whenever $n > N$ a solution of the system

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* The notation here is the same as at the beginning of Part II.

$$\text{I.}\qquad \begin{cases} F_y(i)-\dfrac{F_p(i+1)-F_p(i)}{\Delta}=0, & i=1,2,\ldots,n-1,\\ y_0=y(x_0) \end{cases}$$

exists, satisfying the relations

$$\text{a)}\qquad \begin{cases} |y_i-y(x_i)|<\varepsilon, \\ \left|\dfrac{y_i-y_{i-1}}{\Delta}-y'(x_i)\right|<\varepsilon, & i=1,2,\ldots,n-1, \end{cases}$$

$$\text{b)}\qquad y_n=y(x_n).$$

We shall show first that for n sufficiently large the system

$$\text{I}'.\qquad \begin{cases} F_y(i)-\dfrac{F_p(i+1)-F_p(i)}{\Delta}=0, & i=1,2,\ldots,n-1,\\ y_0=y(x_0), \\ \dfrac{y_1-y_0}{\Delta}=y'(x_0) \end{cases}$$

has a unique solution satisfying conditions a).

Let x, y, p be taken as the rectangular coördinates of a point in space. Since F_{pp} is positive along $y(x)$, we can construct a "tube" S including as interior points all the points $[x, y(x), y'(x)]$ and such that $F_{pp}(x, y, p)$ is positive for all points of this tube.* Let

$$\text{1)}\qquad v=F_p(x, y+p\cdot\Delta, p)$$

define a transformation of the points of the four-space $(x, y, p; \Delta)$ such that (x, y, p) is a point in S and Δ is a small positive quantity, into points of $(x, y, v; \Delta)$ space, and let L denote the image of $y(x)$, *i. e.*, of $[x, y(x), y'(x); 0]$. We shall denote the locus $[x, y(x), y'(x); 0]$ by k. The function $F_p(x, y+p\cdot\Delta, p)$ is of class C' in a region of $(x, y, p; \Delta)$ space including as interior points the points of the locus k. Differentiating this function with respect to p, we have

$$F_{pp}(x, y+p\cdot\Delta, p)+\Delta\cdot F_{py}(x, y+p\cdot\Delta, p),$$

which is positive along the closed interior locus k. All the hypotheses of the extended theorem of inversion† are fulfilled, so that

a) A positive quantity ρ can be found so small that to each point $(x, y, v; \Delta)$ in (L, ρ), the image of the ρ-neighborhood of k, k_ρ, there corresponds one and but one point $(x, y, p; \Delta)$ of k_ρ.

b) The inverse function ϕ,

$$\text{2)}\qquad p=\phi(x, y, v; \Delta),$$

is of class C' in (L, ρ).

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* See Bolza, "Vorlesungen über Variationsrechnung," p. 157, Lemma II.

† See Bolza, "Vorlesungen," p. 160, b; p. 163, c.

c) Corresponding to ρ we can find a positive quantity σ such that to each point of the σ-neighborhood of L, L_σ there corresponds, through the transformation 2), one and only one point in k_ρ.

If there is a solution of I′ satisfying conditions a) as desired, choose Δ so small that $[x_i\ y,\ (y_i-y_{i-1})/\Delta;\ \Delta]$ lies in k_ρ, $i=1, 2, \ldots, n-1$. Let $F_p(x_i,\ y_{i-1}+\Delta p_i,\ p_i)=v_{i-1}$, where $p_i\equiv(y_i-y_{i-1})/\Delta$. Then from 2) we have $p_i=\phi(x_i,\ y_{i-1},\ v_{i-1};\ \Delta)$, and the solution of system I′ lying in k_ρ becomes a solution of the following system lying in (L, ρ):

$$\text{II.}\quad \begin{cases} v_i-v_{i-1}=(x_{i+1}-x_i)F_y\{x_i,\ y_{i-1}+\Delta\cdot\phi(x_i,\ y_{i-1},\ v_{i-1};\ \Delta), \\ \qquad\qquad \phi(x_i,\ y_{i-1},\ v_{i-1};\ \Delta)\}\equiv\Delta\cdot f_1(x_i,\ y_{i-1},\ v_{i-1};\ \Delta), \\ y_i-y_{i-1}=(x_{i+1}-x_i)\phi(x_i,\ y_{i-1},\ v_{i-1};\ \Delta), \\ y_0=y(x_0);\ \ (y_1-y_0)/\Delta=y'(x_0). \end{cases}$$

Conversely, if II has a solution in L_σ, then I′ has a solution in k_ρ. Thus the system II is equivalent to system I′.

Euler's differential equation,

$$F_y(x, y, y')-\frac{d}{dx}F_p(x, y, y')=0,$$

is equivalent to the system

$$\text{III.}\quad \begin{cases} \dfrac{dv}{dx}=F_y[x, y, \Phi(x, y, v)]\equiv g(x, y, v), \\ \dfrac{dy}{dx}=\Phi(x, y, v)\equiv f(x, y, v), \end{cases}$$

in which $\Phi(x, y, v)\equiv\phi(x, y, v; 0)$. Then our hypothesis is that $[v(x), y(x)]$ is a set of solutions of III, $y(x)$ being the original extremal and $v(x)= F_p[x, y(x), y'(x)]$. The functions $g(x, y, v)$, $f(x, y, v)$ satisfy a Lipschitz condition in the sub-region of (L, ρ) in which $\Delta=0$, since they are of class C' in this region. Then the solution $[v(x),\ y(x)]$ of III is approached by a solution of the difference equation system,

$$\text{IV.}\quad \begin{cases} v_i-v_{i-1}=(x_i-x_{i-1})F_y[x_{i-1},\ y_{i-1},\ \Phi(x_{i-1},\ y_{i-1},\ v_{i-1})], \\ y_i-y_{i-1}=(x_i-x_{i-1})\Phi(x_{i-1},\ y_{i-1},\ v_{i-1}), \qquad i=1, 2, \ldots, n-1. \end{cases}$$

Comparing this system with II, we note that since $\phi(x, y, v; \Delta)$ is of class C' in (L, ρ) and F_y is continuous in its three arguments in the region of interest, the functions on the right in II approach uniformly to the corresponding functions on the right in IV as n increases indefinitely, $i=1, 2, \ldots, n-1$. Then the generalization of the Cauchy–Lipschitz theorem, specified in Part II, informs us that for n sufficiently large system II has a solution as near as we

please to $[v(x), y(x)]$; *i. e.*, given $\varepsilon>0$, N can be found such that whenever $n>N$,

$$|y_i-y(x_i)|<\varepsilon, \quad |v_i-v(x_i)|<\varepsilon.$$

Furthermore, N can be chosen independent of i. Here we use the fact that the Cauchy–Lipschitz method gives, for n sufficiently large, solutions of a difference equation system uniformly close to the solution of the differential equation system in any interval which, together with its end points, lies in an interval in which the solutions of the differential system are continuous. We have assumed $y(x)$ to be of class C' for $x_0-\delta \overline{\gtrless} x \overline{\gtrless} x_n+\delta$, $\delta>0$. That $v(x)$ is continuous in such an interval follows from this restriction on $y(x)$ and the continuity of F_p for (x, y) in R and for all values of p.

We have, also,

$$\frac{y_i-y_{i-1}}{\Delta}-y'(x_i)=\phi(x_i, y_{i-1}, v_{i-1}; \Delta)-\phi(x_i, y(x_i), v(x_i); 0).$$

Since ϕ is continuous in (L, ρ), N exists independent of i such that

$$\left|\frac{y_i-y_{i-1}}{\Delta}-y'(x_i)\right|<\varepsilon, \qquad i=1, 2, \ldots . n,$$

whenever $n>N$. It is evident that the solutions of systems II and I′ are unique.

Having established the existence of a unique solution of I′ satisfying conditions a), we are in a position to use the idea of a set of extremals through a point to prove the theorem stated above. Since we assume that the conjugate point of x_0 is beyond x_n or non-existent, $y(x)$ is one of a set of extremals* $y(x, \gamma)$ through (x_0, y_0), γ being the slope of the curve $y=y(x, \gamma)$ for $x=x_0$, which do not intersect in the interval $x_0<x<x_n$. Let us write $\gamma_0=y'(x_0)$. Then there exists a positive number d_1 such that when $|\gamma-\gamma_0| \overline{\gtrless} d_1$ the corresponding extremals do not intersect in the interval $x_0<x \overline{\gtrless} x_n$. Also a positive number d_2 exists such that when $|\gamma-\gamma_0| \overline{\gtrless} d_2$, F_{pp} is positive along $y(x, \gamma)$. Then the hypotheses for the application of the Cauchy-Lipschitz theorem are satisfied in the neighborhood of any one of the curves $y(x, \gamma')$, where $|\gamma_0-\gamma'|<d$, d being the smaller of d_1 and d_2. Furthermore, an examination of the Cauchy–Lipschitz proof shows that the theorem applies when the initial value of v, v_0 is made a function of the parameter γ, $|\gamma_0-\gamma|<d$, and therefore we have that for a given $\varepsilon>0$, N_1 can be found independent of i and γ such that, whenever $n>N_1$,

$$|y_i(\gamma)-y(x_i, \gamma)|<\varepsilon,$$
$$\left|\frac{y_i(\gamma)-y_{i-1}(\gamma)}{\Delta}-y'(x_i, \gamma)\right|<\varepsilon, \qquad i=1, 2, \ldots, n; \; |\gamma-\gamma_0|<d,$$

in which the set $y_1(\gamma), \ldots, y_n(\gamma)$ is the solution of I′ for $y'(x_0)=\gamma$.

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* See Bolza, "Lectures," p. 61.

Our conclusion is that for n sufficiently large we are able to get the solution of the difference system I′ uniformly close to the corresponding extremal throughout the interval $x_0 \overline{\gtrless} x \overline{\gtrless} x_n$ and for $|\gamma-\gamma_0|<d$, γ being the initial slope of the extremal in question. We are now in a position to prove the original theorem; *i. e.*, to find a solution of the system I satisfying conditions a) and b).

As a function of γ alone, $y(x,\gamma)$ and $y'(x,\gamma)$ are continuous for $|\gamma-\gamma_0|<d$. Then a positive number d_3 exists such that

$$|y\,(x,\gamma)-y\,(x,\gamma_0)\,|<\epsilon/2,$$
$$|y'(x,\gamma)-y'(x,\gamma_0)\,|<\epsilon/2,$$

whenever $|\gamma-\gamma_0|\overline{\gtrless} d_3$. Let γ_1, γ_2 be such that $\gamma_1<\gamma_0<\gamma_2$, $\gamma_2-\gamma_0<d_3$, $\gamma_0-\gamma_1<d_3$. As a function of γ alone, $y(x,\gamma)$ is an increasing function, since the set of extremals through (x_0, y_0) do not intersect in the interval in question, and there exists a positive number $h<\epsilon/2$ such that

$$y(x_n, \gamma_2)-y(x_n, \gamma_0)>h,$$
$$y(x_n, \gamma_0)-y(x_n, \gamma_1)>h.$$

Now choose N_2 so large that for $n>N_2$

$$|\,y_i(\gamma)-y(x_i, \gamma)\,|<h,$$
$$\left|\frac{y_i(\gamma)-y_{i-1}(\gamma)}{\Delta}-y'(x_i,\gamma)\right|<h, \qquad \gamma_1 \overline{\gtrless} \gamma \overline{\gtrless} \gamma_2.$$

Choose $n=n'>N_2$ and hold it fast. Then the inequalities $y_{n'}(\gamma_1)<y_{n'}(\gamma_0)<y_{n'}(\gamma_2)$ hold. Also $y_{n'}(\gamma)$ is a continuous function of γ, $|\gamma-\gamma_0|<d_2$. Then for some value of γ, $\gamma=\gamma'$ such that $\gamma_1<\gamma'<\gamma_2$, $y_{n'}(\gamma')=y(x_n, \gamma_0)$. Furthermore, $|y_i(\gamma')-y(x_i, \gamma')\,|<\epsilon/2$ and $|y(x_i, \gamma')-y(x_i, \gamma_0)\,|<\epsilon/2$.

$$\therefore\ |y_i(\gamma')-y(x_i, \gamma_0)\,|<\epsilon,$$

and similarly,

$$\left|\frac{y_i(\gamma')-y_{i-1}(\gamma')}{\Delta}-y'(x_i, \gamma_0)\right|<\epsilon.$$

Thus, in $y_1(\gamma'), \ldots, y_{n'}(\gamma')$ we have a solution of system I satisfying conditions a) and b).

To carry out the plan of Part III we wish now to show that for n sufficiently large a minimum of the algebraic problem exists. We shall show that $(y_1, \ldots, y_{n-1})$ exist satisfying the sufficient conditions I′, II′ III″ of Part II for the desired minimum. We already have the existence for n sufficiently large of a solution $(y_1, \ldots, y_{n-1})$ of

$$\text{I.} \qquad F_y(i)-\frac{F_p(i+1)-F_p(i)}{\Delta}=0, \qquad i=1, 2, \ldots, n-1,$$

going through (x_0, y_0), (x_n, y_n) and as near to the extremal $y(x)$ as we please. To show that, for n sufficiently large, condition

$$\text{II.} \qquad \Delta^2 P_i + 2\Delta Q_i + R_{i+1} + R_i > 0, \qquad i = 1, 2, \ldots, n-1,$$

is satisfied, we note that we can make R_i, R_{i+1} as near to $F_{pp}[x_i, y(x_i), y'(x_i)]$ as we please and therefore positive, and that P_i, Q_i remain finite. It is sufficient now to show that the difference equation system B, Part II, has no solution $(\eta_0, \eta_1, \ldots, \eta_n)$ not vanishing identically, such that $\eta_0 = \eta_n = 0$ for $\lambda \overline{\leqq} 0$. (It should be noticed that the equality sign in $\lambda \overline{\leqq} 0$ is especially important.)

We have in Part II that for n sufficiently large a solution of the difference system B such that $\eta_0 = 0$, $(\eta_1, \ldots, \eta_{n-1}) \not\equiv 0$ is uniformly close to a corresponding solution of the system

$$3) \qquad \begin{cases} \dfrac{d}{dx}(R\eta') + (Q' - P + \lambda)\eta = 0, \\ \eta(x_0) = 0; \qquad \eta(x) \not\equiv 0. \end{cases}$$

But by hypothesis the conjugate point to x_0 is beyond x_n or non-existent. This means that if solutions of 3) for $\lambda = 0$ vanish for any x greater than x_0, the next root, x', is greater than x_n; then a theorem* of Sturm's tells us that for $\lambda < 0$ the next root would be still further to the right. It is therefore evident that for n sufficiently large no solution of the difference system exists for $\lambda \overline{\leqq} 0$ such that $\eta_0 = \eta_n = 0$ and $(\eta_1, \ldots, \eta_{n-1}) \not\equiv 0$. Thus for n sufficiently large we have the existence of $(y_1, \ldots, y_{n-1})$ satisfying the sufficient conditions stated in Part II for a relative minimum of the algebraic sum. Furthermore, it is well to note that for n sufficiently large the ordinates $(y_1, \ldots, y_{n-1})$ are as close as we please to the extremal giving the assumed minimum of the transcendental problem.

IV. The Transcendental Problem Following Formal Analogies to the Algebraic Problem.

There is evidently a very close correspondence between the algebraic and the transcendental problems which we have considered. The purpose of this part of the paper is to make this correspondence still more striking by bringing out the formal analogy involved.

Let $F(x, y, p)$ be a function of class C''' in a certain simply connected region R of the xy plane including as interior points the points $(0,0)$ and $(1,0)$, and for all values of p. Let $y(x)$ be a function of class C'' in the interval $0 \overline{\leqq} x \overline{\leqq} 1$, such that the curve $y = y(x)$ lies entirely in the interior of R and goes through the points $(0,0)$ and $(1,0)$. Let

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* See Theorem b), Part II.

$$L(y) = p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)\cdot y$$

be a self-adjoint differential expression such that the system

$$\begin{cases} L(y)=0, \\ y(0)=y(1)=0 \end{cases}$$

is incompatible, and let $G(x,\xi)$ be the corresponding Green's function.* Suppose we note that $L[y(x)]=r(x)$. Let $\eta(x)$ be continuous in the interval $0 \overline{\gtrless} x \overline{\gtrless} 1$, and let $\bar{y}(x)$ be the function determined by

$$L(\bar{y}) = r(x) + \varepsilon\eta(x),$$
$$\bar{y}(0) = \bar{y}(1) = 0,$$

where ε is a constant chosen small enough that $\bar{y}(x)$ lies in the interior of R. Making use of the Green's function, we may write

$$y(x) = \int_0^1 r(\xi)G(x,\xi)\,d\xi; \quad y'(x) = \int_0^1 r(\xi)G_1(x,\xi)\,d\xi,$$

where $G_1(x,\xi) = \dfrac{\partial}{\partial x}G(x,\xi)$;

$$\bar{y}(x) = \int_0^1 [r(\xi)+\varepsilon\eta(\xi)]G(x,\xi)\,d\xi; \quad \bar{y}'(x) = \int_0^1 [r(\xi)+\varepsilon\eta(\xi)]G_1(x,\xi)\,d\xi;$$

a) $\quad \bar{y}(x) - y(x) = \varepsilon\int_0^1 \eta(\xi)G(x,\xi)\,d\xi; \quad \bar{y}'(x) - y'(x) = \varepsilon\int_0^1 \eta(\xi)G_1(x,\xi)\,d\xi.$

Using Taylor's series with a remainder,

$$\begin{aligned} F(x,\bar{y},\bar{y}') = F(x,y,y') &+ (\bar{y}-y)F_y + (\bar{y}'-y')F_p \\ &+ \frac{1}{2!}\{(\bar{y}-y)^2F_{yy} + 2(\bar{y}-y)(\bar{y}'-y')F_{yp} + (\bar{y}'-y')^2F_{pp}\} \\ &+ \frac{1}{3!}\{(\bar{y}-y)^3F_{yyy} + \ldots + (\bar{y}'-y')^3F_{ppp}\}, \end{aligned}$$

the arguments of all partial derivatives except in the last group being (x,y,y'), those of the last group being of the form $[x,\ y+\theta_1(\bar{y}-y),\ y'+\theta_2(\bar{y}'-y')]$, θ_1, θ_2 being proper fractions. Substituting the values of $\bar{y}-y$, $\bar{y}'-y'$ from a), we have

$$\begin{aligned} F(x,\bar{y},\bar{y}') &= F(x,y,y') + \varepsilon\{F_y\textstyle\int_0^1\eta(\xi)G(x,\xi)\,d\xi + F_p\int_0^1\eta(\xi)G_1(x,\xi)\,d\xi\} \\ &\quad + \frac{\varepsilon^2}{2!}\{F_{yy}(\textstyle\int_0^1\eta(\xi)G(x,\xi)\,d\xi)^2 + 2F_{yp}\int_0^1\eta(\xi)G(x,\xi)\,d\xi\int_0^1\eta(\xi)G_1(x,\xi)\,d\xi \\ &\qquad + F_{pp}(\textstyle\int_0^1\eta(\xi)G_1(x,\xi)\,d\xi)^2\} + \varepsilon^2(\varepsilon) \\ &= F(x,y,y') + \varepsilon\textstyle\int_0^1\eta(\xi)\,[G(x,\xi)F_y + G_1(x,\xi)F_p]\,d\xi \\ &\quad + \frac{\varepsilon^2}{2!}\textstyle\int_0^1\int_0^1\eta(\xi_1)\eta(\xi_2)\,[G(x,\xi_1)G(x,\xi_2)F_{yy} + 2G(x,\xi_1)G_1(x,\xi_2)F_{yp} \\ &\qquad + G_1(x,\xi_1)G_1(x,\xi_2)F_{pp}]\,d\xi_1\,d\xi_2 + \varepsilon^2(\varepsilon). \end{aligned}$$

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* For the definition of the *Green's function* see an article by Bôcher, *Annals of Mathematics*, Vol. XIII, p. 71.

Suppose $y(x)$ makes $\int_0^1 F(x,y,y')\,dx$ as small as or smaller than does any other curve of class C'' in a certain neighborhood of $y=y(x)$ and going through the points $(0,0)$ and $(1,0)$. Then for ε sufficiently small

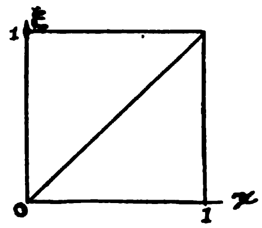
$$\int_0^1 [F(x,\bar{y},\bar{y}')-F(x,y,y')]\,dx \geqq 0.$$

Employing reasoning which was useful in Parts I, II, we obtain conditions exactly analogous to conditions there derived,

$$\text{I}''.\quad \int_0^1\Big\{\int_0^1 \eta(\xi)\,[G(x,\xi)F_y+G_1(x,\xi)F_p]\,d\xi\Big\}\,dx=0,$$

$$\text{II}''.\quad \int_0^1\Big\{\int_0^1\int_0^1 \eta(\xi_1)\eta(\xi_2)\,[G(x,\xi_1)G(x,\xi_2)F_{yy}+2G(x,\xi_1)G_1(x,\xi_2)F_{yp} \\ +G_1(x,\xi_1)G_1(x,\xi_2)F_{pp}]\,d\xi_1\,d\xi_2\Big\}\,dx \geqq 0.$$

Up to this point we have made no essential use of the fact that the third argument in F is the derivatives of $y(x)$, or in any way related to $y(x)$. We should have come to conditions analogous to I″ and II″ if the third argument in F had been any other function of x and had been varied similarly to the way in which $y(x)$ was varied. So far the important fact has been the possibility of expansion by means of Taylor's series with a remainder, *i. e.*, an expansion in ascending powers of a quantity ε which can be made as small as we please. From the very definition of maximum and minimum it is evident that such an expansion is fundamental in a general analytic investigation of such problems, whether the variable be a point or a curve.



Since $G(x,\xi)$ is continuous in the square $0\leqq x\leqq 1$, $0\leqq \xi\leqq 1$, and $G_1(x,\xi)$ is continuous in this square except along the diagonal $x=\xi$, where it has a unit discontinuity, the order of integration may be reversed in I″ and II″, giving

$$\text{I}'.\quad \int_0^1 \eta(\xi)\Big[\int_0^1\{G(x,\xi)F_y+G_1(x,\xi)F_p\}\,dx\Big]\,d\xi=0,$$

$$\text{II}'.\quad \int_0^1\int_0^1 \eta(\xi_1)\eta(\xi_2)\Big[\int_0^1\{G(x,\xi_1)G(x,\xi_2)P \\ +2G(x,\xi_1)G_1(x,\xi_2)Q+G_1(x,\xi_1)G_1(x,\xi_2)R\}\,dx\Big]\,d\xi_1\,d\xi_2 \\ =\int_0^1\int_0^1 \eta(\xi_1)\eta(\xi_2)\,\phi(\xi_1,\xi_2)\,d\xi_1\,d\xi_2 \geqq 0,$$

where $\phi(\xi_1,\xi_2)$ represents the integral in brackets [ ].

Since I′ is an expression for the first variation of $\int_0^1 F(x,y,y')\,dx$, we should expect to be able to show from it that $y(x)$ is a solution of Euler's equation. To do this we proceed as follows:

$$\int_0^1 G_1(x,\xi)F_p\,dx=\int_0^\xi G_1(x,\xi)F_p\,dx+\int_\xi^1 G_1(x,\xi)F_p\,dx.$$

Since in each sub-interval $G_1(x,\xi)$ is continuous, we may integrate by parts thus:

$$\int_0^\xi G_1(x,\xi)F_p\,dx=G(x,\xi)F_p\,\Big|_0^\xi-\int_0^\xi G(x,\xi)\frac{d}{dx}(F_p)\,dx,$$

$$\int_\xi^1 G_1(x,\xi)F_p\,dx=G(x,\xi)F_p\,\Big|_\xi^1-\int_\xi^1 G(x,\xi)\frac{d}{dx}(F_p)\,dx.$$

Since $G(x,\xi)$ is continuous and $G(0,\xi)=G(1,\xi)=0$,

$$\int_0^1 G_1(x,\xi)F_p\,dx = -\int_0^1 G(x,\xi)\frac{d}{dx}(F_p)\,dx.$$

Then I′ may be written,

$$1)\qquad \int_0^1 \eta(\xi)\left[\int_0^1 G(x,\xi)\left\{F_y-\frac{d}{dx}F_p\right\}dx\right]d\xi=0.$$

Since 1) holds for $\eta(\xi)$ any continuous function, it holds if $\eta(\xi)$ is of class C' in the interval $0\overline{\gtrless}\xi\overline{\gtrless}1$ and $\eta(0)=\eta(1)=0$. The coefficient of $\eta(\xi)$ in the integrand involved in 1), $\int_0^1 G(x,\xi)\left\{F_y-\frac{d}{dx}F_p\right\}dx$, is a continuous function of ξ. Then, by the fundamental lemma* of the Calculus of Variations,

$$\int_0^1 G(x,\xi)\left\{F_y-\frac{d}{dx}F_p\right\}dx\equiv 0.$$

From this it follows that

$$\text{I.}\qquad F_y[x,\,y(x),\,y'(x)]-\frac{d}{dx}F_p[x,\,y(x),\,y'(x)]\equiv 0.$$

For, suppose $F_y-\dfrac{d}{dx}F_p=\phi(x)\not\equiv 0$. Since $G(x,\xi)$ is symmetric, we have $\int_0^1 G(x,\xi)\phi(x)\,dx\equiv 0$; but this would be the solution of the system

$$\begin{cases} L(u)=\phi(\xi), \\ u(0)=u(1)=0, \end{cases}$$

which can not vanish identically since $\phi(x)\not\equiv 0$. The contradiction proves the truth of I.

What we have really proved is that of all curves of class C'' through the points $(0,0)$, $(1,0)$, those giving a minimum must satisfy Euler's equation. We can not expect more of this method since we consider no more general curves. However, we shall assume in the discussion of the second variation that $F_{pp}\neq 0$ along $y(x)$; and it has been shown† that in this case an extremal of class C' is of class C''.

It is interesting to notice that the first variation, I′, is a simple integral with respect to ξ; that the second variation, $\int_0^1\int_0^1 \eta(\xi_1)\,\eta(\xi_2)\,\phi(\xi_1,\,\xi_2)\,d\xi_1\,d\xi_2$, can be considered as a double integral; and it is evident that the k-th variation would be a k-fold integral. Here we have a complete formal analogy to the simple and multiple sums in the corresponding variations in the algebraic problem.

In examining the second variation, II′, we shall prove first that $\phi(\xi,\xi)\overline{>}0$, $0\overline{\gtrless}\xi\overline{\gtrless}1$. This condition is analogous to the condition that $\phi_{ii}\overline{>}0$ in the algebraic problem,‡ and the method of proof is very similar.

* See Bolza, "Lectures," p. 20. † Bolza, "Lectures," pp. 25, 26. ‡ See Part I, condition 2).

Suppose that for $\xi = x'$, $\phi(\xi, \xi) = -c$, where c is positive, x' being a point in the interval $0 < x < 1$. The function $\phi(\xi_1, \xi_2)$ is continuous in the square $0 \leqq \xi_1 \leqq 1$, $0 \leqq \xi_2 \leqq 1$, as can be seen by breaking up the integral representing $\phi(\xi_1, \xi_2)$ into integrals over intervals in which the integrand is continuous. Then we can find a positive quantity h such that $\phi(\xi_1, \xi_2) < 0$ in the square, $x' - h \leqq \xi_1 \leqq x' + h$, $x' - h \leqq \xi_2 \leqq x' + h$. Now choose $\eta(x)$ positive in the interval $x' - h < x < x' + h$, zero outside of this interval, and continuous in the interval $0 \leqq x \leqq 1$. Then we have the inequality

$$\int_0^1 \int_0^1 \eta(\xi_1)\, \eta(\xi_2)\, \phi(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 < 0,$$

which is contrary to condition II′.

$$\therefore\ \phi(\xi, \xi) \geqq 0, \qquad 0 \leqq \xi \leqq 1.$$

From this condition we shall show that if $R(x) \neq 0$ in the interval $0 \leqq x \leqq 1$, then $R(x)$ is positive in this interval. For this purpose we need the following properties of $G(x, \xi)$, $G_1(x, \xi)$:

1) $G(x, \xi)$ as a function of x satisfies the boundary conditions $u(0) = u(1) = 0$; *i. e.*, $G(0, \xi) = G(1, \xi) = 0$.

2) $G(x, \xi)$ is symmetric. $\therefore G(x, 0) = G(x, 1) = 0$.

3) $G_1(x, \xi)_{x=\xi_+} - G_1(x, \xi)_{x=\xi_-} = 1$. ($x = \xi_+$ is read, "as x approaches ξ from the right.")

These facts enable us to show by direct computation that $\phi(0, 0) = 0$ and $\phi'(0, 0) = R(0)$, where $\phi'(\xi, \xi) \equiv d\phi(\xi, \xi)/d\xi$. We see that $\phi'(0, 0)$ is not negative, since if it were, $\phi(\xi, \xi)$ would be negative for very small values of ξ. $\therefore R(0) > 0$, since we assume $R(\xi) \neq 0$ in the interval $0 \leqq \xi \leqq 1$; then the continuity of $R(\xi)$ gives us the desired result,

$$\text{II.} \qquad R(x) > 0,$$

in the interval $0 \leqq x \leqq 1$.

In the further consideration of the second variation we shall show that as the kernel* of an integral, $\phi(\xi_1, \xi_2)$ has only positive characteristic numbers. Suppose $\lambda = \lambda_0$ is a characteristic number and that

$$\eta(x) = \lambda_0 \int_0^1 \phi(x, \xi)\, \eta(\xi)\, d\xi.$$

Multiplying both sides by $\eta(x)$ and integrating,

$$\int_0^1 \eta^2(x)\, dx = \lambda_0 \int_0^1 \int_0^1 \phi(x, \xi)\, \eta(\xi)\, \eta(x)\, d\xi\, dx.$$

Since the left-hand side is positive and the double integral is not negative, and therefore actually positive in this case, λ_0 must be positive.

* The number λ is a characteristic number of a kernel $F(x, y)$ if a continuous function $u(x)$ exists such that $u(x) = \lambda \int_0^1 F(x, \xi)\, u(\xi)\, d\xi$. See also any book on integral equations.

To get Jacobi's condition we proceed as follows: Let $\bar{y}(x)-y(x)=\varepsilon u(x)$, where $u(x)$ is of class C'' and $u(0)=u(1)=0$. It is well known* that for such a variation on $y(x)$ the second variation of the integral can be written

$$2)\qquad J(u)\equiv\int_0^1 u\psi(u)\,dx,$$

where† $\psi(u)\equiv(P-Q')u-d(Ru')/dx$, *i. e.*,

$$J(u)=\int_0^1[(P-Q')u^2-u\frac{d}{dx}(Ru')]\,dx\gtreqless 0.$$

It is now convenient to consider only such variations $u(x)$ that $\int_0^1 u^2(x)\,dx=1$. This restriction is exactly analogous to those made in Parts I, II in dealing with the second variation in the algebraic problems, the restrictions $\sum_{i=1}^{n} v_i^2=1$, $\sum_{i=1}^{n}\eta_i^2\Delta=1$, and it should be noted that in no case is this an essential restriction. Then, if ε is a positive constant, $J(u)+\varepsilon$ is positive:

$$J(u)+\varepsilon=\int_0^1[(P-Q'+\varepsilon)u^2-u\frac{d}{dx}(Ru')]\,dx>0.$$

Introducing the notation

$$L(u)\equiv\frac{d}{dx}(Ru')+(Q'-P-\varepsilon)u,$$

we have

$$-\int_0^1 uL(u)\,dx>0.$$

Hence, we conclude that the system

$$3)\qquad \begin{cases} L(u)=0,\\ u(0)=u(1)=0\end{cases}$$

is incompatible, for if it were compatible, $u(x)$ being a solution, a positive constant c could be found such that $\int_0^1\bar{u}^2\,dx=1$ and $\int_0^1\bar{u}L(\bar{u})\,dx=0$, where $\bar{u}=cu$. Let $G(x,\xi)$ be the Green's function of system 3), and let the system

$$\begin{cases} L(u)=\eta(x),\\ u(0)=u(1)=0,\end{cases}$$

$\eta(x)$ being a continuous function, $0\overline{\gtrless}x\overline{\gtrless}1$, define a variation of the curve $y(x)$; (*i. e.*, a function u such that $\bar{y}-y=\varepsilon u$). Then

$$u(x)=\int_0^1\eta(\xi)G(x,\xi)\,d\xi$$

and

$$\int_0^1 uL(u)\,dx=\int_0^1\int_0^1\eta(\xi)\,\eta(x)\,G(x,\xi)\,d\xi\,dx<0.$$

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* See Bolza, "Lectures," p. 52.

† $\psi(u)$ as here used is a standard notation; it should be noted that it is the negative of the $\psi(u)$ used in Part II.

The characteristic numbers of the kernel $G(x, \xi)$ are all negative. For suppose that one, λ_0, were positive, and that

$$\eta(x) = \lambda_0 \int_0^1 G(x, \xi)\, \eta(\xi)\, d\xi.$$

Multiplying both sides by $\eta(x)$ and integrating, we have

$$\int_0^1 \eta^2(x)\, dx = \lambda_0 \int_0^1 \int_0^1 \eta(\xi)\, \eta(x)\, G(x, \xi)\, d\xi\, dx.$$

The left-hand side is positive, while the right-hand side is negative, and the contradiction proves the statement. But the characteristic numbers of the kernel $G(x, \xi)$ are the same as for the system*

$$\begin{cases} L(u) = \lambda u, \\ u(0) = u(1) = 0. \end{cases}$$

Then the system

$$4)\qquad \begin{cases} L(u) + \lambda u = 0, \\ u(0) = u(1) = 0 \end{cases}$$

has a solution $u(x) \not\equiv 0$ only for positive values of λ, no matter how small a positive constant ε be chosen. Then the system

$$5)\qquad \begin{cases} \dfrac{d}{dx}(Ru') + (Q' - P)u + \lambda u = 0, \\ u(0) = u(1) = 0 \end{cases}$$

has no negative characteristic numbers, for if $\lambda_0 < 0$ were a characteristic number and we choose $\varepsilon < -\lambda_0$, then 4) would have a negative characteristic number, namely, $\lambda_0 + \varepsilon$.

There is one and but one value† of λ for which system 5) has a solution not vanishing in the interval $0 < x < 1$, and we have shown that this value is positive. Then there is no non-identically vanishing solution of the system

$$\begin{cases} -\psi(u) \equiv \dfrac{d}{dx}(Ru') + (Q' + P)u = 0, \\ u(0) = 0; \quad u(x') = 0, \qquad 0 < x' < 1, \end{cases}$$

for if there were, system 5) would have a solution vanishing nowhere in the interval $0 < x < 1$ for λ negative. ‡

* See, for instance, Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Göttingische Nachrichten*, 1904, p. 225.

† See Theorem a), Part II.

‡ See Theorem b), Part II.

*On the Conformal Geometry of Analytic Arcs.**

By G. A. Pfeiffer.

Introduction.

Conformal geometry is the geometry associated with the group of conformal transformations. When only the plane of real points is considered, a non-singular conformal transformation is given by any analytic function of a single complex variable,

$$Z=f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\dots,$$

where $a_1\neq 0$. When the plane of complex points is the fundamental domain, the general non-singular conformal transformation is of the form

$$U=f(u)=a_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots,$$
$$V=g(v)=b_0+b_1(v-v_0)+b_2(v-v_0)^2+\dots,$$

where $u=x+iy$ and $v=x-iy$, x and y, which may be real or complex, being the Cartesian coordinates of a point; also, $f(u)$ and $g(v)$ have radii of convergence different from zero and $a_1\neq 0$, $b_1\neq 0$. Both the real and complex, or general, conformal groups are infinite continuous groups in the Lie terminology. Real and complex conformal geometries consist of the codifications of the invariants and covariants of configurations under these respective groups.

In the consideration of harmonic continuation in function theory the notion of general or Schwarzian symmetry plays an important rôle. This notion was introduced by Schwarz in his work, "Ueber die Integration der partiellen Differentialgleichung $\Delta u=0$ unter vorgeschriebenen Grenz- und Unstetigkeitsbedingungen," which was written in 1870.† This notion is, naturally, also important in conformal geometry for the same reason that it is in the above discussion, viz.: Schwarzian symmetry is a covariant property of an analytic curve under the conformal group. The usefulness of this property is well illustrated in the method employed by Kasner‡ in finding conformal invariants

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* Read before the American Mathematical Society, October 31, 1914.

† H. A. Schwarz, "Werke," Vol. II, p. 151.

‡ E. Kasner, "Conformal Geometry," *Proceedings of the International Congress of Mathematicians*, p. 82, Cambridge, 1912. See p. 422 below.

of two intersecting analytic arcs which form an angle whose magnitude is commensurable with π.

A good part of the following is devoted to an investigation of the properties of the transformation of general symmetry, or, briefly, symmetry, *i. e.*, the pairing of points symmetric with respect to an analytic arc, and the associated functional equations. In particular, the existence of an analytic arc, called a *symmetric bisector*, such that one of two intersecting analytic arcs is the symmetric image of the other with respect to the bisector is considered. This is called the *bisection problem.** Taking the two intersecting analytic arcs to be given by

$y=y_0+a_1(x-x_0)+a_2(x-x_0)^2+\dots$ and $y=y_0+b_1(x-x_0)+b_2(x-x_0)^2+\dots$,

it is found that if $\left|\frac{1-a_1 i}{1+a_1 i}\cdot\frac{1+b_1 i}{1-b_1 i}\right| \neq 0$ or 1, there exist just two symmetric bisectors. If $\left|\frac{1-a_1 i}{1+a_1 i}\cdot\frac{1+b_1 i}{1-b_1 i}\right| =1$, it is found necessary to consider two cases: firstly, where $\left(\frac{1-a_1 i}{1+a_1 i}\cdot\frac{1+b_1 i}{1-b_1 i}\right)^n \neq 1$, $n=$ any positive integer; and secondly, where $\left(\frac{1-a_1 i}{1+a_1 i}\cdot\frac{1+b_1 i}{1-b_1 i}\right)^n =1$, n being some positive integer. In the former case just two formal solutions for a bisector exist, while in the latter it is shown that, in case the given configuration is not conformally equivalent to a pair of intersecting rectilinear segments, no more than two formal solutions exist and the conditions which determine the number of formal solutions are given; if, in this latter case, the given configuration is conformally equivalent to a pair of intersecting rectilinear segments, it is shown that the configuration has an infinity of symmetric bisectors. The case when any of the numbers $1\pm a_1 i$, $1\pm b_1 i$ is zero is not considered; in this case either a_1 or b_1 is $\pm i$; *i. e.*, the slope of at least one of the arcs at the common point is minimal, and the symmetric transformation concerned is singular. The uniqueness of a symmetric n-sector of two intersecting analytic arcs with a definite coefficient of the first degree term, *i. e.*, with a definite slope at the point of intersection of the given arcs, when n is a power of 2, is also demonstrated.

Out of the above investigation certain other important theorems in conformal geometry are obtained, two relating to the conformal equivalence of a pair of intersecting analytic arcs to a pair of intersecting rectilinear segments, and another concerning the existence of a unique absolute conformal invariant

* So called by Kasner, *loc. cit.*

of a pair of intersecting analytic arcs which are such that some positive integral power of $\frac{1-a_1 i}{1+a_1 i} \cdot \frac{1+b_1 i}{1-b_1 i}$ is unity, or, in case the real plane is fundamental, which are such that the magnitude of the angle formed by them is commensurable with π. Two cases of this theorem, first, in which the two arcs are tangent, and second, in which the configuration, in normal form, consists of the analytic arc C: $x=b_2y^2+b_3y^3+\ldots$, $b_3 \neq 0$, and the segment of the axis of x in the vicinity of the origin, have been proved by Kasner.*

Properties of transformations which are the products of two and three symmetric transformations are investigated; and theorems concerning necessary and sufficient conditions that the product of two symmetric transformations be a non-singular direct conformal transformation, and that the product of three symmetric transformations be another symmetric transformation, are given.

§ 1. *Symmetry in the Plane of Real Points.*

Definition. Let C be a real curve, analytic in the neighborhood of the point (x_0, y_0), and let T be a real non-singular conformal transformation which transforms the arc of the curve C in the neighborhood of the point (x_0, y_0) into the rectilinear segment Γ. If P and P' are two points which are symmetric with respect to the straight line segment Γ, then the points Q and Q' which correspond to the points P and P' under the transformation T are said to be *symmetric with respect to the analytic arc*.†

The above definition is the function-theoretic definition of Schwarz, and the real plane of analysis is understood to be the fundamental domain. In the discussion immediately following, the real plane is that under consideration. The above definition is of value because it is shown ‡ that if Q and Q' are symmetric with respect to an analytic arc C for the non-singular conformal transformation T, they are also symmetric with respect to the analytic arc C for any non-singular conformal transformation which transforms the analytic arc C into a rectilinear segment. Thus, given an analytic arc, we have in a certain vicinity of a point on the arc an involutoric correspondence in which each point in that vicinity corresponds to the point which is symmetric to it with respect to the arc. This correspondence is called the symmetric transformation or, more

* Kasner, *loc. cit.*, pp. 84, 86.

† A real curve is analytic in the neighborhood of the point (x_0, y_0) if the equation of the locus of its points in that neighborhood is of the form $y=y_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots$ or $x=x_0+a_1(y-y_0)+a_2(y-y_0)+\ldots$, the power series having a radius of convergence different from zero. The portion of the curve given by either development is called an *analytic arc*.

‡ See Osgood, "Lehrbuch der Funktionentheorie," 2d ed., p. 672.

briefly, the symmetry determined by the analytic arc. It is then seen that if T is a non-singular conformal tranformation which transforms an analytic arc into a segment of a straight line, the symmetry determined by the arc is TST^{-1}, where S is the reflection with respect to the straight line which may and will be taken as the x-axis, in which case S is given by the equation

$$Z=\bar{z},$$

where $\bar{z}$ denotes the conjugate of z. Consequently, a symmetry is a non-singular reversed conformal transformation and, further, each point of the determining analytic arc is transformed into itself by the transformation of symmetry. Conversely, any non-singular reversed conformal transformation which transforms each point of an analytic arc into itself is a symmetry determined by the arc, for any non-singular reversed conformal transformation T' may be written $T'=TST^{-1}$, where T is a given non-singular conformal transformation and S is a non-singular reversed conformal transformation. Now, if T be taken as a non-singular conformal transformation which transforms the analytic arc C into the axis of abscissas, then, in order that each point of C be transformed into itself, it is necessary that S be the reflection with respect to the x-axis; *i. e.*, T' must be the symmetric transformation determined by the analytic arc C.

The equation of the symmetry determined by the analytic arc C: $y=a_1x+a_2x^2+\ldots.$* is readily gotten from the fact just stated that the symmetry determined by a given analytic arc is the non-singular reversed conformal transformation which leaves each point of the given analytic arc invariant. Thus, let the symmetry determined by the analytic arc C be given by

$$Z=A_1\bar{z}+A_2\bar{z}^2+A_3\bar{z}^3+\ldots., \qquad A_1\neq 0. \tag{1}$$

Since the point (x, y) of the arc C corresponds to itself under the transformation (1), we have the identity

$$x+iy=A_1(x-iy)+A_2(x-iy)^2+\ldots.,$$

or

$$\begin{aligned}(1+a_1i)x+a_2ix^2+a_3ix^3+\ldots.\\ &=A_1[(1-a_1i)x-a_2ix^2-a_3ix^3-\ldots.]\\ &+A_2[(1-a_1i)x-a_2ix^2-a_3ix^3-\ldots.]^2\\ &+A_3[(1-a_1i)x-a_2ix^2-a_3ix^3-\ldots.]^3\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

* In this and all of the following work the developments are all made about the origin, as no generality is lost and much simplification results.

Putting $1+a_1 i=a$, $1-a_1 i=\bar{a}$, the following equalities are obtained upon equating coefficients of like powers of x:

$$\left.\begin{aligned}&A_1\bar{a}=a,\\ -&A_1a_2i+A_2\bar{a}^2=a_2i,\\ -&A_1a_3i-2A_2\bar{a}a_2i+A_3\bar{a}^3=a_3i,\\ -&A_1a_4i-A_2(2\bar{a}a_3i+a_2^2)-3A_3\bar{a}^2a_2i+A_4\bar{a}^4=a_4i,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \qquad (2)$$

Or, solving for $A_1, A_2, A_3, A_4, \ldots.$, we have

$$\begin{aligned}A_1&=\frac{a}{\bar{a}},\\ A_2&=\frac{2a_2i}{\bar{a}^3},\\ A_3&=\frac{-4a_2^2+2\bar{a}a_3i}{\bar{a}^5},\\ A_4&=\frac{-10\bar{a}a_2a_3+2(\bar{a}^2a_4-5a_2^3)i}{\bar{a}^7},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Now, let $x=b_1y+b_2y^2+\ldots.$ be the inverse function of $y=\bar{a}x-ia_2x^2-ia_3x^3-\ldots..$ Then $x=b_1(\bar{a}x-ia_2x^2-\ldots.)+b_2(\bar{a}x-ia_2x^2-\ldots.)+\ldots.$ is an identity and upon equating coefficients of like powers of x the following equations result:

$$\begin{aligned}&b_1\bar{a}=1,\\ -&ib_1a_2+b_2C_1^2=0,\\ -&ib_1a_3+b_2C_2^2+b_3C_1^3=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ -&ib_1a_n+b_2C_{n-1}^2+b_3C_{n-2}^3+\ldots.+b_nC_1^n=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}$$

where the symbol C_{n+1}^m denotes the coefficient of x^{m+n} in the expansion of $(\bar{a}x-ia_2x^2-ia_3x^3-\ldots.)^m$, $n>0$, $m>1$, and C_1^m denotes $\bar{a}^m$. Solving this system of equations for b_n, we obtain for $n>1$

$$b_n=\frac{(-1)^{n-1}}{\bar{a}^{\frac{n}{2}(n+1)}}\begin{vmatrix}\bar{a} & 0 & 0 & \ldots. & 0 & 1\\ -ia_2 & C_1^2 & 0 & \ldots. & 0 & 0\\ -ia_3 & C_2^2 & C_1^3 & \ldots. & 0 & 0\\ -ia_4 & C_3^2 & C_2^3 & \ldots. & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ -ia_n & C_{n-1}^2 & C_{n-2}^3 & \ldots. & C_2^{n-1} & 0\end{vmatrix}.$$

Likewise, on solving the system of equations (2) for A_n, we obtain for $n>1$

$$A_n=\frac{(-1)^{n-1}}{\bar{a}^{\frac{n}{2}(n+1)}}\begin{vmatrix} \bar{a} & 0 & 0 & \dots & 0 & a \\ -ia_2 & C_1^2 & 0 & \dots & 0 & ia_2 \\ -ia_3 & C_2^2 & C_1^3 & \dots & 0 & ia_3 \\ -ia_4 & C_3^2 & C_2^3 & \dots & 0 & ia_4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -ia_n & C_{n-1}^2 & C_{n-2}^3 & \dots & C_2^{n-1} & ia_n \end{vmatrix}.$$

It is readily seen that $A_n=2b_n$, $n>1$. Consequently, we have the

THEOREM: *The symmetry determined by the analytic arc C: $y=a_1x+a_2x^2+\dots$ is given by*

$$Z=2w-\bar{z},$$

where $z=w-i(a_1w+a_2w^2+\dots)$.

It is obvious that an analytic arc C which intersects the analytic arc C_1 which determines the symmetry S_1 is transformed by the symmetry S_1 into another analytic arc C' such that the magnitude of the angle between the arcs C and C_1 is equal to the magnitude of the angle between the arcs C_1 and C'. C' is said to be the symmetric image of C with respect to the analytic arc C_1. Obviously, C is also the symmetric image of C' with respect to the analytic arc C_1.

As an application of the above formula, we prove the following

THEOREM: *If C_1' is the symmetric image of the analytic arc C_1 with respect to the analytic arc C, C_2' the symmetric image of the analytic arc C_2 with respect to arc C, and arc C_2 the symmetric image of the analytic arc C with respect to the analytic arc C_1, then the arc C_2' is the symmetric image of the arc C with respect to the arc C_1'.*

Without loss of generality we take the arc C as the segment of the x-axis in the neighborhood of the origin; for the above configuration may be conformally transformed so that the arc C is transformed into the segment of the x-axis in the vicinity of the origin. Let the arc C_2 be given by $y=a_1x+a_2x^2+\dots$; then the arc C_2' is given by $y=-(a_1x_1+a_2x^2+\dots)$. Any point $(x, 0)$ on C in the vicinity of the origin is transformed into $Z_1=2w-x$, where $x=w-i(a_1w+a_2w^2+\dots)$ by the symmetry determined by the arc C_2. This point is transformed into $Z_2=2\overline{w}-x$ by the reflection with respect to the x-axis and is, of course, on the arc C_1'. But this latter point is the point which corresponds to the point $(x, 0)$ under the symmetry determined by the arc C_2'; for if we call this point Z_2', we have $Z_2'=2w_1-x$, where $x=w_1+i(a_1w_1+a_2w_1^2+\dots)$. Since $\overline{w}=w_1$, we have $Z_2=Z_2'$, whence the theorem.

Another such theorem is the following:

THEOREM: *If the arc C_2 is the symmetric image of the analytic arc C with respect to the analytic arc C_1, the arc C_3 the symmetric image of the arc C_1 with respect to the arc C_2, and the arc C_4 the symmetric image of the arc C_2 with respect to the arc C_3, then the arc C_4 is the symmetric image of the arc C with respect to the arc C_2.*

§ 2. *In the Plane of Complex Points.*

In the extended complex plane of analysis the coordinates x, y of the point (x, y) may be real or complex numbers. In that plane the general non-singular conformal transformation which leaves the origin fixed, and which is defined in the neighborhood of the origin, is given by

$$U=f(u)=\alpha_1 u+\alpha_2 u^2+\ldots, \qquad \alpha_1 \neq 0,$$
$$V=g(v)=\beta_1 v+\beta_2 v^2+\ldots, \qquad \beta_1 \neq 0,$$

where the coefficients are real or complex and where $u=x+iy$ and $v=x-iy$. If β_i is the conjugate of α_i, the transformation is then a real non-singular conformal transformation which, however, is defined for complex values of the coordinates of a point as well as for real values.

Definition. In the complex plane a complex analytic arc is the set of points (x, y), $[(x, y)]$, whose coordinates satisfy the equation $y=y_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots$ or $x=x_0+a_1(y-y_0)+a_2(y-y_0)^2+\ldots$, where the power series has a radius of convergence different from zero and the coefficients are real or complex. If all the coefficients are real, the set of points which satisfy the above equation is said to be a real analytic arc in the complex plane. Unless the contrary is stated, an analytic arc will hereafter mean a complex analytic arc.

The definition given above of two real points which are symmetric with respect to a real analytic arc is extended to apply to the complex plane in that the arc C may be a complex analytic arc and the transformation T a general non-singular conformal transformation. In all that follows, it is assumed that the slope of the arc at the point about which the power series which represents the arc is developed is not minimal. It may now readily be shown that two points which are symmetric with respect to an analytic arc are such that the two minimal lines on each point intersect on the analytic arc and, conversely, two points which are such that the two minimal lines on each intersect on the analytic arc are symmetric with respect to the analytic arc.* From this fact,

* This fact, as was pointed out by Kasner, may be taken as the definition of two points symmetric with respect to an analytic arc.

and from the theorem that under any general non-singular conformal transformation minimal lines are transformed into minimal lines, it follows that two points which respectively correspond under the non-singular conformal transformation T to two points which are symmetric with respect to an analytic arc C are symmetric with respect to the analytic arc into which the arc C is transformed by the transformation T.

§ 3. *The (u, v)-plane.*

Now, given the complex plane, we may consider the set of points (u, v), $[(u, v)]$, such that $u=x+iy$ and $v=x-iy$. This set of points is called the *(u, v)-plane.* It is found more convenient to work in this plane and then to interpret the results so obtained to apply to the complex plane by means of the transformation

$$u=x+iy,$$
$$v=x-iy.$$

In the (u, v)-plane the transformation, called a pseudo-conformal transformation, which corresponds to the general non-singular conformal transformation in the complex plane defined, say, in the neighborhood of the origin, is simply given by two convergent power series in u and v respectively; if we assume that the origin is invariant, as we may for our purpose without loss of generality, the transformation is given by

$$U=\alpha_1 u+\alpha_2 u^2+\ldots., \qquad \alpha_1\neq 0,$$
$$V=\beta_1 v+\beta_2 v^2+\ldots., \qquad \beta_1\neq 0.$$

Definition. An analytic arc in the (u, v)-plane is the set of points (u, v), $[(u, v)]$, which satisfy the equation $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\ldots.$, or $u=u_0+a_1(v-v_0)+a_2(v-v_0)^2+\ldots.$, the power series having a radius of convergence different from zero and the a_i being real or complex.

Obviously an analytic arc in the (u, v)-plane corresponds to an analytic arc in the complex plane and conversely. Minimal lines in the complex plane correspond to lines parallel to the coordinate axes in the (u, v)-plane. Hence:

Definition. Two points in the (u, v)-plane are said to be symmetric with respect to an analytic arc if the two lines on each which are parallel to the coordinate axes intersect on the analytic arc.

For these points correspond to two points in the complex plane which are symmetric with respect to the analytic arc which corresponds to the given analytic arc, and conversely. The equations for the symmetric transformation determined by a given analytic arc are readily derived and are of simple form.

Let (u, v) and (U, V) be any two points which are symmetric with respect to the analytic arc C: $v=a_1u+a_2u^2+\ldots$, $a_1\neq 0$. Let the lines on the point (u, v) parallel to the u- and v-axes intersect the arc C in the points (u_1, v_1) and (u_2, v_2) respectively, and let the lines on the point (U, V) parallel to the u- and v-axes intersect the arc C in the points (u_2, v_2) and (u_1, v_1) respectively, where $u_1=U$, $v_1=v$, $u_2=u$, and $v_2=V$. But

$$v_1=a_1u_1+a_2u_1^2+\ldots,$$
$$v_2=a_1u_2+a_2u_2^2+\ldots,$$

whence

$$\left.\begin{aligned} v&=a_1U+a_2U^2+\ldots,\\ V&=a_1\,u+a_2\,u^2+\ldots,\end{aligned}\right\} \tag{10}$$

or

$$\left.\begin{aligned} U&=\alpha_1v+\alpha_2v^2+\ldots,\\ V&=a_1u+a_2v^2+\ldots,\end{aligned}\right\} \tag{11}$$

where α_i, $i=1, 2, \ldots$, are the coefficients of the power series which is the inverse of the power series with the coefficients a_i, $i=1, 2, \ldots$, and where $a_1\neq 0$. When the analytic arc in the complex plane which corresponds to the arc C is a real analytic arc, then the α_i, $i=1, 2, \ldots$, in the above transformation are also the conjugates of the a_i, $i=1, 2, \ldots$.

The problem of finding the bisector of two intersecting analytic arcs may again be simplified somewhat by considering the normal form in which one analytic arc is taken as a rectilinear segment. In the (u, v)-plane the rectilinear segment will be taken as the segment of the line $v=u$ in the neighborhood of the origin, as this line corresponds to the line $y=0$ in the complex plane.

Let, then, $v=f(u)=a_1u+a_2u^2+\ldots$, $a_1\neq 0$, be any analytic arc, taken as passing through the origin. Let the symmetric image of the segment of the line $v=u$ in the neighborhood of the origin with respect to the analytic arc C: $v=f(u)$ be $v=g(u)=b_1u+b_2u^2+\ldots$. Then by aid of the transformation (10) or (11) we obtain the equation

$$f[f(u)]=g(u). \tag{12}$$

Hence, the problem of finding the bisector of two intersecting analytic arcs is equivalent to solving the functional equation (12) for the unknown function $f(u)$ which must be analytic in the neighborhood of the origin, $g(u)$ being a given function of u, analytic in the vicinity of the origin, and the coefficient of its first-degree term not vanishing.

§ 4. *Conformal Equivalence of a Pair of Intersecting Arcs to a Pair of Rectilinear Segments.*

Obviously, any two intersecting analytic arcs which are conformally equivalent to two intersecting rectilinear segments must have two bisectors which are the conformal transforms of the rectilinear bisectors of these two rectilinear segments. For a large class of configurations consisting of two intersecting analytic arcs, a non-singular conformal transformation may be found which transforms any one of these configurations into two intersecting rectilinear segments.

Consider, now, the normal form of a configuration consisting of two intersecting analytic arcs, the arcs

$$C_1: \quad V=U,$$
$$C_2: \quad V=A_1U+A_2U^2+\ldots, \qquad A_1\neq 0.$$

Then, there exists an infinite number of non-singular pseudo-conformal transformations which transform the above configuration into the two intersecting rectilinear segments in the neighborhood of the origin of the lines

$$C_3: \quad v=u,$$
$$C_4: \quad v=A_1u,$$

such that the arc C_1 is transformed into the arc C_3 and the arc C_2 into the arc C_4, provided $|A_1|\neq 1$.

Any of the required transformations must be of the form

$$\left.\begin{aligned} U&=c_1'u+c_2'u^2+\ldots, & c_1'&\neq 0,\\ V&=d_1'v+d_2'v^2+\ldots, & d_1'&\neq 0,\end{aligned}\right\} \tag{13}$$

where the power series have radii of convergence different from zero. It will be more convenient to consider the transformation which is the inverse of the transformation (13) and which, therefore, transforms the arc C_3 into the arc C_1 and the arc C_4 into the arc C_2. Let this transformation be

$$U=c_1u+c_2u^2+\ldots, \qquad c_1\neq 0,$$
$$V=d_1v+d_2v^2+\ldots, \qquad d_1\neq 0.$$

Since the segment of the line $v=u$ in the vicinity of the origin is transformed into itself, we have identically

$$c_1u+c_2u^2+\ldots=d_1u+d_2u^2+\ldots,$$

i. e.,

$$c_n=d_n, \qquad n=1, 2, \ldots.$$

Thus the required transformations are of the form

$$\left.\begin{aligned} U&=c_1u+c_2u^2+\ldots, & c_1&\neq 0,\\ V&=c_1u+c_2u^2+\ldots. & & \end{aligned}\right\} \tag{14}$$

Since the rectilinear segment in the neighborhood of the origin of the line $v=A_1u$ is transformed into the analytic arc $V=A_1U+A_2U^2+\ldots.$ by the transformation (14), the following identity exists:

$$\begin{aligned} c_1A_1u+c_2A_1^2u^2+c_3A_1^3u^3+\ldots. \\ &=A_1(c_1u+c_2u^2+c_3u^3+\ldots.) \\ &+A_2(c_1u+c_2u^2+c_3u^3+\ldots.)^2 \\ &+A_3(c_1u+c_2u^2+c_3u^3+\ldots.)^3 \\ &+\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned} \tag{15}$$

whence the following set of equalities is obtained on equating coefficients of like powers of the argument:

$$\begin{aligned} c_1A_1&=c_1A_1, \\ c_2A_1^2&=c_2A_1+c_1^2A_2, \\ c_3A_1^3&=c_3A_1+2c_1c_2A_2+c_1^3A_3, \\ \ldots\ldots&\ldots\ldots\ldots\ldots, \end{aligned}$$

or c_1 is arbitrary and

$$\left.\begin{aligned} (A_1^2-A_1)c_2&=c_1^2A_2, \\ (A_1^3-A_1)c_3&=2c_1c_2A_2+c_1^3A_3, \\ \ldots\ldots&\ldots\ldots\ldots\ldots, \\ (A_1^n-A_1)c_n&=G_n(c_1, c_2, \ldots., c_{n-1}), \\ \ldots\ldots&\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \tag{16}$$

where $G_n(c_1, c_2, \ldots., c_{n-1})$ is a linear function of the A_i, $i=2, 3, \ldots.$, and the coefficients of A_i are integral functions of c_i, $i=1, 2, \ldots.$; further, only the plus sign appears. The c_i may, therefore, be found formally, but it remains to prove the convergence of the resulting power series.

Let m be a positive number for which the power series $A_1U+A_2U^2+\ldots.$ converges, and let g be an upper bound of the moduli of the terms of this series when $U=m$. Then

$$|A_n| \leqq \frac{g}{m^n}.$$

Let

$$\begin{aligned} D_1&=\frac{g}{m}, \\ D_2&=\frac{1}{|A_1^2-A_1|}\frac{g}{m}D_1^2, \\ D_3&=\frac{1}{|A_1^3-A_1|}g\left(\frac{2D_1D_2}{m^2}+\frac{D_1^3}{m^3}\right), \\ \ldots\ldots&\ldots\ldots\ldots\ldots\ldots, \\ D_n&=\frac{1}{|A_1^n-A_1|}H_n(D_1, D_2, \ldots., D_{n-1}), \\ \ldots\ldots&\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

where $H_n(D_1, D_2, \ldots, D_{n-1})$ denotes the value of $G_n(c_1, c_2, \ldots, c_{n-1})$ when $\frac{g}{m^n}$ and D_n are substituted respectively for A_n and c_n.

Obviously each D_n is positive, and it may be shown that

$$|c_n| \leqq D_n.$$

Let $\Gamma_n(c_1, c_2, \ldots, c_{n-1})$ denote the result obtained when A_n is replaced by its absolute value in $G_n(c_1, c_2, \ldots, c_{n-1})$. Then

$$\left.\begin{aligned} |c_n| &= \frac{1}{|A_1^n - A_1|} \, |G_n(c_1, c_2, \ldots, c_{n-1})| \\ &\leqq \frac{1}{|A_1^n - A_1|} \, \Gamma(|c_1|, |c_2|, \ldots, |c_{n-1}|) \\ &\leqq \frac{1}{|A_1^n - A_1|} \, H_n(|c_1|, |c_2|, \ldots, |c_{n-1}|), \end{aligned}\right\} \qquad (\alpha)$$

since $|A_n| \leqq \frac{g}{m^n}$.

Now take c_1, such that

$$|c_1| \leqq \frac{g}{m} = D_1;$$

then

$$|c_2| \leqq \frac{1}{|A_1^2 - A_1|} \, \frac{g}{m} D_1^2,$$

and by complete induction and the above inequality (α) it follows that

$$|c_n| \leqq \frac{1}{|A_1^n - A_1|} \, H_n(D_1, D_2, \ldots, D_{n-1}) = D_n.$$

Further, from the inequality

$$\big| |A_1|^n - |A_1| \big| \leqq |A_1^n - A_1|$$

it follows that for p sufficiently large, $r > p$,

$$|A_1^r - A_1| > 1 \text{ if } |A_1| > 1, \quad \text{and} \quad |A_1^r - A_1| > \frac{|A_1|}{2} \text{ if } |A_1| < 1 \text{ but } \neq 0.$$

Consequently, the set of numbers $|A_1^n - A_1|$, $n = 2, 3, 4, \ldots$, has a lower bound which is greater than zero. Let b be this lower bound. Put

$$\left.\begin{aligned} D_1' &= \frac{g}{m}, \\ D_2' &= \frac{g}{bm^2} D_1'^2, \\ D_3' &= \frac{g}{b}\left(\frac{2D_1'D_2'}{m^2} + \frac{D_1'^3}{m^3}\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ D_n' &= \frac{1}{b} H_n(D_1', D_2', \ldots, D_{n-1}^1), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (17)$$

Obviously, $D_n' \geqq D_n \geqq |c_n|$.

Further, consider the equation

$$y = \frac{g}{m}x + \frac{g}{b}\left(\frac{y^2}{m^2} + \frac{y^3}{m^3} + \frac{y^4}{m^4} + \dots\right). \tag{18}$$

The right-hand member is convergent for all values of x and for values of y such that $|y| < m$; and, further, the coefficient of the first-degree term in y appearing in the equation is not zero and there is no absolute term. Consequently, there exists a unique solution of this equation giving y explicitly in terms of x as a power series convergent in some circle about the origin with a non-vanishing radius and vanishing there. The solution is

$$y = D_1'x + D_2'x^2 + D_3'x^3 + \dots;$$

for by comparing the original identity (15) and the derived set of equalities (16) and (17) with the equation (18) it will be seen that

$$\begin{aligned} bD_1'x + bD_2'x^2 + bD_3'x^3 + bD_4'x^4 + \dots \\ = bD_1'x + \frac{g}{m^2}(D_1'x + D_2'x^2 + \dots)^2 \\ + \frac{g}{m^3}(D_1'x + D_2'x^2 + \dots)^3 \\ + \dots\dots\dots\dots\dots\dots \end{aligned}$$

is a formal identity.

Since $|c_n| \leqq D_n'$, it follows that the series $y = c_1x + c_2x^2 + \dots$ has a circle of convergence with a radius different from zero. Consequently, the transformation found is convergent.*

When the analytic arc C_2 corresponds to a real analytic arc in the real or complex plane, the modulus of A_1 is unity. In this case, no transformation as the above may exist, even formally. However, when the angle between the two real analytic arcs is incommensurable with π, which implies that $|A_1^n - A_1| \neq 0$, an infinite number of formal solutions exist. The question of convergence of the formal solutions obtained when $|A_1| = 1$ and $A_1^n \neq 1$ for all positive integral values of n is not settled.† If $A_1^n = 1$, where n is the smallest such positive integer and >1, then in the formal solution the coefficients of $c_{n+1}, c_{2n+1}, \dots, c_{mn+1}, \dots$ vanish. If c_{mn+1}, $m = 1, 2, \dots,$ are taken equal to zero, there exists an infinite set of polynomials of the A_i, obtained from the set of equations (16), which must vanish in order that a formal solu-

* This result is contained in the work by G. Koenigs, "Nouvelles recherches sur les équations fonctionnelles," *Annales de l'École Normale*, 1884. However, the above proof is much more direct and more simple.

† Following a suggestion of Professor Birkhoff, I have shown that in this case, for certain arcs $V = A_1U + A_2U^2 + \dots$, the formal solutions referred to are divergent. This proof will appear in a later paper.

tion be possible. Suppose that all these polynomials do vanish. The c's thus determined, c_{mn+1}, $m=1, 2, \ldots.$, being taken to equal zero, are the coefficients of a convergent power series, for the above proof may be easily modified to apply in this case, *i. e.*, the D_{mn+1}, $m=1, 2, \ldots.$, may be taken equal to zero and b may be taken as the smallest of the $|A_1^m - A_1|$, $m=2, 3, \ldots., n$. Thus, in this case, the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments. Conversely, if the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments, then the equalities involving the A_i referred to above exist, for it may be shown that if the non-singular pseudo-conformal transformation T transforms the pair of intersecting rectilinear segments $v=A_1u$, $v=u$, in the vicinity of the origin, into the given configuration, then there exists a non-singular pseudo-conformal transformation T' which also transforms the pair of rectilinear segments into the given configuration, such that the coefficients of u^{mn+1} and v^{mn+1}, $m=1, 2, \ldots.$, vanish.

Let T be given by

$$U=c_1u+c_2u^2+\ldots.,$$
$$V=c_1v+c_2v^2+\ldots., \qquad c_1 \neq 0,$$

and T' by

$$U=\gamma_1u+\gamma_2u^2+\ldots.,$$
$$V=\gamma_1v+\gamma_2v^2+\ldots., \qquad \gamma_1 \neq 0, \gamma_{mn+1}=0, m=1, 2, \ldots..$$

Then a transformation T_1, given by

$$U=d_1u+d_{n+1}u^{n+1}+d_{2n+1}u^{2n+1}+\ldots.,$$
$$V=d_1v+d_{n+1}v^{n+1}+d_{2n+1}v^{2n+1}+\ldots., \qquad d_1 \neq 0,$$

may be found such that $T_1T=T'$. The transformation T_1 leaves the pair of rectilinear segments invariant, and consequently T', the product of T_1 and T, transforms the pair of rectilinear segments into the given configuration. In order that this be so, it is sufficient that we have identically

$$c_1(d_1u+d_{n+1}u^{n+1}+\ldots.)+c_2(d_1u+d_{n+1}u^{n+1}+\ldots.)^2+\ldots.=\gamma_1u+\gamma_2u^2+\ldots..$$

On equating coefficients of u^{mn+1}, $m=0, 1, 2, \ldots.$, we have

$$c_1d_1=\gamma_1,$$
$$c_1d_{n+1}+c_{n+1}d_1^{n+1}=0,$$
$$c_1d_{2n+1}+(n+1)c_{n+1}d_1^nd_{n+1}+c_{2n+1}d_1^{2n+1}=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$c_1d_{mn+1}+G(d_1, \ldots., d_{(m-1)n+1}, c_1, \ldots., c_{mn+1})=0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

whence the d's are determined uniquely for γ_1 given.

Consider the equation $c_1y+c_{n+1}y^{n+1}+c_{2n+1}y^{2n+1}+\ldots\ldots=\gamma_1x$. The left-hand member is convergent in a circle about the origin with a non-vanishing radius, and since $c_1\neq 0$ and since there is no absolute term, there exists a unique solution, $y=f(x)$, which is analytic about the origin and vanishes there. This solution is precisely the power series $y=d_1x+d_{n+1}x^{n+1}+\ldots\ldots$ Consequently, the existence of the transformation T_1 is demonstrated, and hence that of T'.

Such a set of polynomials of the A_i as obtained above may also be gotten as follows: Let the configuration consisting of the analytic arc $V=A_1U+A_2U^2+\ldots\ldots$, $A_1^n=1$, n being the smallest such positive integer, and the segment of the line $V=U$ in the vicinity of the origin be pseudo-conformally equivalent to a pair of intersecting rectilinear segments; then we have the identity, in symbolic form, $cA_1=Ac$, where A_1 denotes the function $v=A_1u$, A the function $v=A_1u+A_2u^2+\ldots\ldots$ and c the function $U=c_1u+c_2u^2+\ldots\ldots$ It then follows that $cA_1c^{-1}=A$, where c^{-1} denotes the inverse function of $U=c_1u+c_2u^2+\ldots\ldots$, and consequently $(cA_1c^{-1})^n=1=A^n$, since $A_1^n=1$, where F^n denotes the n-th iteration of the function F. From the latter identity an infinite number of polynomials of the A_i is obtained and the vanishing of each is necessary for the pseudo-conformal equivalence of the given configuration to a pair of intersecting rectilinear segments. Conversely, if these conditions are satisfied, that is, if $A^n=1$, then the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments. Suppose that the given configuration is not pseudo-conformally equivalent to a pair of intersecting rectilinear segments. Then the non-singular pseudo-conformal transformation, given by

$$U=d_1u+d_2u^2+\ldots\ldots,$$
$$V=d_1v+d_2v^2+\ldots\ldots,\qquad d_1\neq 0,$$

which is the inverse of a transformation of which the first rn coefficients, with the exception of the first one, are determined by the first $rn-1$ equations of the set (16) when c_{mn+1}, $m=1, 2, \ldots\ldots, r$, are taken equal to zero, and the remaining coefficients arbitrary, transforms the segment of the line $V=U$ in the vicinity of the origin into itself and the analytic arc $V=A_1U+A_2U^2+\ldots\ldots$ into the analytic arc $V=A_1U+B_{rn+1}U^{rn+1}+B_{rn+2}U^{rn+2}+\ldots\ldots$, where $B_{rn+1}\neq 0$. (The converse of this is obvious; that is, a configuration consisting of the analytic arc $V=B_1U+B_{rn+1}U^{rn+1}+\ldots\ldots$, $B^n=1$ and $B_{rn+1}\neq 0$, and the segment of the line $V=U$ in the vicinity of the origin, is not pseudo-conformally equivalent to a pair of intersecting rectilinear segments; for if this were so, then we would have the identity

$$B_1(c_1u+c_2u^2+\ldots\ldots)+B_{rn+1}(c_1u+c_2u^2+\ldots\ldots)^{rn+1}+\ldots\ldots=c_1B_1u+c_2B_1^2u^2+\ldots\ldots$$

or $c_{rn+1}B_1+c_1^{rn+1}B_{rn+1}=c_{rn+1}B_1^{rn+1}$, and since $B_1^n=1$ and $c_1\neq 0$, B_{rn+1} must vanish, which is contradictory. From the fact just proved follows readily another proof of the necessity of the conditions referred to on p. 409.) We have now, symbolically, the identity $dA=Bd$, where B denotes the function $V=A_1U+B_{rn+1}U^{rn+1}+\dots$ and d the function $U=d_1u+d_2u^2+\dots$. Suppose that $A^n=1$; then we have the identity $(dAd^{-1})^n=1=B^n$. However, B^n is given by $V=U+nB_1^{n-1}B_{rn+1}U^{rn+1}+\dots$, and since $n>1$ and $B_{rn+1}\neq 0$, B^n cannot be identity. Consequently, if $A^n=1$, the given configuration is pseudo-conformally equivalent to a pair of intersecting rectilinear segments. In particular, we may state the

THEOREM: *A necessary and sufficient condition that a configuration consisting of two real analytic arcs which intersect at an angle which is commensurable with π,* but which are not tangent at the point of intersection, be conformally equivalent to a pair of intersecting rectilinear segments, is the vanishing of each of an infinite number of rational functions, determined as indicated above, of the coefficients of the equations of the arcs.*

Let C_1: $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots$, $a_1\neq 0$, and C_2: $v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\dots$, $b_1\neq 0$, be two analytic arcs intersecting in the point (u_0, v_0). If the arc C_1 is transformed into the analytic arc C_1': $v=\alpha_1u+\alpha_2u^2+\dots$ and the arc C_2 into the analytic arc C_1': $v=\beta_1u+\beta_2u^2+\dots$ by the translation

$$U=u-u_0,$$
$$V=v-v_0,$$

then $a_1=\alpha_1$, $b_1=\beta_1$. If the arcs C_1', C_2' are transformed by a non-singular pseudo-conformal transformation into the analytic arc $v=A_1u+A_2u^2+\dots$ and the segment in the vicinity of the origin of the line $v=u$, respectively, then $\frac{\alpha_1}{\beta_1}=A_1$, and conversely. Therefore $\left|\frac{a_1}{b_1}\right|=\left|\frac{\alpha_1}{\beta_1}\right|=|A_1|$. We may now state the

THEOREM: *The two intersecting analytic arcs*

$$C_1:\quad v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\dots,\qquad a_1\neq 0,$$
$$C_2:\quad v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\dots,\qquad b_1\neq 0,$$

where $\left|\frac{a_1}{b_1}\right|\neq 1$, are pseudo-conformally equivalent to a pair of intersecting rectilinear segments.

---

* This corresponds to the above condition that $A^n=1$.

We also have the

THEOREM: *The two intersecting analytic arcs*

$$C_1: \qquad v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\ldots, \qquad a_1\neq 0,$$
$$C_2: \qquad v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\ldots, \qquad b_1\neq 0,$$

where $\left|\frac{a_1}{b_1}\right|\neq 1$ *have just two symmetric bisectors.*

That there are just two bisectors follows from the fact that there are just two formal solutions for the coefficients of the power series which represents the bisector, as will be seen below.

We have the two corresponding theorems in the complex plane when the corresponding condition on the two analytic arcs, using the same notation as above, $\left|\frac{1+a_1b_1-(a_1-b_1)i}{1+a_1b_1+(a_1+b_1)i}\right|\neq 1$ is verified. Further, from the above developments we have the

THEOREM: *The functional equation*

$$f[f(u)]=g(u),$$

where $g(u)$ *is an analytic function defined in the vicinity of the origin and vanishing there has just two solutions,* $f(u)$, *which are analytic in the vicinity of the origin and which vanish there also, provided the modulus of the first-degree term of* $g(u)\neq 0, 1$.

Another result* also follows from the above developments. It is the

THEOREM: *Let* S *and* S' *be any two real non-singular pseudo-conformal transformations, defined in the vicinity of the point* $z_0=x_0+iy_0$, *which is left invariant by both, such that the coefficients of* $(z-z_0)$ *of each transformation, given as a development in powers of* $(z-z_0)$, *are equal and have a modulus not equal to unity. Then* S *and* S' *are equivalent in the sense that* S *may be transformed into* S' *by a non-singular pseudo-conformal transformation.*

Let S and S' be given by

$$S: \qquad Z=z_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots,$$
$$S': \qquad Z=z_0+a_1(z-z_0)+b_2(z-z_0)^2+\ldots, \qquad z=x+iy,\ a_1\neq 0,\ |a_1|\neq 1.$$

In order that

$$T: \qquad Z=z_0+c_1(z-z_0)+c_2(z-z_0)^2+\ldots, \qquad c_1\neq 0,$$

be a transformation such that $T^{-1}ST=S'$ or $ST=TS'$, it is necessary and sufficient that we have identically

* This interpretation was suggested by Professor Kasner.

$$\begin{aligned} z_0 &+ c_1(a_1\xi + a_2\xi^2 + \dots) \\ &+ c_2(a_1\xi + a_2\xi^2 + \dots)^2 \\ &+ \dots\dots\dots\dots \\ &\quad = z_0 + a_1(c_1\xi + c_2\xi^2 + \dots) \\ &\qquad + b_2(c_1\xi + c_2\xi^2 + \dots)^2 \\ &\qquad + \dots\dots\dots\dots, \end{aligned}$$

where $\xi = z - z_0$. This is the same identity which is obtained when the analytic arc $v = z_0 + a_1(u - z_0) + a_2(u - z_0)^2 + \dots$ is transformed pseudo-conformally into the analytic arc $v = z_0 + a_1(u - z_0) + b_2(u - z_0)^2 + \dots$ and the segment of the line $v = u$ in the vicinity of the point (z_0, z_0) into itself. Thus, by the theorem at the bottom of p. 409 concerning the pseudo-conformal equivalence of a pair of intersecting analytic arcs to a pair of intersecting rectilinear segments, the existence of the transformation T follows.

§ 5. *Direct Discussion of the Bisection Problem.*

As stated above, the problem of finding a bisector of two intersecting analytic arcs is equivalent to solving the functional equation

$$a_1(a_1u + a_2u^2 + \dots) + a_2(a_1u + a_2u^2 + \dots)^2 + \dots = b_1u + b_2u^2 + \dots,$$

where $b_1u + b_2u^2 + \dots$ is the given function, analytic about the origin and $b_1 \neq 0$, and $a_1u + a_2u^2 + \dots$ the unknown function, also required to be analytic about the origin. Upon equating coefficients of like powers of the argument, the following set of equalities is obtained:

$$\begin{aligned} a_1^2 &= b_1, \\ a_1a_2 + a_2a_1^2 &= b_2, \\ a_1a_3 + 2a_1a_2^2 + a_3a_1^3 &= b_3, \\ \dots\dots\dots&\dots, \end{aligned}$$

or

$$\left.\begin{aligned} a_1^2 &= b_1, \\ (a_1 + a_1^2)a_2 &= b_2, \\ (a_1 + a_1^3)a_3 + 2a_1a_2^2 &= b_3, \\ \dots\dots\dots&\dots, \\ (a_1 + a_1^m)a_m + P(a_1, a_2, \dots, a_{m-1}) &= b_m, \\ \dots\dots\dots\dots\dots&\dots, \end{aligned}\right\} \qquad (19)$$

where $P_m(a_1, a_2, \dots, a_{m-1})$ is a polynomial in a_i, $i = 1, 2, \dots, (m-1)$. If $a_1^m \neq -1$ for all positive integral values of m, a_i, $i = 2, 3, \dots$, may be found formally, and hence if $b_1^m \neq 1$ for all positive integral values, there exist two formal solutions which, if $|b_1| \neq 1$, are both convergent in a circle of radius

different from zero; if $|b_1|=1$, the question of convergence, in general, of either of the two formal solutions is unsettled.*

Consider the configuration consisting of the segments of the lines $v=u$ and $v=b_1u$ in the vicinity of the origin. Let this configuration have the segment in the vicinity of the origin of the line $v=a_1u$ as a bisector, where $a_1^n=-1$, n being the smallest such positive integer. Then $b^n=1$. The given configuration may be transformed into itself by an infinitude (of potency c) of non-singular pseudo-conformal transformations; for let

$$U=c_1u+c_2u^2+\dots,$$
$$V=c_1v+c_2v^2+\dots, \qquad c_1\neq 0,$$

be any non-singular pseudo-conformal transformation which transforms the segment in the vicinity of the origin of the line $v=u$ into itself. Then, in order that this transformation transform the segment in the vicinity of the origin of the line $v=b_1u$ into itself, it is necessary and sufficient that the following identity exist:

$$c_1b_1u+c_2(b_1u)^2+c_3(b_1u)^3+\dots=b_1(c_1u+c_2u^2+c_3u^3+\dots),$$

or

$$c_1b_1=c_1b_1,$$
$$c_2b_1^2=c_2b_1,$$
$$\dots\dots\dots,$$
$$c_nb_1^n=c_nb_1,$$
$$\dots\dots\dots,$$

whence $c_1, c_{n+1}, c_{2n+1}, \dots, c_{mn+1}, \dots$ are arbitrary and $c_r=0$, $r\neq mn+1$, $m=0, 1, 2, \dots$.

Let the transformation

$$U=c_1u+c_{n+1}u^{n+1}+c_{2n+1}u^{2n+1}+\dots,$$
$$V=c_1v+c_{n+1}v^{n+1}+c_{2n+1}v^{2n+1}+\dots, \qquad c_1\neq 0,$$

transform the segment in the vicinity of the origin of the line $v=a_1u$ into the analytic arc C: $V=A_1U+A_2U^2+A_3U^3+\dots$; then we have identically

$$\begin{aligned} c_1(a_1u)+c_{n+1}(a_1u)^{n+1}+c_{2n+1}(a_1u)^{2n+1}+\dots \\ =A_1(c_1u+c_{n+1}u^{n+1}+c_{2n+1}u^{2n+1}+\dots) \\ +A_2(c_1u+c_{n+1}u^{n+1}+c_{2n+1}u^{2n+1}+\dots)^2 \\ +\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned} \tag{20}$$

from which it follows that

* The second foot-note on p. 407 has reference here also.

$$\left.\begin{aligned}
&A_1=a_1,\\
&A_2=A_3=\ldots\ldots=A_n=0,\\
&c_{n+1}a_1^{n+1}=a_1c_{n+1}+A_{n+1}c_1^{n+1},\\
&A_{n+2}=A_{n+3}=\ldots\ldots=A_{2n}=0,\\
&c_{2n+1}a_1^{2n+1}=a_1c_{2n+1}+(n+1)A_{n+1}c_1^nc_{n+1}+c_1^{2n+1}A_{2n+1},\\
&A_{2n+2}=A_{2n+3}=\ldots\ldots=A_{3n}=0,\\
&c_{3n+1}a_1^{3n+1}=a_1c_{3n+1}+(n+1)A_{n+1}c_1^nc_{2n+1}+\frac{(n+1)n}{2}A_{n+1}c_1^{n-1}c_{n+1}^2\\
&\qquad\qquad\qquad\qquad+(2n+1)A_{2n+1}c_1^{2n}c_{n+1}+A_{3n+1}c_1^{3n+1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}\quad(21)$$

As on p. 408, it can readily be shown that the set of A's which satisfy (21) also satisfy a similar set of equations in which $c_{2mn+1}=0$, $m=1, 2, \ldots\ldots$ We may then assume without loss of generality that $c_{2mn+1}=0$, and we have, in general,

$$c_1^{(2m-1)n+1}A_{(2m-1)n+1}=-2a_1c_{(2m-1)n+1}+G_{(2m-1)n+1}(A_1, A_n, \ldots\ldots, A_{(2m-2)n+1}), \quad (22)$$

$$A_{2mn+1}=G_{2mn+1}(A_1, A_n, \ldots\ldots, A_{(2m-1)n+1}), \quad (23)$$

$$A_r=0; \qquad r\neq kn+1;\ k=0, 1, 2, \ldots\ldots \quad (24)$$

The equalities (23) are obviously necessary conditions that $V=A_1U+A_2U^2+\ldots\ldots$ be a bisector of the given configuration; they can also be shown to be sufficient. In other words, a non-singular pseudo-conformal transformation

$$\begin{aligned}
&U=c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots,\\
&V=c_1v+c_{n+1}v^{n+1}+c_{3n+1}v^{3n+1}+\ldots\ldots, \qquad c_1\neq 0,
\end{aligned}$$

may be found such that the given pair of intersecting rectilinear segments is transformed into itself and the arc C: $V=A_1U+A_2U^2+\ldots\ldots$ into the segment in the vicinity of the origin of the line $v=a_1u$. The proof of this is similar to the convergence proof on pp. 404–407. Corresponding to the identity (15), we have the identity

$$\begin{aligned}
c_1a_1u+c_{n+1}(a_1u)^{n+1}+c_{3n+1}(a_1u)^{3n+1}+\ldots\ldots&\\
=A_1(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)&\\
+A_n(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)^n&\\
+A_{2n+1}(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)^{2n+1}&\\
+A_{3n+1}(c_1u+c_{n+1}u^{n+1}+c_{3n+1}u^{3n+1}+\ldots\ldots)^{3n+1}&\\
+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,&
\end{aligned}$$

whence the following equalities to determine the $c_{(2m-1)n+1}$:

$$c_1 a_1 = A_1 c_1,$$
$$-2A_1 c_{n+1} = c_1^{n+1} A_{n+1},$$
$$-2A_1 c_{3n+1} = \frac{(n+1)n}{2} A_{n+1} c_1^{n-1} c_{n+1}^2 + (2n+1) c_1^{2n} A_{2n+1} + c_1^{3n+1} A_{3n+1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$-2A_1 c_{(2m-1)n+1} = G_{(2m-1)n+1}(c_1, c_{n+1}, \ldots, c_{(2m-3)n+1}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where $G_{(2m-1)n+1}(c_1, c_{n+1}, \ldots, c_{(2m-3)n+1})$ is a linear function of the A_{kn+1} and the coefficients of the A_{kn+1} are integral functions of the c's. Corresponding to the set of equations (17), we have

$$D_1 = \frac{g}{\mu},$$
$$D_{n+1} = \frac{1}{|2A_1|} \frac{g}{\mu} D_1^{n+1},$$
$$D_{2n+1} = \frac{1}{|2A_1|} g \left[\frac{n+1}{\mu^{n+1}} D_1^n D_{n+1} + \frac{D_1^{2n+1}}{\mu^{2n+1}} \right],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$D_{mn+1} = \frac{1}{|2A_1|} H_{mn+1}[D_1, D_{n+1}, \ldots, D_{(m-1)n+1}],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \qquad m=0, 1, 2, \ldots,$$

where μ is a positive number for which the series $A_1 U + A_{n+1} U^{n+1} + \ldots$ converges and g is an upper bound of the moduli of the terms of the series when $U=\mu$. Corresponding to the equation (18), we have the equation

$$V = \frac{g}{\mu} U + \frac{g}{|2A_1|} \left[\frac{V^{n+1}}{\mu^{n+1}} + \frac{V^{2n+1}}{\mu^{2n+1}} + \ldots + \frac{V^{mn+1}}{\mu^{mn+1}} + \ldots \right],$$

the left-hand member of which is convergent for all values of U and for $|V| < \mu$.

From (22) and (23) it is seen that $A_{(2m-1)n+1}$, $m=1, 2, \ldots, p=$ any positive integer, are arbitrary and that if $A_{(2m-1)n+1}$, $m=1, 2, \ldots, p$, are given, then A_{2mn+1}, $m=1, 2, \ldots, p$, are determined uniquely. The set of relations (23) is equivalent to the set of relations between the coefficients of the equation of the bisector gotten directly from (19) when $b_n=0$, $n \neq 1$.

We may now state the two closely related theorems:

THEOREM: *The configuration consisting of the two intersecting segments $v=u$ and $v=b_1 u$, where $b_1^n=1$, $n=$ a positive integer >1, has an uncountable infinity of bisectors.*

THEOREM: *The functional equation $f[f(u)]=b_1 u$, $b^n=1$, $n=$ smallest such positive integer >1, has an uncountable infinity of solutions, analytic about the*

origin and vanishing there, all of which may be determined as above and are of the form $V=A_1U+A_{n+1}U^{n+1}+A_{2n+1}U^{2n+1}+\ldots$*; further,* $A_{(2m-1)n+1}$, $m=1, 2, \ldots, p=$ *any positive integer, are arbitrary when* $A_1^n=-1$.*

It is obvious that there is only one solution, viz., $V=A_1U$, when n is odd, which has A_1 equal to that square root of b_1 which has the largest amplitude θ, where θ is taken such that $0<\theta\leqq 2\pi$; in all other cases $A_1^n=-1$, n being the smallest such positive integer.

Now, consider the configuration consisting of the segment in the vicinity of the origin of the line $v=u$ and the analytic arc C: $v=B_1u+B_2u^2+\ldots$, and let a bisector be given by $v=a_1u+a_2u^2+\ldots$, where $a_1^n=-1$, n being the smallest such positive integer. It may be shown that this configuration has not more than one bisector such that the coefficient of u in its equation is a definite value of a_1 if the configuration is not pseudo-conformally equivalent to a pair of intersecting rectilinear segments.

We may, without losing generality, consider the configuration as consisting of the analytic arc C: $v=b_1u+b_{rn+1}u^{rn+1}+\ldots+b_{(r+j)n+1}u^{(r+j)n+1}+\ldots+b_qu^q+\ldots$, $b_{rn+1}\neq 0$, $q=$ any positive integer $\neq kn+1$, $k=0, 1, 2, \ldots$, and the segment in the vicinity of the origin of the line $v=u$.†

Obviously, if $r=1$, there exists no formal solution for a bisector as required. Let r be greater than unity; then the coefficients $b_{(r+m)n+1}$, $m=1, 2, \ldots, r-1$, may be taken equal to zero as well as the coefficients b_s, $s\neq kn+1$, $k=0, 1, 2, \ldots, 2r-1, \ldots, t$. For it may be shown that there exists a non-singular pseudo-conformal transformation which transforms the segment of the line $v=u$ in the vicinity of the origin into itself and the analytic arc

$$C: \qquad v=b_1u+b_{rn+1}u^{rn+1}+b_{(r+1)n+1}u^{(r+1)n+1}+\ldots+b_{(r+m)n+1}u^{(r+m)n+1}+\ldots$$

into the analytic arc

$$C': \qquad V=b_1U+B_{rn+1}U^{rn+1}+B_{(r+1)n+1}U^{(r+1)n+1}+\ldots+B_{(r+m)n+1}U^{(r+m)n+1}+\ldots,$$

such that $B_{(r+m)n+1}=0$, $m=1, 2, \ldots, r-1$.

Let

$$U=c_1u+c_2u^2+\ldots,$$
$$V=c_1v+c_2v^2+\ldots, \qquad c_1\neq 0,$$

be a transformation which transforms the segment of the line $v=u$ in the vicinity of the origin into itself. Then, in order that the analytic arc C be transformed into the analytic arc C' by the above transformation, it is necessary and sufficient that we have identically

* Leau and Lémeray have given some theorems on this functional equation (*Bull. de la Soc. Math. de France*, Vol. XXVI, 1898). More details of the nature of the solutions are given above.

† See p. 409.

$$
\begin{aligned}
& b_1(c_1u+c_2u^2+\ldots\ldots)\\
& +B_{rn+1}(c_1u+c_2u^2+\ldots\ldots)^{rn+1}\\
& +B_{(r+1)n+1}(c_1u+c_2u^2+\ldots\ldots)^{(r+1)n+1}\\
& +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
& +B_{(r+m)n+1}(c_1u+c_2u^2+\ldots\ldots)^{(r+m)n+1}\\
& +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
& \quad =c_1(b_1u+b_{rn+1}u^{rn+1}+b_{(r+1)n+1}u^{(r+1)n+1}+\ldots\ldots+b_{(r+m)n+1}u^{(r+m)n+1}+\ldots\ldots)\\
& \quad +c_2(b_1u+b_{rn+1}u^{rn+1}+b_{(r+1)n+1}u^{(r+1)n+1}+\ldots\ldots+b_{(r+m)n+1}u^{(r+m)n+1}+\ldots\ldots)^2\\
& \quad +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

The following equalities then result:

$$
\begin{aligned}
b_1c_1 &= b_1c_1,\\
b_1c_2 &= b_1^2c_2,\\
&\ldots\ldots\ldots,\\
b_1c_{rn} &= b_1^{rn}c_{rn},
\end{aligned}
$$

from which it follows that $c_1, c_{n+1}, c_{2n+1}, \ldots\ldots, c_{mn+1}, \ldots\ldots, c_{(r-1)n+1}$ are arbitrary $(c_1\neq 0)$, while

$$c_2=\ldots\ldots=c_n=c_{n+2}=\ldots\ldots=c_{2n}=c_{2n+2}=\ldots\ldots=c_{rn}=0, \qquad n>1.^*$$

Further,

$$
\begin{aligned}
& b_1c_{rn+1}+c_1^{rn+1}B_{rn+1}=c_1b_{rn+1}+c_{rn+1}b_1^{rn+1},\\
& b_1c_{(r+1)n+1}+(rn+1)c_1^{rn}c_{n+1}B_{rn+1}+c_1^{(r+1)n+1}B_{(r+1)n+1}\\
& \qquad\qquad =c_1b_{(r+1)n+1}+(n+1)c_{n+1}b_1^{n}b_{rn+1}+c_{(r+1)n+1}b_1^{(r+1)n+1},\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
& b_1c_{(r+m)n+1}+(rn+1)c_1^{rn}c_{mn+1}B_{rn+1}+\ldots\ldots+c_1^{(r+m)n+1}B_{(r+m)n+1}\\
& \qquad\qquad =c_1b_{(r+m)n+1}+\ldots\ldots+(mn+1)b_1^{mn}b_{rn+1}c_{mn+1}+b_1^{(r+m)n+1}c_{(r+m)n+1},\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
$$

where the terms omitted in the last equality do not involve c_{mn+1} or $B_{(r+m)n+1}$. Also it is readily seen that $c_{(r+m)n+l}$, $l=2, 3, \ldots\ldots, n-1$, $n>1$, vanish. Solving for B_i, we find

$$c_1^{rn}B_{rn+1}=b_{rn+1}$$

and

$$
\begin{aligned}
c_1^{(r+m)n+1}B_{(r+m)n+1} &= (m-r)nb_{rn+1}c_{mn+1}\\
&\quad +G(c_1, c_{n+1}, \ldots\ldots, c_{(m-1)n+1}, b_{(r+1)n+1}, \ldots\ldots, b_{(r+m)n+1}).
\end{aligned}
$$

Since $b_{rn+1}\neq 0$ and c_{mn+1} is arbitrary, the latter may be so taken that $B_{(r+m)n+1}$, $m=1, 2, \ldots\ldots, r-1$, vanishes. Hence the equation of the analytic arc C may be taken to be of the form

$$v=b_1u+b_{rn+1}u^{rn+1}+b_{2rn+1}u^{2rn+1}+\ldots\ldots.$$

---

* Of course, when $n=1$ this reduction is uncalled for.

Let $r=2p$, where p is a positive integer. Now, from the set of equations (19) the following two equations are obtained:

$$(a_1+a_1^{2pn+1})\,a_{2pn+1}+P_1(a_1,\,\ldots.,\,a_{(2p-1)n+1})=b_{2pn+1},$$
$$(a_1+a_1^{(2p+1)n+1})\,a_{(2p+1)n+1}+(2p-1)\,na_{n+1}a_{2pn+1}+P_{n+1}(a_1,\,\ldots.,\,a_{(2p-1)n+1})=0,$$

since $a_2=\ldots.=a_n=0$. Similarly, if, instead of the analytic arc C: $v=b_1u+b_{2pn+1}u^{2pn+1}+\ldots.$, we have the segment of the line $v=b_1u$ in the vicinity of the origin, the following equations result by using the last theorem on p. 415:

$$2a_1a'_{2pn+1}+P_1(a_1,\,\ldots.,\,a_{(2p-1)n+1})=0,$$
$$(2p-1)\,na_{n+1}a'_{2pn+1}+P_{n+1}(a_1,\,\ldots.,\,a_{(2p-1)n+1})=0.$$

From the above two pairs of equations it follows that

$$2a_1(a_{2pn+1}-a'_{2pn+1})=b_{2pn+1}\neq 0$$

and

$$(2p-1)\,na_{n+1}(a_{2pn+1}-a'_{2pn+1})=0,$$

whence $a_{n+1}=0$. Assuming that $a_{n+1}=\ldots.=a_{(2m+1)n}=0$, we obtain likewise from (19) the equation

$$(a_1+a_1^{(2p+2m+1)n+1})\,a_{(2p+2m+1)n+1}+(2p-2m-1)\,na_{(2m+1)n+1}a_{2pn+1}$$
$$+P_{(2m+1)n+1}(a_1,\,\ldots.,\,a_{(2p-1)n+1})=0, \qquad \text{where } m=1,\,2,\,\ldots.,\,p-1,$$

the first term of the left-hand member vanishing; and in the case that the segment in the vicinity of the origin of the line $v=u$ is taken instead of the analytic arc C: $v=b_1u+b_{2pn+1}u^{2pn+1}+\ldots.$, we have correspondingly the equation

$$(2p-2m-1)\,a_{(2m+1)n+1}a'_{2pn+1}+P_{(2m+1)n+1}(a_1,\,\ldots.,\,a_{(2p-1)n+1})=0,$$

whence it follows that

$$(2p-2m-1)\,a_{(2m+1)n+1}(a_{2pn+1}-a'_{2pn+1})=0.$$

Therefore, $a_{(2m+1)n+1}=0$, and thus, by complete induction, it follows that $a_s=0$, $s=2,\,3,\,\ldots.,\,2pn$.

The set of equations (19) then gives further the following equations:

$(a_1+a_1^{2pn+1})\,a_{2pn+1}=b_{2pn+1}\neq 0,$ (a_{2pn+1} is here uniquely determined and $\neq 0$.)
$(a_1+a_1^{2pn+2})\,a_{2pn+2}=0,$
$\ldots\ldots\ldots\ldots\ldots\ldots,$
$(a_1+a_1^{2pn+n})\,a_{2pn+n}=0,$
$(a_1+a_1^{2pn+n+1})\,a_{2pn+n+1}=0,$ ($a_{2pn+n+1}$ is not determined here.)
$(a_1+a_1^{2pn+n+2})\,a_{2pn+n+2}=0,$
$\ldots\ldots\ldots\ldots\ldots\ldots,$
$(a_1+a_1^{2pn+3n+1})\,a_{2pn+3n+1}=0,$ ($a_{2pn+3n+1}$ is not determined here.)
$\ldots\ldots\ldots\ldots\ldots\ldots,$
$(a_1+a_1^{4pn})\,a_{4pn}=0,$
$(a_1+a_1^{4pn+1})\,a_{4pn+1}=b_{4pn+1}-(2pn+1)\,a_{2pn+1}^2,$

$$(a_1+a_1^{4pn+2})\,a_{4pn+2}=b_{4pn+2}\,,$$
$$\dots\dots\dots\dots\dots\dots,$$
$$(a_1+a_1^{4pn+n})\,a_{4pn+n}=b_{4pn+n}\,,$$
$$(a_1+a_1^{4pn+n+1})\,a_{4pn+n+1}-na_{2pn+1}a_{2pn+n+1}=b_{4pn+n+1}\,,$$

(The first term in this equation vanishes; and since $a_{2pn+1}\neq 0$, $a_{2pn+n+1}$ is uniquely determined.)

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$(a_1+a_1^{2(p+m)n+n+1})\,a_{2(p+m)n+n+1}=0,$$
$$\begin{cases}(a_1+a_1^{2(p+m)n+n+2})\,a_{2(p+m)n+n+2}=C_{2(p+m)n+n+2}\,,\\ \dots\dots\dots\dots\dots\dots\dots\dots,\\ (a_1+a_1^{2(p+m)n+3n})\,a_{2(p+m)n+3n}=C_{2(p+m)n+3n}\,,\end{cases}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$(a_1+a_1^{2(2p+m)n+n+1})\,a_{2(2p+m)n+n+1}-(2m+1)\,na_{2pn+1}a_{2(p+m)n+n+1}=C_{2(2p+m)n+n+1}\,,$$

(The first term here vanishes and $a_{2(p+m)n+n+1}$ is uniquely determined.)

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$m=0,\,1,\,2,\,\dots,$$

where the C's do not involve any a_t, $t\geqq 2(p+m)n+n+1$, and are therefore definite. The coefficients of the a's in the bracketed set of equations are different from zero, and a_t, whose coefficient $(a_1+a_1^t)$ vanishes, is determined uniquely by the equation obtained by equating the coefficients of u^{t+2pn}. It is thus seen that the determination of the a's is unique.

If b_{2pn+1} vanishes, then it is readily seen by an argument similar to that employed on p. 418 that $b_{2pn+n+1}$ must also vanish in order that a formal solution be possible, for it is found that $b_{2pn+n+1}=\dfrac{(2p-1)n}{2}\,\dfrac{a_{n+1}}{a_1}\,b_{2pn+1}$. Hence, if the arc C: $v=b_1u+b_{rn+1}u^{rn+1}+\dots+b_qu^q+\dots$ is such that r is an even positive integer, there exists a unique formal solution, while if r is an odd positive integer there exists no formal solution with the given value of a_1. If n is even, the above applies to both values which a_1 may take for a given b_1, while if n is odd, it is evident that there exists always a unique formal solution having a_1 equal to that square root of b_1 which has the largest amplitude θ, where θ is taken such that $0<\theta\leqq 2\pi$. In particular, in the case that the given curvilinear analytic arc is tangent to the segment $v=u$ at the origin, *i. e.*, $b_1=1$ and some $b_r\neq 0$, $r>1$, there exists one formal solution having $a_1=-1$ in the case that the first b_r, $r>1$, which does not vanish is the coefficient of u^{2p+1}, $p=1$, $2,\dots$, and no solution in the case that the first coefficient b_r, $r>1$, which does not vanish is the coefficient of u^{2p}, $p=1,2,\dots$. Of course, there exists one formal solution having $a_1=+1$ in either case. Hence, we have the related theorems:

THEOREM: *The configuration consisting of the two intersecting analytic arcs which in normal form are the segment in the vicinity of the origin of the*

line $v=u$ and the analytic arc $v=b_1u+b_2u^2+\ldots$, $b_1 \neq 0$, and which, if $b_1^n=1$, $n=$ a positive integer, are not pseudo-conformally equivalent to a pair of intersecting rectilinear segments, has at most one bisector, given by $v=a_1u+a_2u^2+\ldots$, corresponding to each of the two values, $+\sqrt{b}$, $-\sqrt{b}$, for a_1.

THEOREM: *The functional equation*

$f[f(u)]=b_1u+b_{mn+1}u^{mn+1}+b_{mn+2}u^{mn+2}+\ldots$, $b_1^n=1$, $b_{mn+1} \neq 0$, m, $n=1, 2, \ldots$, *has at most one solution, analytic about the origin and vanishing there, corresponding to each of the two possible values of the coefficient of the first-degree term of $f(u)$.*

Definitions. A configuration which is pseudo-conformally equivalent to the configuration consisting of the segment $v=u$ in the vicinity of the origin and the analytic arc C: $v=b_1u+b_2u^2+\ldots$, $b_1^n=1$, is called a *rational angle.* If n is the smallest such positive integer, the angle is said to be of the n-th order. If $b_1^n \neq 1$ for all positive integral values of n, the configuration is called an *irrational angle.*

Consider the configuration in the complex plane consisting of the segment $y=0$ and the real analytic arc C: $y=a_1x+a_2x^2+\ldots$. Let the configuration in the (u, v)-plane which corresponds to the above consist of the segment $v=u$ and the analytic arc $v=b_1u+b_2u^2+\ldots$. Then

$$\frac{1-a_1i}{1+a_1i}=b_1,$$

from which it follows that the angle formed by the given real analytic arc and the x-axis is commensurable with π if $b_1^n=1$, and conversely. Thus we may state the

THEOREM: *If two intersecting analytic arcs have three bisectors, then the configuration is a rational angle and is conformally equivalent to a pair of intersecting rectilinear segments.*

As noted above, the problem of the conformal equivalence of two intersecting analytic arcs to a rectilinear angle is identical with the problem of the solution of the functional equation

$$\phi[g(u)]=a\phi(u), \qquad a= \text{coefficient of first-degree term of } g(u),$$

where $g(u)$ is a given function, analytic about the origin and vanishing there, and $\phi(u)$ is unknown and required to be analytic about the origin and to vanish there. This equation is known as Schroeder's equation. We have the

THEOREM: *If the functional equation*

$$f[f(u)]=g(u),$$

where $g(u)$ is a given function, analytic about the origin and vanishing there,

and having a non-vanishing coefficient of its first-degree term, and $f(u)$ is unknown and required to be analytic about the origin and to vanish there also, has at least three solutions, then Schroeder's equation, as given above, has an infinity of solutions; in this case it is necessary that some positive integral power of the coefficient of the first-degree term of $g(u)$ be unity.

We may also state the

THEOREM: *The functional equation*

$$f[f(u)]=g(u)=b_1u+b_2u^2+\ldots., \qquad b_1\neq 0,$$

where $f(u)$ is the unknown function which is required to be analytic about the origin and to vanish there, has at most two solutions if $b_1^n\neq 1$ for all positive integral values of n; and if $b_1^n=1$, $n=$ a positive integer, there must be either no solution, one solution, two solutions or an infinite number of solutions.

§ 6. *The n-Section of a Curvilinear Angle, where $n=2^p$.*

Let the analytic arc C_2: $v=B_1u+B_2u^2+\ldots.=G(u)$ be the symmetric image of the analytic arc C_1: $v=b_1u+b_2u^2+\ldots.=g(u)$ with respect to the analytic arc C: $v=a_1u+a_2u^2+\ldots.=f(u)$. The symmetry determined by the arc C is given by

$$U=\alpha_1v+\alpha_2v^2+\ldots.=f^{-1}(v),$$
$$V=a_1u+a_2u^2+\ldots.=f(u).$$

Then we have identically

$$a_1u+a_2u^2+\ldots.=B_1(\alpha_1v+\alpha_2v^2+\ldots.)+B_2(\alpha_1v+\alpha_2v^2+\ldots.)^2+\ldots.,$$

where $u=\beta_1v+\beta_2v^2+\ldots.=g^{-1}(v)$, or, symbolically,

$$f=Gf^{-1}g$$

or

$$fg^{-1}f=G; \tag{1'}$$

and if $g(u)=u$, *i. e.*, $g=1$, (1′) becomes $ff=G$.

If $g=f_1$ and $f=f_1f_1$, then $f_1f_1f_1^{-1}f_1f_1=G$ or $f_1f_1f_1=G$, and, in general, if G represents an analytic arc C_n which is such that there exists an analytic arc C_1, given by $v=f_1(u)$, such that the image of the segment $v=u$ with respect to the arc C_1 is the arc C_2 and the image of the arc C_1 with respect to the arc C_2 is the C_3, and so on until the image of the arc C_{n-2} with respect to the arc C_{n-1} is the arc C_n, then

$$f_1^n=G.$$

If the configuration consisting of the segment $v=u$ and the analytic arc C_n is not conformally equivalent to a rectilinear angle, then the functional equation

$$\phi^2=G$$

has at most one solution, as required, corresponding to a given value of the coefficient of the first-degree term (which can only have one of two values). Consequently, if $n=2r$, $r=$ a positive integer,

$$f_1^r=\phi.$$

Obviously, the configuration consisting of the segment $v=u$ and the analytic arc C: $v=\phi(u)$ is not conformally equivalent to a rectilinear angle, and therefore if $r=2t$, $t=$ a positive integer, we have again

$$f_1^t=\psi.$$

Consequently, if $n=2^p$, f_1 is uniquely determined, if it exists at all, for a given value of the coefficient of the first-degree term.

Consider now a rational angle consisting of the analytic arcs C_0, C_1 which is not conformally equivalent to a rectilinear angle. There then exists a finite number of analytic arcs $C_2, C_3, \ldots, C_n$ through the intersection, P, of the arcs C_0, C_1, such that the arc C_n is tangent to the arc C_0 at P, *i. e.*, the arcs C_0 and C_n form a horn angle, and such that the image of the arc C_m with respect to the arc C_{m+1} is the arc C_{m+2}, $m=0, 1, \ldots, n-2$. If the set of arcs $C_1, C_2, \ldots, C_{n-1}$ is uniquely determined when the arcs C_0 and C_n and the magnitude of the angle formed by the arcs C_0 and C_1 are given, then every absolute conformal invariant of the related horn angle is an absolute conformal invariant of the given rational angle and conversely.* Kasner has shown that a horn angle has only one absolute conformal invariant of finite order besides the magnitude of the angle formed.† It follows from the preceding that a rational angle consisting of the arcs C: $v=v_0+a_1(u-u_0)+a_2(u-u_0)^2+\ldots$ and C': $v=v_0+b_1(u-u_0)+b_2(u-u_0)^2+\ldots$, not conformally equivalent to a rectilinear angle, has one and only one absolute conformal invariant of finite order besides the magnitude of the angle formed if $\left(\frac{a_1}{b_1}\right)^{2^p}=1$, $p=0, 1, 2, \ldots$. This result is established for all rational angles in the next section by a direct method.

The result stated on p. 409 may be restated as follows, if we call the arc C_{m+1} in the above the m-th symmetric image of the arc C_0 with respect to the arc C_1:

THEOREM: *A necessary and sufficient condition that a rational angle of the n-th order be conformally equivalent to a pair of intersecting rectilinear segments is that the $(n-1)$-st symmetric image of one of the analytic arcs with respect to the other be the first analytic arc.*

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* This is the "method of successive symmetries" employed by Kasner, *loc. cit.*, p. 84.

† Kasner, *loc. cit.*, p. 84.

§ 7. *Conformal Invariants of Rational Angles.*

That a rational angle has a unique absolute conformal invariant of higher finite order than the first may be shown directly by using the normal form of such an angle given on p. 417; *i. e.*, the rational angle may be taken to consist of the segment of the line $v=u$ in the vicinity of the origin and the analytic arc

$$C: \quad v=a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+a_{(2r+1)n+1}u^{(2r+1)n+1}+\dots \\ +a_{(2r+m)n+1}u^{(2r+m)n+1}+\dots+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\dots,$$

where $a_1^n=1$, $n=$ smallest such positive integer; $a_{rn+1}\neq 0$; and $m=2, 3, \dots, p-1$, $p=$ any positive integer.

Let the analytic arc C be transformed into the analytic arc

$$C': \quad V=A_1U+A_{n+1}U^{n+1}+A_{2n+1}U^{2n+1}+\dots \\ +A_{mn+1}U^{mn+1}+\dots+A_{(2r+p)n+1}U^{(2r+p)n+1} \\ +A_{(2r+p)n+2}U^{(2r+p)n+2}+\dots, \qquad m=3, 4, \dots, 2r+p-1,$$

by the non-singular pseudo-conformal transformation given by

$$U=c_1u+c_2u^2+\dots, \\ V=c_1v+c_2v^2+\dots, \qquad c_1\neq 0.$$

Then the identity follows:

$$c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\dots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\dots \\ +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\dots) \\ +c_2(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\dots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\dots \\ +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\dots)^2 \\ +\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ =A_1(c_1u+c_2u^2+\dots) \\ +A_{n+1}(c_1u+c_2u^2+\dots)^{n+1} \\ +\dots\dots\dots\dots \\ +A_{mn+1}(c_1u+c_2u^2+\dots)^{mn+1} \\ +\dots\dots\dots\dots \\ +A_{(2r+p)n+1}(c_1u+c_2u^2+\dots)^{(2r+p)n+1} \\ +A_{(2r+p)n+2}(c_1u+c_2u^2+\dots)^{(2r+p)n+2} \\ +\dots\dots\dots\dots,$$

whence the equalities

$$c_1a_1=A_1c_1 \quad \text{or} \quad a_1=A_1, \\ c_2a_1^2=A_1c_2 \quad \text{or} \quad c_2=0, \\ \dots\dots\dots\dots, \\ c_na_1^n=A_1c_n \quad \text{or} \quad c_n=0, \\ c_{n+1}a_1^{n+1}=A_1c_{n+1}+A_{n+1}c_1^{n+1}, \text{ or } c_{n+1} \text{ is arbitrary and } A_{n+1}=0.$$

Assuming that $c_s=0$, $s<kn+1$ and $\neq jn+1$, $k<r$, $j=1, 2, \ldots, k$, and that $A_{jn+1}=0$, it follows that

$$c_{kn+l}a_1^{kn+l}=A_1c_{kn+l}, \qquad l=2, 3, \ldots, n,$$

and

$$c_{(k+1)n+1}a_1^{(k+1)n+1}=A_1c_{(k+1)n+1}+A_{(k+1)n+1}c^{(k+1)n+1}.$$

Thus, in general, $c_s=0$, $s<rn+1$ and $\neq jn+1$, $j=1, 2, \ldots$, and $A_{kn+1}=0$, $k=1, 2, \ldots, (r-1)$; c_{kn+1}, $k=1, 2, \ldots, (r-1)$, are arbitrary thus far.

Further, we have, on equating the coefficients of u^{rn+1},

$$c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=A_1c_{rn+1}+A_{rn+1}c_1^{rn+1}$$

or

$$c_1^{rn}A_{rn+1}=a_{rn+1}, \quad i.\ e., \quad A_{rn+1}\neq 0.$$

Hence, we may take the pseudo-conformal transform of the analytic arc C to be the analytic arc

$$C'': \quad V=a_1U+A_{rn+1}U^{rn+1}+A_{2rn+1}U^{2rn+1}+\ldots+A_{(2r+m)n+1}U^{(2r+m)n+1}+\ldots$$
$$+A_{(2r+p)n+1}U^{(2r+p)n+1}+A_{(2r+p)n+2}U^{(2r+p)n+2}+\ldots,$$
$$m=1, 2, \ldots, p-1.$$

Now, let the non-singular pseudo-conformal transformation, given by

$$U=c_1u+c_2u^2+\ldots,$$
$$V=c_1v+c_2v^2+\ldots, \qquad c_1\neq 0,$$

transform the analytic arc C into the analytic arc C''; then we have the identity

$$\begin{aligned}
&c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad\qquad +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)\\
&+c_2(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad\qquad +a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^2\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad =a_1(c_1u+c_2u^2+\ldots)\\
&\quad +A_{rn+1}(c_1u+c_2u^2+\ldots)^{rn+1}\\
&\quad +A_{2rn+1}(c_1u+c_2u^2+\ldots)^{2rn+1}\\
&\quad +\ldots\ldots\ldots\ldots\\
&\quad +A_{(2r+m)n+1}(c_1u+c_2u^2+\ldots)^{(2r+m)n+1}\\
&\quad +\ldots\ldots\ldots\ldots\\
&\quad +A_{(2r+p)n+1}(c_1u+c_2u^2+\ldots)^{(2r+p)n+1}\\
&\quad +A_{(2r+p)n+2}(c_1u+c_2u^2+\ldots)^{(2r+p)n+2}\\
&\quad +\ldots\ldots\ldots\ldots,
\end{aligned}$$

whence it follows that

$$c_1a_1=a_1c_1,$$
$$c_2a_1^2=a_1c_1,$$
$$\ldots\ldots,$$
$$c_{rn}a_1^{rn}=a_1c_{rn},$$
$$c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=a_1c_{rn+1}A_{rn+1}c_1^{rn+1},$$

from which it follows that $c_s=0$, $s\neq kn+1$ and $<rn+1$, while c_{kn+1}, $k=0, 1, 2, \ldots, r$, are arbitrary thus far (of course, $c_1\neq 0$). Further, assuming that $c_s=0$, $s<(r+k)n+1$ and $\neq jn+1$, $k<r-1$, $j=0, 1, 2, \ldots$, we have upon equating coefficients of $u^{(r+k)+l}$, $l=2, 3, \ldots, n$,

$$c_{(r+k)n+l}a_1^{(r+k)n+l}=a_1c_{(r+k)n+l}.$$

Consequently, $c_s=0$, $s<(2r+p)+1$ and $\neq jn+1$, $j=0, 1, 2, \ldots$.

The last identity then reduces to

$$\begin{aligned}
&c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)\\
&+c_{n+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{n+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+c_{mn+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{mn+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad=a_1(c_1u+c_{n+1}u^{n+1}+\ldots+c_{mn+1}u^{mn+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)\\
&\quad+A_{rn+1}(c_1u+c_{n+1}u^{n+1}+\ldots+c_{mn+1}u^{mn+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{rn+1}\\
&\quad+A_{2rn+1}(c_1u+c_{n+1}u^{n+1}+\ldots+c_{mn+1}u^{mn+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{2rn+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad+A_{(2r+m)n+1}(c_1u+c_{n+1}u^{n+1}+\ldots+c_{mn+1}u^{mn+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+m)n+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad+A_{(2r+p)n+1}(c_1u+c_{n+1}u^{n+1}+\ldots+c_{mn+1}u^{mn+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+p)n+1}\\
&\quad+A_{(2r+p)n+2}(c_1u+c_{n+1}u^{n+1}+\ldots+c_{mn+1}u^{mn+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+p)n+2}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

From this identity the following equalities result:

$$\begin{gathered}
c_1a_1=a_1c_1,\\
c_{n+1}a_1^{n+1}=a_1c_{n+1},\\
\ldots\ldots\ldots\ldots,\\
c_{(r-1)n+1}a_1^{(r-1)n+1}=a_1c_{(r-1)n+1},\\
c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=a_1c_{rn+1}+A_{rn+1}c_1^{rn+1},\\
(n+1)c_{n+1}a_1^na_{rn+1}+c_{(r+1)n+1}a_1^{(r+1)n+1}=a_1c_{(r+1)n+1}+(rn+1)A_{rn+1}c_1^{rn}c_{n+1},
\end{gathered}$$

or

$$(r-1)na_{rn+1}c_{n+1}=0,\quad i.\ e.,\quad c_{n+1}=0.$$

Assuming that $c_{kn+1}=0$, $0<k<r-1$, we get, upon equating coefficients of $u^{(r+k+1)n+1}$,

$$[(k+1)n+1]c_{(k+1)n+1}a_1^{(k+1)n}a_{rn+1}+c_{(r+k+1)n+1}a_1^{(r+k+1)n+1}$$
$$=a_1c_{(r+k+1)n+1}+(rn+1)A_{rn+1}c_1^{rn}c_{(k+1)n+1},$$

or

$$(r-k-1)na_{rn+1}c_{(k+1)n+1}=0, \quad i.\ e., \quad c_{(k+1)n+1}=0, \qquad k=0, 1, 2, \ldots, r-2.$$

The above identity may now be written

$$\begin{aligned}
&c_1(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)\\
&+c_{rn+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{rn+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+c_{(r+h)n+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(r+h)n+1}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+c_{(2r+p)n+1}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+p)n+1}\\
&+c_{(2r+p)n+2}(a_1u+a_{rn+1}u^{rn+1}+a_{2rn+1}u^{2rn+1}+\ldots+a_{(2r+m)n+1}u^{(2r+m)n+1}+\ldots\\
&\qquad+a_{(2r+p)n+1}u^{(2r+p)n+1}+a_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+p)n+2}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad=a_1(c_1u+c_{rn+1}u^{rn+1}+\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)\\
&\quad+A_{rn+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{rn+1}\\
&\quad+A_{2rn+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{2rn+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad+A_{(2r+m)n+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+m)n+1}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\quad+A_{(2r+p)n+1}(c_1u+c_{rn+1}u^{rn+1}+\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+p)n+1}\\
&\quad+A_{(2r+p)n+2}(c_1u+c_{rn+1}u^{rn+1}+\ldots+c_{(r+h)n+1}u^{(r+h)n+1}+\ldots\\
&\qquad+c_{(2r+p)n+1}u^{(2r+p)n+1}+c_{(2r+p)n+2}u^{(2r+p)n+2}+\ldots)^{(2r+p)n+2}\\
&\quad+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

$$h=1, 2, \ldots, r+p-1; \; m=1, 2, \ldots, p-1.$$

The following equalities result from this identity:

(α) $c_1a_{rn+1}+c_{rn+1}a_1^{rn+1}=a_1c_{rn+1}+A_{rn+1}c_1^{rn+1}$,

$$(\beta)\quad c_1 a_{2rn+1} + (rn+1) c_{rn+1} a_1^{rn} a_{rn+1} + c_{2rn+1} a_1^{2rn+1}$$
$$= a_1 c_{2rn+1} + (rn+1) A_{rn+1} c_1^{rn} c_{rn+1} + A_{2rn+1} c_1^{2rn+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$(\gamma)\quad c_1 a_{(2r+m)n+1} + [(r+m)n+1] c_{(r+m)n+1} a_1^{(r+m)n} a_{rn+1} + \ldots\ldots + c_{(2r+m)n+1} a_1^{(2r+m)n+1}$$
$$= a_1 c_{(2r+m)n+1} + (rn+1) A_{rn+1} c_1^{rn} c_{(r+m)n+1} + \ldots\ldots + A_{(2r+m)n+1} c_1^{(2r+m)n+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$m = 1, 2, \ldots\ldots, p,$$

where the terms omitted in the last equality do not involve $c_{(r+m)n+1}$ or $A_{(2r+m)n+1}$.* From (α) and (β) it follows that

$$\frac{a_{2rn+1}}{a_{rn+1}^2} = \frac{A_{2rn+1}}{A_{rn+1}^2}.$$

Further, from (γ) we have

$$c_1^{(2r+p)n+1} A_{(2r+p)n+1} = pna_{rn+1} c_{(r+p)n+1} + P(c_1, \ldots\ldots, c_{(r+p-1)n+1}, A_1, \ldots\ldots, A_{(r+p-1)n+1}).$$

Since $c_{(r+p)n+1}$ is arbitrary, and since $pna_{rn+1} \neq 0$, $A_{(2r+p)n+1}$ is arbitrary and, therefore, there exists no absolute pseudo-conformal invariant of order $(2r+p)$; *i. e.*, since p is any positive integer, there is no other higher invariant of finite order besides a_1 and $\dfrac{a_{2rn+1}}{a_{rn+1}^2}$.

We have, in the complex plane, the

THEOREM: *Any rational angle not conformally equivalent to a pair of intersecting rectilinear segments has just one absolute conformal invariant of finite order besides the magnitude of the angle.*

§ 8. *Products of Symmetries.*

Consider the symmetry determined by the analytic arc C_1: $v = v_0 + a_1(u - u_0) + a_2(u - u_0)^2 + \ldots\ldots$, and thus given by

$$S_1:\quad \begin{cases} v = v_0 + a_1(U - u_0) + a_2(U - u_0)^2 + \ldots\ldots, \\ V = v_0 + a_1(u - u_0) + a_2(u - u_0)^2 + \ldots\ldots, \end{cases}$$

and also the reflection with respect to the line $v = b_0 + b_1 u$,

$$S:\quad \begin{cases} U = \dfrac{b_0}{b_1} + \dfrac{v}{b_1}, \\ V = b_0 + b_1 u; \end{cases}$$

then $S_1 = TST^{-1}$, where T is a non-singular pseudo-conformal transformation which transforms the arc C_1 into a segment of the line $v = b_0 + b_1 u$.

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* By equating the coefficients of $u^{(r+h)n+1}$, $h = 1, 2, \ldots\ldots, n-1$, only the trivial equalities

$$c_{(r+h)n+1} a_1^{(r+h)n+1} = a_1 c_{(r+h)n+1}$$

are obtained.

THEOREM: *If S_1 is a symmetry, then $T^{-1}S_1T$ is a symmetry, where T is a non-singular conformal transformation which leaves a point of the determining arc of S_1 invariant.*

For $S_1=T_1ST_1^{-1}$, where T_1 is a non-singular conformal transformation which transforms the determining arc of S_1 into a rectilinear segment through the invariant point and S is the reflection with respect to this segment. Then $T^{-1}S_1T=T^{-1}T_1ST_1^{-1}T=(T_1^{-1}T)^{-1}S(T_1^{-1}T)$, which is a symmetry.

We have also the following simple

THEOREM: *If a non-singular conformal transformation T' which leaves a point invariant is the product of two symmetries S_1, S_2 which leave the same point invariant, then the transform of T' by a non-singular conformal transformation T is the product of the reflection S with respect to a line through the invariant point and a symmetry S_3, where T and S_3 are determined as below.*

For

$$T^{-1}T'T=T^{-1}S_1S_2T=T^{-1}S_1TT^{-1}S_2T=SS_3,$$

where T is a non-singular conformal transformation which transforms the analytic arc which determines the symmetry S_1 into a segment in the vicinity of the invariant point of the line referred to above and S is the reflection with respect to this line.

If the invariant point is the origin, SS_3 is of the form

$$\begin{aligned} U&=a_1u+a_2u^2+\dots, \\ V&=b_1v+b_2v^2+\dots, \qquad a_1\neq 0, \end{aligned}$$

where the b's are the coefficients of the power series which is the inverse of the power series whose coefficients are the a's, when S is taken as the reflection with respect to the line $v=u$. Again, let S_1 and S_2 be given by

$$S_1:\quad \begin{cases} U=b_1v+b_2v^2+\dots=f^{-1}(v), \\ V=a_1u+a_2u^2+\dots=f(u), \qquad a_1\neq 0, \end{cases}$$

$$S_2:\quad \begin{cases} U=d_1v+d_2v^2+\dots=g^{-1}(v), \\ V=c_1u+c_2u^2+\dots=g(u), \qquad c_1\neq 0; \end{cases}$$

then $T'=S_1S_2$ is given by

$$\begin{aligned} U&=g^{-1}[f(u)], \\ V&=g[f^{-1}(v)]. \end{aligned}$$

It is seen that the product of two symmetries which leave the origin invariant is by no means the general non-singular conformal transformation which leaves the origin fixed. In particular, if both symmetries are real, then the real conformal transformation, expressed as a power series of a complex variable, which

is their product has unity as the modulus of the coefficient of the first-degree term.

Consider the product of three symmetries S_1, S_2, S_3, determined by three analytic arcs passing through the origin: $S'=S_1S_2S_3$. S' is a reversed non-singular conformal transformation, but not, in general, a symmetry. Let $S=T^{-1}S_1T$, where S is the reflection with respect to the line $u=v$ and T a non-singular conformal transformation which transforms the determining arc of S_1 into the segment in the vicinity of the origin of the line $v=u$, leaving the origin invariant. Then

$$T^{-1}S'T=T^{-1}S_1S_2T=T^{-1}S_1TT^{-1}S_2TT^{-1}S_3T,$$

or $S''=SS_2'S_3'$, where S'', S_2', S_3' are symmetries leaving the origin invariant. Let S_2' be given by

$$U=f^{-1}(v),$$
$$V=f(u)=a_1u+a_2u^2+\ldots, \qquad a_1\neq 0.$$

and S_3' by

$$U=g^{-1}(v),$$
$$V=g(u)=b_1u+b_2u^2+\ldots, \qquad b_1\neq 0.$$

Then $SS_2'S_3'$ is given by

$$U=g^{-1}[f(v)],$$
$$V=g[f^{-1}(u)].$$

Hence, in order that S'' be a symmetry, it is necessary and sufficient that

$$g^{-1}f=fg^{-1},$$

or

$$fg=gf,$$

whence the

THEOREM: *In order that the product of three symmetries determined by three analytic arcs intersecting in the origin be a symmetry, it is necessary and sufficient that the transforms of the second and third factors by a non-singular conformal transformation which transforms the determining analytic arc of the first symmetry into the segment $v=u$, and leaves the origin invariant, have determining analytic arcs which are such that the substitutions represented by the equations of these arcs are commutative.*

Since any symmetry may be transformed by a translation into a symmetry determined by an arc through the origin, the above theorem may be readily modified to apply in the case that the invariant point is not the origin.

The last theorem implies the existence of an infinite set of necessary conditions involving the coefficients of the power series which represent the analytic arcs which determine the symmetries. The most simple necessary condi-

tion that the product of three real symmetries which leave a point invariant be a symmetry in the real plane has an intersecting geometrical interpretation. Taking the origin as the invariant point, we have as the three symmetries

$$\begin{aligned} S_1: &\quad Z=\alpha_1\bar{z}+\alpha_2\bar{z}^2+\dots, \\ S_2: &\quad Z=\beta_1\bar{z}+\beta_2\bar{z}^2+\dots, \\ S_3: &\quad Z=\gamma_1\bar{z}+\gamma_2\bar{z}^2+\dots. \end{aligned}$$

Then $S_1S_2S_3$ is given by

$$Z=\alpha_1\bar{\beta}_1\gamma_1\bar{z}+(\alpha_1^2\bar{\beta}_1^2\gamma_2+\alpha_1^2\bar{\beta}_2\gamma_1+\alpha_2\bar{\beta}_1\gamma_1)\bar{z}^2+\dots.$$

Let

$$\begin{aligned} C_1: &\quad y=a_1x+a_2x^2+\dots, \\ C_2: &\quad y=b_1x+b_2x^2+\dots, \\ C_3: &\quad y=c_1x+c_2x^2+\dots \end{aligned}$$

be the analytic arcs which determine S_1, S_2 and S_3 respectively. Then

$$\alpha_1=\frac{1+a_1i}{1-a_1i},\qquad \beta_1=\frac{1+b_1i}{1-b_1i},\qquad \gamma_1=\frac{1+c_1i}{1-c_1i},$$
$$\alpha_2=\frac{2a_2i}{(1-a_1i)^3},\quad \beta_2=\frac{2b_2i}{(1-b_1i)^3},\quad \gamma_2=\frac{2c_2i}{(1-c_1i)^3}.$$

Then

$$\frac{2d_2i}{(1-a_1i)^3(1-b_1i)^3(1-c_1i)^3}=\alpha_1^2\bar{\beta}_1^2\gamma_2+\alpha_1^2\bar{\beta}_2\gamma_1+\alpha_2\bar{\beta}_1\gamma_1,$$

where d_2 is the coefficient of the second-degree term of the power series which represents the analytic arc which determines the resulting symmetry and is, of course, real. On separating into real and imaginary parts, it is found that

$$\frac{a_2(b_1-c_1)}{1+a_1^2}+\frac{b_2(c_1-a_1)}{1+b_1^2}+\frac{c_2(a_1-b_1)}{1+c_1^2}=0,$$

or

$$\Gamma_1\sin(C_2, C_3)+\Gamma_2\sin(C_3, C_1)+\Gamma_3\sin(C_1, C_2)=0,$$

where Γ_1, Γ_2, Γ_3 denote the curvatures at the origin of the arcs C_1, C_2, C_3 respectively, and $\sin(C_2, C_3)$ denotes the sine of the angle between the arcs C_2 and C_3, etc. We state the

THEOREM: *If the product of three real symmetries determined by arcs through a common point is also a symmetry, then the sum of the three products of the curvature at the common point of the arc determining one factor by the sine of the angle between the two arcs determining the other symmetries, and oriented according to the order of the factors in the product, is equal to zero.*

*The Non-homogeneous Differential Equation of Parabolic Type.**

By Griffith C. Evans.

1. *Introduction.*

It is characteristic of the partial differential equation of parabolic type that on one side it partakes much of the nature of the equation of elliptic type, and on the other, that of hyperbolic type. To be more precise, if we are considering an equation in two variables, x and t, the solution is something like an harmonic function with respect to the variable x, in regard to which the differentiation is of the second order, but more like a solution of an equation of hyperbolic type with respect to the variable t. For instance, if in the equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=f(x,t) \tag{1}$$

the function $f(x, t)$ is analytic in the variable x in the neighborhood of the point (x_1, t_1), every solution u of equation (1), which is continuous together with its derivatives $\partial u/\partial x$, $\partial^2 u/\partial x^2$, $\partial u/\partial t$, is an analytic function of x in that same neighborhood.† We know, moreover, that a regular solution of equation (1) exists, under suitable boundary conditions, provided that $f(x, t)$ and its derivative in regard to x are finite and continuous.‡ In the corresponding equation of elliptic type a solution exists if the function with its first derivatives with respect to both variables are finite and continuous (Gauss).

We know that the second-order derivatives of the definite integral which represents the potential function exist in the neighborhood of a point P_1, if the density ρ at any point P in that neighborhood satisfies the condition

$$|\rho-\rho_1|\leq A r^{\nu},$$

where A is a constant, r is the distance from P_1 to P, and ν is a positive number

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† E. E. Levi, "Sull' equazione del calore," *Annali di Matematica*, Vol. XIV (1907–1908), p. 239.

‡ W. A. Hurwitz, "Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung," p. 85, Göttingen, 1910.

not zero (Kronecker). This is not a necessary condition, of course, but it is one which is convenient in actual use. If we look for a similar condition with respect to the definite integral u, which is used to represent the principal solution of equation (1), and ask when the derivatives $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ exist, we find Levi's theorem that there are regular solutions of (1), under given boundary conditions, provided that in the neighborhood of any point (x_1, t_1), $f(x, t)$ satisfies a condition of the form

$$\left| \frac{f(x,t)-f(x_1, t_1)}{[(x-x_1)^2+(t-t_1)^2]^\nu} \right| < N.\text{*}$$

If we write the expression analogous to the potential function, for equation (1), the derivatives of that quantity exist in the neighborhood of (x_1, t_1) if the function $f(x, t)$ is finite and integrable in the given region, and at (x_1, t_1) satisfies Levi's condition. But this condition is symmetrical in x and t, and therefore misses in part the essential nature of the parabolic equation. We are thus led to look for an asymmetrical condition more analogous to Kronecker's but not so much like it.

2. *The Principal Solution of Equation* (1).

Let us consider a region ${}_{t_0}R_{t_1}$ (notation of Hurwitz) bounded on the left by the line $t=t_0$, on the right by the line $t=t_1$, above by the curve $x=\xi_2(t)$ and below by the curve $x=\xi_1(t)$. The functions ξ_1 and ξ_2 are to be continuous with their first derivatives, and are to have only a finite number of maxima and minima in the interval under consideration. Moreover, we are to have $\xi_2(t)>\xi_1(t)$, for $t \geq t_0$.

We shall say that the function $u(x, t)$ is regular at a point (x_0, t_0) if u and $\partial u/\partial x$ are finite and continuous in the neighborhood of (x_0, t_0), and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ exist at (x_0, t_0). We shall say that $u(x, t)$ is regular in a region ${}_{t_0}R_{t_1}$ if it is regular at every point inside the region, continuous with its derivative $\partial u/\partial x$ on the boundary, and if $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are linearly integrable in regard to x and t respectively. In this paper we shall not deal extensively with regularity at a point; we shall, however, touch on the matter again, in the last section.

Making use of the notation (Levi)

$$h_{\alpha,\beta}(x, t \mid x_1, t_1) = \frac{(x-x_1)^\alpha}{(t-t_1)^\beta} e^{-\frac{(x-x_1)^2}{4(t-t_1)}}, \tag{2}$$

* E. E. Levi, *loc. cit.*, p. 239.

we have the following theorem for the existence of the principal solution of equation (1).

THEOREM I. *The function*

$$u(x,t)=\frac{1}{2\sqrt{\pi}}\iint_{({}_{t_0}R_{t_1})} h_{0,\frac{1}{2}}(\xi,\tau\,|\,x,t)\,f(\xi,\tau)\,d\xi\,d\tau \tag{3}$$

is regular in the neighborhood of the point (x_1, t_1), and satisfies there the equation (1), *if $f(x,t)$ is finite and integrable in ${}_{t_0}R_{t_2}\,(t_2>t_1)$, is continuous in the neighborhood of (x_1, t_1), and satisfies the condition*

$$|f(x,t)-f(x',t)|\leq A\,|x-x'|^{\nu}, \tag{4}$$

where (x,t) and (x',t') are any two points in that neighborhood, and A and ν are positive constants, not zero.

3. *The Functions u and $\partial u/\partial x$.*

From the finiteness and integrability of $f(x,t)$ follows at once the continuity of u and the existence and continuity of $\partial u/\partial x$, within and on the boundary of R; for we may differentiate under the integral sign once in regard to x.* But we may not differentiate twice in regard to x or once in regard to t, under the integral sign, and we must therefore establish the existence of these derivatives in other ways.

4. *The Function $\partial^2 u/\partial x^2$.*

Let us consider now the point (x_1, t_1) within R, and divide the region R into two parts, of which one shall be a small open rectangle ${}_a r_t$, wholly within the above-mentioned neighborhood of (x_1, t_1), and the other the rest of ${}_{t_0}R_t$. If we separate the integral in (3) into the corresponding two parts, denoting by u' the result of extending the integral over r, and by u'' the result due to the rest of R, we shall have for $\partial^2 u''/\partial x^2$ a continuous function of x and t. It remains then to investigate u'.

For this purpose let us write

$$\left.\begin{aligned} f(x,t)&=f(x',t)+\phi(x,t,x'),\\ u'(x,t)&=u_1(x,t,x')+u_2(x,t,x'),\\ u_1(x,t,x')&=\frac{1}{2\sqrt{\pi}}\iint_{({}_a r_t)} h_{0,\frac{1}{2}}(\xi,\tau\,|\,x,t)\,f(x',\tau)\,d\xi\,d\tau, \end{aligned}\right\} \tag{5}$$

thus defining the functions $\phi(x,t,x')$, $u_1(x,t,x')$, $u_2(x,t,x')$. We shall have, then,

* E. E. Levi, *loc. cit.*, p. 234.

$$u_2(x, t, t') = \frac{1}{2\sqrt{\pi}} \iint_{(a\,r\,t)} h_{0,\frac{1}{2}}(\xi, \tau | x, t)\, \phi(\xi, \tau, x')\, d\xi\, d\tau. \tag{6}$$

To u_1 we may apply the theorem of Hurwitz already mentioned, from which it follows that if the function $f(x, t)$ is finite and integrable in R, and continuous at (x_1, t), with its first derivative in regard to x, then the function $u(x, t)$, defined by (3), is regular at and in the neighborhood of (x_1, t_1). Therefore, since the derivative, with respect to x, of $f(x', t)$ is continuously zero, the function u_1, defined by (5), is regular at (x', t'). In particular, we have

$$\frac{\partial^2 u_1(x, t, x')}{\partial x^2} = \frac{1}{4\pi} \int_a^{t'} [h_{1,\frac{1}{2}}(\xi_2(\tau), \tau | x, t) - h_{1,\frac{1}{2}}(\xi_1(\tau), \tau | x, t)]\, f(x', \tau)\, d\tau, \tag{7}$$

a function which inside r is continuous in x, t and x'.*

Let us now actually calculate $\partial^2 u_2/\partial x^2 = \lim\limits_{\Delta x = 0} \Delta[\partial u_2/\partial x]/\Delta x$, when $x = x'$ and $t = t'$. To do this we may follow closely the analysis of E. E. Levi,† replacing his condition by ours, and obtain the result

$$\left(\frac{\partial^2 u_2(x, t, x')}{\partial x^2}\right)_{\substack{x=x'\\ t=t'}} = -\frac{1}{2\pi} \iint_{(a\,r\,t)} h_{0,\frac{1}{2}}(\xi, \tau | x', t')\, \phi(\xi, \tau, x')\, d\xi\, d\tau$$
$$+ \frac{1}{4\pi} \iint_{(a\,r\,t)} h_{2,\frac{1}{2}}(\xi, \tau | x', t')\, \phi(\xi, \tau, x')\, d\xi\, d\tau. \tag{8}$$

The presence of the ϕ makes the above integrals convergent. Each of them, moreover, represents a continuous function of x' and t' in the neighborhood of (x_1, t_1), as we see by dividing r into two parts, one of them being a very small neighborhood of (x_1, t_1). For, if we form the expression (8) for (x'', t''), and subtract from it its value for (x', t'), keeping both points well within the very small neighborhood, we can make the contributions due to this neighborhood as small as we please, on account of the uniformity of the condition (4); then we can make the rest of the expression as small as we please by taking the point (x'', t'') close enough to the point (x', t').

We see from the results just obtained that the function

$$\left(\frac{\partial^2 u'(x, t)}{dx^2}\right)_{x=x',\, t=t'} = \left(\frac{\partial^2 u_1(x, t, x')}{\partial x^2}\right)_{x=x',\, t=t'} + \left(\frac{\partial^2 u_2(x, t, x')}{\partial x^2}\right)_{x=x',\, t=t'} \tag{9}$$

represents a continuous function of x' and t' in the neighborhood of the point (x_1, t_1), and is therefore linearly integrable in regard to x' and t' in that neighborhood. Accordingly it follows that we have established those same properties for the function $\partial^2 u/\partial x^2$.

* W. A. Hurwitz, *loc. cit.*, p. 86. † E. E. Levi, *loc. cit.*, p. 236.

5. *The Function $\partial u/\partial t$.*

We might in the same way, by means of slightly more complicated expressions, investigate the function $\partial u/\partial t$. It is more convenient, however, to make use of a different method, which is more general, and adopts an entirely different point of view. To do this we shall need to mention one or two introductory definitions and theorems.

A *standard curve* is a closed curve composed of a finite number of pieces, which does not cut itself at any point; along each piece the coordinates of a point are given by two functions $\phi(q)$ and $\psi(q)$, continuous with their first derivatives, throughout a finite interval for q. In this interval $\phi'(q)$ and $\psi'(q)$ shall not vanish together; and neither of them shall vanish at more than a finite number of points unless it vanishes throughout the whole interval. We see, then, that a standard curve can not be cut by a vertical or horizontal straight line in more than a finite number of points unless the straight line includes itself a portion of the curve.

We shall say that a standard curve approaches a point uniformly if, when we are given a circle with center at the point and radius arbitrarily small, the standard curve becomes and remains entirely within the given circle.

We have the following general theorem: *If, in the x, y plane, s is a standard curve, enclosing a region σ, which approaches a point P uniformly, and the functions u and $\partial u/\partial x$ are continuous throughout the neighborhood of P, then*

$$\frac{\partial u}{\partial x} = \lim \frac{1}{\sigma}\left[\int_s u\, dy\right]. \tag{10}$$

On the other hand, if merely u is continuous throughout the neighborhood of P, and lim $[\int_s u\, dy]/\sigma$ exists when σ approaches P uniformly, then $\partial u/\partial x$ exists at P and is equal to that limit.

A similar theorem applies, of course, to

$$\frac{\partial u}{\partial y} = -\lim \frac{1}{\sigma}\left[\int_s u\, dx\right]. \tag{11}$$

The first part of this theorem is proved by integrating $\partial u/\partial x$ over the region σ, and the second part may be obtained by choosing for s a rectangle whose side Δy is conveniently small with reference to its side Δx.

Besides this general theorem, we need the following result which applies particularly to the parabolic equation: *If $f(x, t)$ is finite and continuous within the region ${}_{t_0}R_t$, the function $u(x, t)$, defined by* (3), *is continuous with its derivative in regard to x within and on the boundary of the region, and satisfies the equation*

$$\int_s \left[u(x,t)\,dx + \frac{\partial u(x,t)}{\partial x}\,dt \right] = \iint_\sigma f(x,t)\,dx\,dt, \tag{12}$$

where s is any standard curve entirely within the region (the axis of t being horizontal, as heretofore).* From this theorem it follows that if $f(x,t)$ is finite and integrable in R and continuous in the neighborhood of a point P within R, and s is a standard curve which approaches P uniformly, then

$$\lim \frac{1}{\sigma}\left[\int_s \left[u(x,t)\,dx + \frac{\partial u(x,t)}{\partial x}\,dt \right]\right] = f(x,t). \tag{13}$$

Let us apply these results to our present problem. Since as we have already shown, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ exist and are continuous throughout the neighborhood of (x_1, y_1), it follows that as s approaches P uniformly, P being in that neighborhood, we have the relation

$$\frac{\partial^2 u}{\partial x^2} = -\lim \frac{1}{\sigma}\left[\int_s \frac{\partial u}{\partial x}\,dt\right]. \tag{14}$$

Hence, from (13), $\lim\,[\int_s u\,dx]/\sigma$ exists as s approaches P uniformly, and, since u is continuous in the neighborhood of P, our general theorem tells us that $\partial u/\partial t$ exists at P and is given by the equation

$$\frac{\partial u}{\partial t} = \lim \frac{1}{\sigma}\,[\int_s u\,dx]. \tag{15}$$

If now we substitute (14) and (15) in (13), the equation (13) becomes merely the differential equation (1), and since by that equation $\partial u/\partial t$ may be written, in the neighborhood of (x_1, t_1), as the sum of two continuous functions of x and t, it is itself continuous in that neighborhood. Theorem I is therefore completely established.

6. *Regularity at a Point.*

Regularity throughout a region is necessary if we wish to apply Green's theorem. But if we are interested only in regularity at a point, we can get less restrictive conditions for it than those expressed in Theorem I. A sufficient condition for regularity at a point is given by the following theorem:

THEOREM II. *The function given by equation* (3) *is regular at the point* (x_1, t_1), *and satisfies there the equation* (1), *if* $f(x,t)$ *is finite and integrable*

* G. C. Evans, "On the Reduction of Integrodifferential Equations," *Transactions of the American Mathematical Society*, Vol. XV (1914), p. 477.

in ${}_{t_0}R_{t_2}$ $(t_2>t_1)$, and in the neighborhood of (x_1, t_1) is continuous and satisfies the condition

$$|f(x,t)-f(x_1,t)|\leq N|x-x_1|^{\nu}, \tag{4'}$$

where N aud ν are positive constants, not zero.

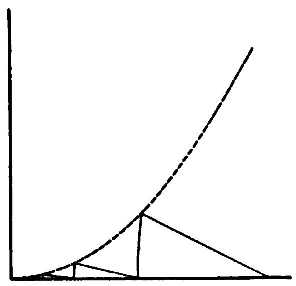
This theorem may be established directly by calculating $\partial^2u/\partial x^2$ and $\partial u/\partial t$ at the point (x_1, t_1). Or section 5 may be applied, provided that the curves s be throughout restricted to rectangles with sides parallel to the axes, and the general theorem there mentioned be extended to the following: If, along the line $x=x_1$, in the neighborhood of the point (x_1, y_1), we have, uniformly,

$$\lim_{\Delta x=0}\frac{u(x_1+\Delta x, y)-u(x_1, y)}{\Delta x}=\frac{\partial u}{\partial x},$$

then at (x_1, y_1), if s approaches (x_1, y_1) uniformly, we have

$$\frac{\partial u}{\partial x}=\lim\frac{1}{\sigma}\left[\int_s u\,dy\right].$$

The condition (4′) might hold for every point of the region, and the condition (4) still fail to hold. Similarly, Levi's condition might hold for every point of a region,* and conditions (4′) and (4) still fail to hold, although for most purposes these conditions are more general than Levi's. In other words, Levi's condition and the condition (4′) are such that they do not necessarily hold uniformly even if they hold at every point. An example will make this clear.



Consider a function $f(x)$, defined as follows: Divide the interval $(0, 1)$ by the points 1/2, 1/4, 1/8, ..., and in each sub-interval draw the curve $y=2(x-p)^p$, where p is 1/2, 1/4, 1/8, ..., connecting the point where each of these curves cuts the curve $y=x^2$ to the next point of division to the right with a straight line. Let $f(x)$ be the curve made up of these portions

* In the development referred to above, Levi considers only regularity at a point. If Levi's condition is made uniform, his theorem becomes a particular case of Theorem I.

of straight lines and curves $y=2(x-p)^\nu$, when $x>0$; when $x\leq 0$, let $f(x)=0$. We notice now that for every point in the interval $0\leq x\leq 1$ we have a condition of the form

$$|f(x)-f(x_1)|\leq N|x-x_1|^\nu,$$

where N and ν are positive constants, not zero, depending in value on the value of x_1. We can not find a positive ν which will hold for all points of the interval, because if we let x_1 approach 0 by coinciding with the points 1/2, 1/4, 1/8, ..., the value of ν approaches 0.

THE RICE INSTITUTE, *July*, 1914.

*On Properties of the Solutions of Linear q-Difference Equations with Entire Function Coefficients.**

By Thomas E. Mason.

§ 1. *Introduction.*

The purpose of this paper is to study the existence of entire function solutions of linear n-th order homogeneous and non-homogeneous q-difference equations with entire function coefficients, and to study the character of such solutions, when they do exist, with respect to the relation of the "ordre apparent"† of the solutions and the coefficients. Entire function solutions are proved to exist, whenever there exist formal expansions in ascending powers of x satisfying the equation. In certain other cases it is shown that there exist solutions made up of entire functions multiplied by rather simple functions which are not entire. The "ordre apparent" of the solutions of equations with coefficients of finite order is not greater than the greatest "ordre apparent" among the coefficients. In the case of equations with coefficients of infinite, but not transfinite, order, the "Vergleichsfunktion"‡ of the solutions is not greater than the greatest among the "Vergleichsfunktionen" of the coefficients.

We shall write our equations in the forms

$$G(q^n x) + B_1(x) G(q^{n-1}x) + B_2(x) G(q^{n-2}x) + \dots + B_n(x) G(x) = B(x), \quad |q| > 1, \tag{1}$$

and

$$G(q^{-n} x) + B_1(x) G(q^{-(n-1)} x) + B_2(x) G(q^{-(n-2)} x) + \dots + B_n(x) G(x) = B(x), \quad |q| < 1,$$

where

$$B_i(x) = \sum_{j=0}^{\infty} b_{ij} x^j \quad \text{and} \quad B(x) = \sum_{j=0}^{\infty} b_j x^j,$$

and where $B(x) \equiv 0$ for the homogeneous equation. If we replace q^{-1} in the equation above for $|q| < 1$ by q', we have a new equation where $|q'| > 1$. In what follows we shall consider only the case $|q| > 1$, since the case $|q| < 1$ reduces so readily to this.

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* Read before the American Mathematical Society at Chicago, April, 1914.

† See Borel, "Leçons sur les fonctions entières," p. 74.

‡ See Kraft, "Über ganze transzendente Funktionen von unendlicher Ordnung," Göttingen, 1903.

Carmichael* has studied the homogeneous form of equation (1) with analytic coefficients of certain forms, and has found fundamental systems of solutions with proper restrictions on the roots of the characteristic equation. There may be entire function solutions when the conditions given by Carmichael are not fulfilled. The various cases are considered in this paper.

§ 2. *Existence Theorems.*

Let us consider the homogeneous form of equation (1). We shall write the roots ρ_i of the characteristic equation in the form

$$\rho_i = q^{m_i}, \qquad i = 1, 2, \ldots, n.$$

If m_i is a positive integer, then the substitution of a power series will show that there exists a formal expansion in ascending powers of x. For m_i not a positive integer, the substitution

$$G(x) = x^{m_i} F_i(x)$$

will give, after dividing by $x^{m_i} q^{m_i n}$, an equation in $F_i(x)$ of the form of equation (1), which will have a formal solution in ascending powers of x. In case the numbers m_i, for two different values of i, differ by an integer, there will be, in general, a solution only for the m_i of greatest algebraic value.

For the case in which all the roots of the characteristic equation are zero, the substitution

$$G(x) = e^{\frac{\beta}{2}\log q\,\{\eta^2+\eta\}} F(x), \qquad \eta = \log x / \log q,$$

will give, after dividing by the proper factor, an equation of the form of equation (1), which will have some of the roots of its characteristic equation different from zero, provided that it is possible to choose β an integer, so that

$$n\beta = (n-i)\beta + \alpha_i$$

for some value of i, $i = 1, 2, \ldots, n$, and so that no other value of i will make

$$(n-i)\beta + \alpha_i < n\beta,$$

where α_i is the exponent of the lowest power of x in $B_i(x)$.

For the proof of convergence it will be sufficient to show that a formal ascending power-series solution of equation (1) is a convergent series; since the other cases referred to above transform to equations of the type of (1) by simple transformations, which do not affect convergence. The proof of convergence will be carried out for the non-homogeneous case, and this can be made to cover the homogeneous case by making the coefficients b_j of the function $B(x)$ identically zero in what follows.

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 147–168.

We shall suppose that equation (1) has formal ascending power-series solutions, which we write

$$G(x)=\sum_{i=0}^{\infty} a_i x^i.$$

Reckoning out coefficients, we shall have

$$a_k=\frac{-a_{k-1}\sum_{i=0}^{n} q^{(k-1)(n-i)}b_{i1}-a_{k-2}\sum_{i=0}^{n} q^{(k-2)(n-i)}b_{i2}-\ldots-a_1\sum_{i=0}^{n} q^{n-i}b_{i,k-1}-a_0\sum_{i=0}^{n} b_{ik}+b_k}{q^{kn}+b_{10}q^{k(n-1)}+b_{20}q^{k(n-2)}+\ldots+b_{n-1,0}q^k+b_{n0}}.$$

Form a comparison series in the following manner: Take absolute values in the numerator of a_k and consider all signs between terms as positive. This will increase the numerator if any change is made in its value. Since $|q|>1$, it will increase the numerator if we consider each term as multiplied by the highest power of q which occurs in any term, namely, $q^{(k-1)(n-1)}$. The coefficients $B_i(x)$ and the function $B(x)$ in equation (1) are entire functions, and hence, there is a positive constant $\bar{b}$ which is greater than the absolute value of each of the coefficients b_{ij} and b_j. The numerator will be increased, therefore, if we replace each b with subscript by $\bar{b}$. The absolute value of the denominator will be decreased* if we replace it by q^{kn}/M, where M is a properly chosen constant. Call the new a_k which we are forming a'_k. As we form each new a'_k, we shall further increase its value if we replace each a, with subscript less than k, by the corresponding a'. We shall have

$$|a_k|<a'_k=\frac{\bar{b}n|q|^{(k-1)(n-1)}\{\sum_{i=0}^{k-1}a'_i+1\}}{|q|^{kn}/M}$$

and

$$a'_{k+1}=\frac{\bar{b}n|q|^{k(n-1)}\{\sum_{i=0}^{k}a'_i+1\}}{|q|^{(k+1)n}/M}=\frac{\bar{b}n|q|^{k(n-1)}\{a'_k+\sum_{i=0}^{k-1}a'_i+1\}}{|q|^{(k+1)n}/M}.$$

Hence,

$$\lim_{k\doteq\infty}\frac{a'_{k+1}}{a'_k}=\lim_{k\doteq\infty}\frac{\dfrac{\bar{b}n|q|^{k(n-1)}\{a'_k+\sum_{i=0}^{k-1}a'_i+1\}}{|q|^{(k+1)n}/M}}{\dfrac{\bar{b}n|q|^{(k-1)(n-1)}\{\sum_{i=0}^{k-1}a'_i+1\}}{|q|^{kn}/M}}$$

$$=\lim_{k\doteq\infty}\left\{\frac{\bar{b}n|q|^{k(n-1)}}{|q|^{(k+1)n}/M}+\frac{|q|^{k(n-1)}|q|^{kn}}{|q|^{(k+1)n}|q|^{(k-1)(n-1)}}\right\}$$

$$=\frac{1}{|q|}.$$

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* As pointed out at the beginning of § 2, we are seeking the solution corresponding to the root $\rho=q^m$ of the characteristic equation for which m is the greatest in algebraic value. Thus the denominator can not equal zero for $k>m$. In carrying out the reckoning of coefficients we see that $a_k=0$ for $k<m$ and that a_m is arbitrary. Hence, the denominator can vanish for no value of k for which the numerator is not also zero.

Hence, for $|x| < |q|$ the series

$$\sum_{k=0}^{\infty} a'_k |x|^k \tag{3}$$

is convergent. The series

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

has its corresponding coefficients less in absolute value than those of the series (3), and hence, it is uniformly convergent for $|x| < r < |q|$ and represents an analytic function in that region. By means of equation (1) $G(x)$ can be continued throughout the plane as an analytic function, since the coefficients $B_i(x)$ of equation (1) introduce no singularities. Therefore, $G(x)$ is an entire function. In the same way $F_i(x)$ or $F(x)$ of the transformed equations mentioned above represents an entire function.

We have thus proved the following theorems:

THEOREM I. *The equation*

$$G(q^n x) + B_1(x) G(q^{n-1}x) + \ldots . + B_{n-1}(x) G(qx) + B_n(x) G(x) = 0, \tag{1}$$

where the coefficients $B_i(x)$ are entire functions, has solutions of the form

$$G_i(x) = x^{m_i} F_i(x),$$

where $F_i(x)$ is an entire function and where the roots of the characteristic equation are written in the form $\rho_i = q^{m_i}$; provided that not all the roots of the characteristic equation are zero. The solution $G_i(x)$ will be an entire function provided that m_i is a positive integer.

THEOREM II. *The equation*

$$G(q^n x) + B_1(x) G(q^{n-1}x) + \ldots . + B_{n-1}(x) G(qx) + B_n(x) G(x) = B(x), \tag{1}$$

where the functions $B_i(x)$ and $B(x)$ are entire, has an entire function solution whenever it has a formal ascending power-series solution.

Partly as a corollary of the preceding theorem we have the theorem:

THEOREM III. *The q-finite integral of an entire function, defined by the equation*

$$G(qx) - G(x) = \sum_{i=0}^{\infty} a_i x^i, \tag{4}$$

can be written in the form

$$G(x) = a_0 \frac{\log x}{\log q} + \sum_{i=1}^{\infty} \frac{a_i}{q^i - 1} x^i + C(x),$$

where $C(x)$ is an arbitrary function satisfying the relation $C(qx) = C(x)$. The integral $G(x)$ will be entire only when $a_0 = 0$ and $C(x)$ is an arbitrary constant.

It is evident that the substitution

$$G(x) = a_0 \frac{\log x}{\log q} + F(x)$$

in equation (4) will give an equation in $F(x)$ which has a formal solution in ascending powers of x; and hence, by Theorem II, $F(x)$ will be an entire function. The values of the coefficients are easily found.

§ 3. *Order of Solutions.*

We come now to a study of the relation of the "ordre apparent" of the entire function solutions of equation (1) as related to the "ordre apparent" of the entire functions which are the known functions of that equation. We shall suppose that the greatest "ordre apparent" of $B_i(x)$ and $B(x)$ is ρ. From the theorems which have grown out of the Borel theory of increase, we have

$$|B_i(x)| < e^{|x|^{\rho+\epsilon/2}} \text{ and } |B(x)| < e^{|x|^{\rho+\epsilon/2}}, \quad |x| > R.$$

Then for any x we have

$$|B_i(x)| < Me^{|x|^{\rho+\epsilon/2}} \text{ and } |B(x)| < Me^{|x|^{\rho+\epsilon/2}},$$

where M is a properly chosen constant.

Now consider the plane divided into fundamental regions by circles of radii r_j, where $r_{j+1} : r_j = q : 1$. Call $\bar{x}$ any x in some one of these circular rings in the neighborhood of the origin. Then for $|x| \leqq |\bar{x}|$

$$|G(x)| < K.$$

Write equation (1) in the form

$$G(x) = -B_1(x/q^n)G(x/q) - \ldots . - B_n(x/q^n)G(x/q^n) + B(x/q^n). \tag{5}$$

Replace each term on the right by the largest at each step, and by means of the equation continue the function out from the origin. We have

$$\begin{aligned}
|G(q\,\bar{x})| &< (n+1)MKe^{|q\bar{x}|^{\rho+\epsilon/2}}, \\
|G(q^2\bar{x})| &< (n+1)^2MKe^{2|q^2\bar{x}|^{\rho+\epsilon/2}}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
|G(q^k\bar{x})| &< (n+1)^kMKe^{k|q^k\bar{x}|^{\rho+\epsilon/2}}.
\end{aligned}$$

As k increases indefinitely, the last inequality can be replaced by

$$|G(q^k\bar{x})| < e^{|q^k\bar{x}|^{\rho+\epsilon}}.$$

Hence,

$$|G(x)| < e^{|x|^{\rho+\epsilon}}, \quad |x| > R',$$

where R' is properly chosen. Therefore, $G(x)$ is of "ordre apparent" not greater than ρ.

When $G(x)$ is not an entire function, as in Theorem I, but is $x^{m_i}F_i(x)$, m_i not a positive integer, then an argument similar to that used above will show that $F_i(x)$ is of "ordre apparent" not greater than the greatest "ordre apparent" of the coefficients.

We have thus completed the proof of the theorem:

THEOREM IV. *The entire function solutions, if any exist, of a linear n-th order homogeneous or non-homogeneous q-difference equation, with coefficients which are entire functions of finite order, are of "ordre apparent" not greater than the greatest "ordre apparent" of the known functions of the equation.*

In the same way in which we proved Theorem IV we could prove the theorem:

THEOREM V. *The entire function solutions, if any exist, of a linear n-th order homogeneous or non-homogeneous q-difference equation, with coefficients which are entire functions of infinite, but not transfinite, order, have their "Vergleichsfunktionen" not greater than the greatest among the "Vergleichsfunktionen" of the known functions of the equation.*

It is easy to construct equations which have solutions which are of "ordre apparent" less than the greatest "ordre apparent" of the coefficients. The same is true for the case of the infinite, but not transfinite, orders. One easy method of doing so is to choose a function for solution which has no zeros, and then choose all but one of the coefficients entire functions, at least one of greater "ordre apparent" than the solution, and solve for the other coefficient. Thus each of the equations

$$G(q^2x) + e^{x^5}\, G(qx) + \left(e^{x^4-x} - e^{x(q-1)+x^5} - e^{x(q^2-1)}\right)\, G(x) = e^{x^4}$$

and

$$G(qx) + \left(e^{e^{e^{x^2}}-x} - e^{x(q-1)}\right) G(x) = e^{e^{e^{x^2}}}$$

has the solution e^x, which is of "ordre apparent" less than the coefficients of the first equation, and is of finite "ordre apparent" while the coefficients of the second equation are of infinite order.

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On Rational Sextic Surfaces Having a Nodal Curve of Order 9.

By C. H. Sisam.

1. It is known that, if the section, by a generic plane, of a surface of order greater than 4 is rational, then the surface is ruled.* If the section is of genus 1, Castelnuovo has shown that the surface is either ruled or rational.† Ruled sextics have been the subject of several investigations.‡ In this paper, the non-ruled sextic surfaces whose plane sections are of the lowest possible genus will be studied. These surfaces have been discussed briefly by Caporali.§ Particular cases have been studied by del Pezzo‖ and Bonicelli.¶

2. Castelnuovo shows, in the memoir cited above, that the parametric equations

$$x_i = f_i(\lambda_1, \lambda_2, \lambda_3), \qquad i = 1, 2, 3, 4, \tag{1}$$

of the given sextic surface can be set up in such a way that the four curves $f_i = 0$, in the λ-plane, are cubics which have three points P_1, P_2, P_3 in common. These three points are thus fundamental points for the linear system $\Sigma u_i f_i = 0$ of cubics which correspond to the plane sections of the given surface.

If we take the triangle** whose vertices are P_1, P_2, P_3 as coordinate triangle in the λ-plane, equations (1) simplify to

$$x_i = a_{i0}\lambda_1^2\lambda_2 + a_{i1}\lambda_1\lambda_2^2 + b_{i0}\lambda_1^2\lambda_3 + 2b_{i1}\lambda_1\lambda_2\lambda_3 + b_{i2}\lambda_2^2\lambda_3 + c_{i0}\lambda_1\lambda_3^2 + c_{i1}\lambda_2\lambda_3^2, \tag{2}$$
$$i = 1, 2, 3, 4.$$

3. To a fundamental point P_i there corresponds, on the surface, a right line g_i, in such a way that to each direction through P_i corresponds†† a point

* Picard, *Crelle's Journal*, Vol. C, p. 71.

† *Rendiconti dei Lincei*, Series 5, Vol. III (1894), p. 59.

‡ Cf. Snyder, American Journal of Mathematics, Vol. XXVII (1905), pp. 77–102. Other references will be found in this article.

§ "Collectanea Mathematica in Memoriam D. Chelini," p. 169, Hoepli, Pisa, 1881.

‖ *Rendiconti di Napoli*, Series 3, Vol III (1897), pp. 196–203.

¶ *Giornali di Mat. di Battaglini*, Series 2, Vol. IX (1897), pp. 184–191.

** For the cases in which P_1, P_2, P_3 do not form a triangle, see Arts. 23–25. Throughout the following discussion, except when the contrary is stated, it should be understood that the general case is the one under discussion.

†† Clebsch, *Math. Ann.*, Vol. I, p. 266.

on g_i. To each direction through $(0,0,1)$, for example, corresponds at once, from (2), a point on the line

$$x_i = c_{i0}\lambda_1 + c_{i1}\lambda_2, \qquad i=1,2,3,4,$$

and conversely. The three lines g_1, g_2, g_3 corresponding to P_1, P_2, P_3 respectively, do not intersect.

To the lines joining P_1, P_2, P_3 in pairs correspond three other lines g_1', g_2', g_3' on the surface. Each of these lines intersects two of the lines g_1, g_2, g_3 so that the six lines form a gauche hexagon.

4. There are no other non-multiple right lines on the surface. For, let g be a line on the surface to which corresponds, in the λ-plane, a curve of order m having an α_i-fold point at P_i $(i=1,2,3)$. Since a generic plane intersects g in a single point, the image cubic of the section of the surface by the plane intersects the image of g in a single non-fundamental point, so that

$$3m - \alpha_1 - \alpha_2 - \alpha_3 = 1.$$

But $\alpha_1 + \alpha_2 + \alpha_3 \leq 2m$, since, otherwise, the curve of order m would be composite. Hence, $m=1$, $\alpha_1+\alpha_2+\alpha_3=2$ and g coincides with one of the lines g_1', g_2', g_3'.

5. To a generic line in the λ-plane through a fundamental point P_i, corresponds a conic on the surface, since it is intersected by a generic plane in two points. To the pencil of lines through P_i thus corresponds a system of conics on the surface, such that, through a generic point on the surface, there passes a single conic of the system.

The planes of such a system of conics envelope a curve of class 4. For, from (2), as the line $k_2\lambda_1 - k_1\lambda_2 = 0$ describes the pencil with vertex at $\lambda_1=\lambda_2=0$, the plane of the corresponding conic describes the developable of class 4 defined by the equation

$$\begin{vmatrix} x_1 & k_1 a_{10} + k_2 a_{11} & k_1^2 b_{10} + 2k_1 k_2 b_{11} + k_2^2 b_{12} & k_1 c_{10} + k_2 c_{11} \\ x_2 & k_1 a_{20} + k_2 a_{21} & k_1^2 b_{20} + 2k_1 k_2 b_{21} + k_2^2 b_{22} & k_1 c_{20} + k_2 c_{21} \\ x_3 & k_1 a_{30} + k_2 a_{31} & k_1^2 b_{30} + 2k_1 k_2 b_{31} + k_2^2 b_{32} & k_1 c_{30} + k_2 c_{31} \\ x_4 & k_1 a_{40} + k_2 a_{41} & k_1^2 b_{40} + 2k_1 k_2 b_{41} + k_2^2 b_{42} & k_1 c_{40} + k_2 c_{41} \end{vmatrix} = 0. \qquad (3)$$

It will be shown (Art. 20) that, if a sextic surface is generated by a system of non-composite conics whose planes envelope a rational curve of class 4, then a generic plane section of the surface is of genus 1.

Four tangents can be drawn from each of the points P_i to a generic curve $\Sigma u_i f_i = 0$. Hence, four conics of each of the three systems touch a generic plane. The cross-ratios of the parameters of the four conics of the three systems are respectively equal.

6. We have seen that three conics, one of each of the above systems, pass through a generic point on the surface. Through such a point, no other conic lying on the surface can pass. For, let C_2 be a non-multiple conic on the surface and let its image on the λ-plane be of order m and have an α_i-fold point at the fundamental point $P_i\,(i=1,2,3)$. We have (cf. Art. 4)

$$3m-\alpha_1-\alpha_2-\alpha_3=2.$$

Since $\alpha_1+\alpha_2+\alpha_3\leq 2m$, we find either $m=1$, $\alpha_1+\alpha_2+\alpha_3=1$, or $m=2$, $\alpha_1+\alpha_2+\alpha_3=4$. But the latter case is impossible, since a proper conic can not have a node. Hence, C_2 belongs to one of the three given systems.

7. Koenigs has shown* that if, through a generic point on a surface, there pass two (or more) conics lying on the surface, then the surface is rational and a generic plane section is of genus not greater than unity. If such a surface is a sextic, it can not be ruled, since it contains infinitely many non-multiple conics. Hence, the plane sections are of genus 1 (Art. 1) and the surface is generated by precisely three systems of conics.

8. There are two systems of cubics, each ∞^2, on the given surface. Those of the first system correspond to the right lines in the λ-plane and intersect the lines g'_1, g'_2, g'_3; those of the second system correspond to the conics through P_1, P_2, P_3 and intersect g_1, g_2, g_3. The two systems are equivalent, since, if we subject the points of the λ-plane to the transformation

$$\lambda_i=\frac{1}{\lambda'_i}, \qquad i=1,2,3,$$

the representations of the two systems on the surface (2) are interchanged.

There are three systems, each ∞^3, of quartics. They correspond to the systems of conics through two fundamental points, and are thus characterized by the non-intersection of one line of either triad g_1, g_2, g_3 or g'_1, g'_2, g'_3.

There are six systems, each ∞^4, of quintics. They correspond to the conics through a fundamental point and to the cubics with a double point at one fundamental point and passing through the other two fundamental points.

There are eight systems, each ∞^5, of rational sextics. They correspond to the conics in the λ-plane, to the quartics with double points at P_1, P_2, P_3, and to the cubics which have a node at one of these points and pass through a

* *Annales de L'École Normale Supérieure*, Series 3, Vol. V. Koenigs' proof expressly excludes the case of a non-rational surface generated by a system of conics such that three (or more) conics of the system pass through a generic point of the surface and two generic conics of the system have just one variable point in common. No such surface exists, however, since it follows from a theorem by Castelnuovo (*Atti di Torino*, Vol. XXVIII (1893), p. 736) that a surface generated by such a system of conics is rational.

second. There is also a system, ∞^6, of sextics of genus 1 corresponding to the cubics through P_1, P_2, P_3 (cf. Art. 13).

9. *The genus a non-multiple curve of order n on the given surface does not exceed the greatest integer in*

$$\frac{n^2-6n+12}{12}.$$

Let C_n be the given curve and let the corresponding curve C_m in the λ-plane be of order m and have an α_i-fold point at $P_i (i=1,2,3)$. We have (cf. Art. 4)

$$3m-\alpha_1-\alpha_2-\alpha_3=n.$$

Let $\alpha_1+\alpha_2+\alpha_3=k$, so that $m=\frac{n+k}{3}$. The genus of C_n, which is equal to that of C_m, does not exceed

$$\frac{1}{18}(n+k-3)(n+k-6)-\frac{1}{6}k(k-3).$$

The maximum value of this expression is $\frac{n^2-6n+12}{12}$.

The limit stated in the above theorem is actually attained for all values of n.

10. The characteristic numbers for the given surface can be determined at once from formulas by Caporali.*

The surface is of class 12. The tangent cone from a generic point to the surface is of order 12 and genus 7. The parabolic curve is of order 24 and genus 25. The tangent planes along the parabolic curve envelop a curve of class 24 and order 48.

If we form the invariants S and T for the cubics $\Sigma u_i f_i=0$ which correspond to the plane sections of the given surface, we determine two surfaces, $S(u_1, u_2, u_3, u_4)=0$ and $T(u_1, u_2, u_3, u_4)=0$, of class 4 and 6 respectively, enveloped by the planes which intersect the surface in curves for which the corresponding invariants are zero. The planes tangent to both of these surfaces touch the given surface along the parabolic curve.

The envelope of the planes which intersect the given surface in curves for which the corresponding absolute invariant is equal to a given constant k is a surface $S^3-kT^2=0$ of class 12. In particular, if $k=6$, so that the section is rational, we have the equation

$$S^3-6T^2=0$$

of the given surface in plane coordinates.

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* "Collectanea Mathematica in Memoriam D. Chelini," 1881. A few of the results for this surface are explicitly stated (p. 169).

11. The double developable of the given surface is of class 24. The section of the surface by a bitangent plane is composite, since it has eleven double points. The developable of these planes consists of the six pencils having the six lines on the surface (Art. 3) as axes, the three developables of class 4 determined by the planes of the three systems of conics on the surface, and a developable of class 6 and genus 1 whose planes intersect the surface in two cubics. To two coplanar cubics correspond a line and a conic in the λ-plane; so two such cubics belong to distinct systems of cubics on the surface (Art. 8). The cubics in the planes of the sextic developable thus constitute two systems (each of genus 1) of rational plane cubics on the surface. Three cubics of each system pass through a generic point on the surface; each cubic has a double point on the double curve and intersects the double curve in seven other points. At each point of the double curve, one cubic of each system has a double point.

12. The double curve C_9 is of order 9 and genus 1. It has four triple points which are also triple on the surface. There are twelve pinch-points on C_9. The corresponding curve in the λ-plane is of order 9. It has a triple point at each fundamental point and no other point singularities.

The double curve C_9 lies on a cubic surface. For, since C_9 is of genus 1, the pencil of cubic surfaces through the triple points and fourteen generic points of C_9 all intersect C_9 in a fixed point. The cubic of the pencil which passes through another generic point of C_9 contains the curve. The double curve constitutes the complete intersection of this cubic surface with the given surface.

13. Every curve of order n on the given surface intersects C_9 in $3n$ points, since its $3n$ intersections with a cubic surface lie on C_9. In particular, the sextics of genus 1 which correspond to the cubics through P_1, P_2, P_3 (Art. 8) intersect C_9 in eighteen points. Hence, each of these curves lies on a quartic surface which contains C_9, since the quartic surface which contains the triple points and twenty-four generic simple points of C_9 and six generic points of such a sextic contains both curves. Each cubic through P_1, P_2, P_3 thus defines a quartic surface which contains C_9, and conversely.

The quartic surfaces which contain C_9 and a fixed cubic C_3 on the surface constitute a bundle of quartics which cut from the surface the system of cubics opposite to the one to which C_3 belongs (Art. 8). Since two cubics of the same system intersect in a single point, the residual intersection of two quartics of the bundle determines a unique point on the surface, and conversely, so that the parametric equations of the surface can be determined as soon as such a bundle of quartics has been found.

14. The cubic curves in the λ-plane which pass through P_1, P_2, P_3 form a linear system ∞^6 and thus determine a surface of order 6 belonging to* an S_6,

$$\xi_i = f_i(\lambda_1, \lambda_2, \lambda_3), \qquad i = 1, 2, 3, \ldots, 7, \tag{4}$$

where $\xi_1, \xi_2, \ldots, \xi_7$) are current coordinates and the curves $f_i(\lambda) = 0$ are cubics through P_1, P_2, P_3. Del Pezzo † has shown that the equations of any non-ruled sextic surface belonging to an S_6 can be written in the above form.

By a suitable choice of coordinate system, the fundamental points P_1, P_2, P_3 being supposed to form a triangle (Art. 2), equations (4) can be reduced to

$$\xi_1 = \lambda_1^2\lambda_2,\ \xi_2 = \lambda_1\lambda_2^2,\ \xi_3 = \lambda_2^2\lambda_3,\ \xi_4 = \lambda_2\lambda_3^2,\ \xi_5 = \lambda_3^2\lambda_1,\ \xi_6 = \lambda_3\lambda_1^2,\ \xi_7 = \lambda_1\lambda_2\lambda_3. \tag{5}$$

The surface of equations (2) is the projection on S_3, from a generic plane π, of the surface (5).

15. Bordiga has shown ‡ that the point of intersection of corresponding S_4 of three projective bundles of S_4 belonging to an S_6 is a non-ruled sextic surface. Such a surface can, in fact, be generated in infinitely many ways by such a correspondence. For, as in Art. 13, it is seen that the S_4 of the bundle in S_6 determined by an S_3 which contains a cubic curve on the surface are in (1, 1) correspondence with the points of the surface, and hence with the points of the λ-plane. Three such bundles are thus in projective correspondence with each other, in such a way that corresponding S_4 intersect on the surface.

In particular, we can take the basis S_3 of each of the three bundles to contain two non-intersecting lines on the surface. The surface can thus be defined by the three projective bundles

$$\begin{matrix} \lambda_3\xi_1 = \lambda_1\xi_7 \\ \lambda_3\xi_2 = \lambda_2\xi_7 \end{matrix},\quad \begin{matrix} \lambda_1\xi_3 = \lambda_2\xi_7 \\ \lambda_1\xi_4 = \lambda_3\xi_7 \end{matrix},\quad \begin{matrix} \lambda_2\xi_5 = \lambda_3\xi_7 \\ \lambda_2\xi_6 = \lambda_1\xi_7 \end{matrix}.$$

16. The ∞^4 bisecants of (5) generate an hypersurface H in S_6. This hypersurface is of order 3 and has the surface (5) as a double surface. For, the plane π' determined by three generic points of the surface (5) contains the three lines, lying on H, which join these three points in pairs. It has no other points in common with H, since the projection of (5) from π' on S_3 is a cubic surface with no multiple points.

The hypersurface H is generated by each of two rational systems of ∞^2 S_3 such that each S_3 contains a cubic curve lying on the surface (5). For, let C_3 be a cubic curve of either of the two systems (cf. Art. 8) lying on (5). The S_3 determined by C_3 lies on H since, through a given point of S_3, a bisecant

* The notation S_r is used to indicate a space of r dimensions. An entity is said to belong to an S_r if it lies entirely in the S_r but not in any S_{r-1}.

† *Rendiconti di Palermo*, Vol. I, p. 441.

‡ *Comptes Rendus*, Vol. CII, pp. 743–745.

of C_3 can be passed. Each generator of H lies in an S_3 of the system determined by C_3, since, through its two points of intersection with (5), a cubic of the given system can be passed.

Two generic S_3 on H, of opposite systems, have a line in common, since their corresponding cubics intersect in two points. Two S_3 of the same system have in common a single point, the point of intersection of their corresponding cubics.

17. The surface (5) has no trisecant lines, since its projection on S_4 from such a line would be a cubic surface belonging to S_4 and having its generic hyperplane sections of genus 1, which is impossible.

18. The points of the double curve of (2) are in (1,1) correspondence with the generators of H which intersect the plane π (Art. 14) from which (5) is projected onto (2). These generators define a ruled surface of order 12 and genus 1 belonging to S_6. Twelve of its generators touch (5), since there are twelve pinch-points of (2) on the double curve (Art. 12). It follows that the four-spread generated by the tangent planes to (5) is of order 12. It has the surface (5) as fourfold surface.

19. It follows* from Art. 5 that the surface (2) can be defined (in any one of three ways) as the locus of the conic of intersection of corresponding surfaces of the two systems

$$L_1 k_1^4 + L_2 k_1^3 k_2 + L_3 k_1^2 k_2^2 + L_4 k_1 k_2^3 + L_5 k_2^4 = 0, \tag{6}$$

$$Q_1 k_1^3 + Q_2 k_1^2 k_2 + Q_3 k_1 k_2^2 + Q_4 k_2^3 = 0, \tag{7}$$

where k_1 and k_2 are parameters, $L_i = 0$ $(i = 1, 2, \ldots, 5)$ is a plane, and $Q_i = 0$ $(i = 1, 2, 3, 4)$ is a quadric surface.

Since the given surface is of order 6, there are five values of the ratio $k_2 : k_1$ for which the plane (6) is a component of the corresponding quadric (7). It is no restriction to suppose that these five values coincide at $k_2 = 0$. For, if

$$Q_1 \alpha^3 + Q_2 \alpha^2 + Q_3 \alpha + Q_4 \equiv L(L_1 \alpha^4 + L_2 \alpha^3 + L_3 \alpha^2 + L_4 \alpha + L_5),$$

where $L = 0$ is a plane, then $k_1 - \alpha k_2$ is a factor of the left member of

$$\begin{aligned} L(L_1 k_1^4 + L_2 k_1^3 k_2 + L_3 k_1^2 k_2^2 + L_4 k_1 k_2^3 + L_5 k_2^4) \\ - k_2(Q_1 k_1^3 + Q_2 k_1^2 k_2 + Q_3 k_1 k_2^2 + Q_4 k_2^3) = 0. \end{aligned}$$

If we remove the factor $k_1 - \alpha k_2$, the resulting equation, with (6), determines the given system of conics. If we replace (7) by the equation so determined, and repeat the above operation successively, we determine, in place of (7), a system of quadrics

$$\begin{aligned} L_1 L_5' k_1^3 + (L_2 L_5' + L_1 L_4') k_1^2 k_2 + (L_3 L_5' + L_2 L_4' + L_1 L_3') k_1 k_2^2 \\ + (L_4 L_5' + L_3 L_4' + L_2 L_3' + L_1 L_2') k_2^3 = 0, \end{aligned} \tag{8}$$

* Cf. also an article by the author in American Journal of Mathematics, Vol. XXX, pp. 99–116.

where $L'_i=0\,(i=1,2,\ldots,5)$ is a plane and the ten planes $L_i=0$, $L'_i=0$ satisfy the identity

$$L_1 L'_1+L_2 L'_2+L_3 L'_3+L_4 L'_4+L_5 L'_5\equiv 0. \tag{9}$$

From (6), (8) and (9) it follows that the given system of conics is also defined by (6) and any one of the systems of quadrics

$$L_1 L'_4 k_1^3+(L_2 L'_4+L_1 L'_3)k_1^2 k_2+(L_3 L'_4+L_2 L'_3+L_1 L'_2)k_1 k_2^2-L_5 L'_5 k_2^3=0, \tag{10}$$
$$L_1 L'_3 k_1^3+(L_2 L'_3+L_1 L'_2)k_1^2 k_2-(L_4 L'_4+L_5 L'_5)k_1 k_2^2-L_5 L'_4 k_2^3=0, \tag{11}$$
$$L_1 L'_2 k_1^3+(L_2 L'_2+L_1 L'_1)k_1^2 k_2-(L_4 L'_5+L_3 L'_4)k_1 k_2^2-L_5 L'_3 k_2^3=0. \tag{12}$$

If we eliminate k_1 and k_2 between (8), (10), (11) and (12), remove extraneous linear factors, and simplify, we find, as the equation of the surface,

$$\begin{vmatrix} L_1 L'_3 & L'_4 & L'_5 & Q_2 \\ L_1 L'_2 & L'_3 & L'_4 & L_5 L'_5 \\ L_1 L'_1 & L'_2 & L'_3 & L_5 L'_4 \\ Q_1 & L'_1 & L'_2 & L_5 L'_3 \end{vmatrix}=0, \tag{13}$$

wherein has been written, for brevity,

$$Q_1\equiv-(L_5 L'_4+L_4 L'_3+L_3 L'_2+L_2 L'_1),\quad Q_2\equiv-(L_4 L'_5+L_3 L'_4+L_2 L'_3+L_1 L'_2).$$

20. At points on the double curve, equations (8), (10), (11) and (12) have two solutions in common. Hence, all the first minors of (13) are zero for points on this curve. The linear system ∞^6 of quartic surfaces which contain the double curve (Art. 13) are determined by the quartic surfaces in the above set.

At the triple points of the surface, all the second minors of (13) are zero.

The equation of the cubic surface (Art. 12) on which the double curve lies is

$$\begin{vmatrix} L'_1 & L'_2 & L'_3 \\ L'_2 & L'_3 & L'_4 \\ L'_3 & L'_4 & L'_5 \end{vmatrix}=0. \tag{14}$$

It follows, in fact, from (9) that the complete intersection of (13) and (14) is a double curve on (13).

21. Denote the left member of (13) by f_6 and of (14) by f_3. The double curve of $f_6=0$ is also double on all the sextic surfaces of the pencil

$$k_1 f_6+k_2 f_3^2=0. \tag{15}$$

Every sextic surface for which this curve is double belongs to the pencil (15), since the double curve forms its complete intersection with the surfaces (15).

Let there be given a space curve C_9 of order 9 and genus 1 which has four triple points (Art. 12). There exists a pencil of sextic surfaces on which

C_9 is a double curve. For, if we take the triple points of C_9 as vertices of the coordinate tetrahedron, and subject C_9 to the cubic transformation

$$x_i = \frac{1}{x_i'}, \qquad i = 1, 2, 3, 4, \tag{16}$$

then C_9 is transformed into a plane cubic C_3. The sextic surfaces on which C_3 is a double curve are of the form

$$\phi_1^2\phi_4 + \phi_1\phi_2\phi_3 + \phi_3^2 = 0,$$

where $\phi_1 = 0$ is the plane of C_3, $\phi_3 = 0$ is the cubic cone projecting C_3 from a fixed point P, $\phi_2 = 0$ is a quadric cone with vertex at P, and $\phi_4 = 0$ is a quartic surface. If we subject the forty-two independent homogeneous parameters in this equation to the forty conditions that the surface shall have a triple point at each vertex of the tetrahedron of reference, we determine a pencil of sextics which are transformed by the involutorial transformation (16) into a pencil of sextics which have C_9 as double curve.

It follows from a theorem by Halphen* that twelve sextics of the pencil (15) touch a generic plane.

22. The condition that a curve of order n, not lying on $f_3 = 0$, shall lie on a surface of the pencil (15) is that its $3n$ intersections with $f_3 = 0$ shall lie on the double curve C_9 (Art. 13). The locus of the right lines which lie on surfaces (15) is thus the scroll of trisecants R of C_9. The surface R has C_9 as eightfold curve, since the projecting cone of C_9 from a generic point on the curve has eight double generators. The order of R is 25, since its complete intersection with a generic surface (15) consists of C_9 and six lines. The triple points of C_9 are twelvefold points of R. In fact, the plane determined by each pair of tangents to C_9 at a triple point P_3 is torsal tangent plane to R along the generators joining P_3 to each of the four distinct intersections of C_9 with the plane.

Through a generic point in space, there pass three conics which intersect C_9 in six points (Art. 5). These three conics lie on a quartic surface which contains C_9. The ∞^1 cubics through a generic point P which have nine points in common with C_9 constitute two systems (Art. 8) each of which generates the surface (15) through P. Cubics through P, of opposite systems, intersect in one variable point.

23. Some particular cases of the surface defined by equations (1) will now be noticed.

We have supposed (Art. 2) that the fundamental points P_1, P_2, P_3 in the λ-plane form a triangle. If two of these points, P_1, P_2, are consecutive, the

* *Bull. Soc. Math.*, Vol. X (1882), p. 166.

surface has an additional double point D determined by the bundle of planes whose corresponding cubics have a double point at P_1. Two lines of each triad on the surface (Art. 3) are consecutive and pass through D. Two of the three generating systems of conics coincide. All the conics of the coincident systems pass through D. Conversely, if all the conics of a system on the surface pass through a point, this point is a double point additional to the double curve and the given system counts for two. For, let P_i be the vertex of the corresponding pencil. The point common to the conics corresponds either to P_i on all lines through P_i or else to a curve intersecting these lines in at least one point distinct from P_i. In the latter case we have, using the notation of Art. 4,

$$3m-\alpha_1-\alpha_2-\alpha_3=0, \qquad \alpha_i<m,$$

which is impossible. Hence, to all directions through P_i, corresponds a fixed point on the surface so that (Cf. Art. 3) a second fundamental point coincides with P_i.

24. If the fundamental points P_1, P_2, P_3 lie on a line l, then all the points of l correspond to a fixed double point D on the surface, since every cubic of the system $\Sigma u_i f_i=0$ which contains a point of l distinct from P_1, P_2, P_3 has l as a component. The six lines on the surface pass through D and coincide in pairs. The systems of generating conics are distinct. The two composite conics of each system are consecutive and pass through D. The two systems of cubic curves on the surface (Art. 8) coincide. All these cubics pass through D.

25. The surfaces of Arts. 23 and 24 are projections of sextic surfaces belonging to S_6 (Art. 14) each of which has a conical double point. A non-ruled sextic surface belonging to S_6 (and hence its sextic projection on S_3) can have at most two (distinct) conical double points. This happens when P_1, P_2, P_3 are collinear and two of them coincide. One of these double points is of the type of Art. 23, the other of Art. 24.

26. If the given surface (2), in S_3, has a double right line, l_2, then, since the surface is not ruled, the residual sections in the planes through l_2 are either (a) rational quartics or (b) pairs of conics. In case (a), to the pencil of quartics on the surface corresponds, in the λ-plane, a pencil of conics through two fundamental points, P_2, P_3. To l_2 thus corresponds a right line through P_1. It is thus a degenerate conic of the system determined by the pencil of lines through P_1. It follows from (3) that the planes of these conics envelope a curve of class 3, and that, conversely, if the planes of such a system envelope a curve of class 3, a conic of the system degenerates into a double line. The line l_2 passes through two triple points on the surface corresponding to

the non-fundamental basis-points of the pencil of conics determined by the residual sections in the planes through l_2. It intersects the residual nodal curve in one other point.

In case (b), the conics in the planes through l_2 all belong to one system, since through each point of the surface there passes a conic of the system. Each plane through l_2 thus determines two lines through a fundamental point P_1, and l_2 is determined by the coincidence of the line corresponding to P_1 with the line corresponding to the join of P_2 and P_3. Since the developable (3) of the conics in the planes through l_2 reduces to a pencil of planes counted twice, there are two double lines of type (a) which intersect l_2 on this surface.

The surface (a) is the projection of the surface (5) from a plane which intersects in a point the plane of a conic on (5). The surface (b) is the projection of (5) from a plane which intersects in a line the S_3 determined by a pair of opposite sides of the hexagon formed by the lines on* (5).

27. The condition that a non-ruled sextic, whose plane sections are of genus 1, have a fourfold line, is that it have three double lines of type (a) (Art. 26) all of which belong to the same system of conics on the surface. To the fourfold line corresponds a conic through two fundamental points. The three double lines, and four of the simple lines, intersect the fourfold line. Each of the other simple lines intersects the three double lines. The double lines may be consecutive to each other or to the fourfold line.

28. If the multiple curve reduces to a triple cubic, this cubic can not have any apparent double points, since the bisecant of the triple curve from a generic point on the surface would lie on it so that the surface would be ruled. The triple curve thus reduces to three lines concurring in a fivefold point of the surface. The tangent cone at the fivefold point intersects the surface in the triple curve and in three lines each of which counts for two consecutive simple lines, so that the surface is of the type of Art. 24. The variable intersection with the surface of the pencil of quadric cones determined by the three triple lines and any one of the simple lines, is a system of conics on the surface.

29. If the given surface has a triple curve of order 2, that curve can not be two skew lines, since the triple curve has no apparent double points (Art. 28). The surface is transformed, by a quadratic transformation which has the triple curve and a simple point on the surface as fundamental elements, into a quintic

* Del Pezzo has discussed [*Rendiconti di Napoli*, Series 3, Vol. III (1897), pp. 196–203)] a sextic surface with nine double lines. Six of these are of type (a) and concur in a fourfold point. The other three are coplanar and of type (b). The equation of the surface is

$$(x_1+x_2+x_3)(x_1+x_2-x_3)(x_1-x_2+x_3)(x_1-x_2-x_3)x_4^2=x_1^2 x_2^2 x_3^2.$$

surface which has the inverse fundamental conic as double conic and an additional double cubic.

30. If the inverse double conic is not composite, it is known* that the residual double cubic on the quintic degenerates into three lines which intersect the double conic and concur at a threefold point on the surface. Hence, if the given sextic has a proper triple conic, the residual nodal curve is three double lines which concur at a triple point.† The pencil of quadrics determined by the triple conic and any two double lines determines a system of conics on the surface to which the third double line belongs. These lines are thus of type (a) (Art. 26).

31. If the inverse double conic is composite, the point of intersection of its components is a triple point, at least, on the quintic. The residual ‡ nodal curve is either a proper cubic through the triple point, a right line through the triple point and a conic, or three lines through the intersection of the components of the fundamental conic. This point is, in this case, a fourfold point on the surface.

It follows at once that if the triple curve on the sextic is two (distinct or consecutive) intersecting lines, the point of intersection of these lines is a fourfold point (at least) on the surface. The residual multiple curve is either a proper cubic through the fourfold point, a line through the fourfold point and a conic, or three lines through a fivefold point.

The multiple curve, in Arts. 30 and 31, is the complete intersection of the given surface with a quadric.

32. If the given surface has a single triple line, the parametric equations of the surface can be chosen so that the curve in the λ-plane corresponding to the triple line is a conic through the fundamental points. To the residual cubics of section by the planes through the triple line correspond the lines of a pencil. To the vertex of this pencil corresponds a fourfold point of the surface which lies on the triple line. The double curve of order 6 is rational. It has, at the fourfold point, a triple point such that the three tangents are coplanar. The cubic surface (Art. 20) on which the multiple curve lies is a cone with vertex at the fourfold point. It has the triple line as double generator.

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* Cf. an article by the author in AMERICAN JOURNAL OF MATHEMATICS, Vol. XXX, p. 110.

† This surface was studied by Bonicelli, *Giornali di Mat. di Battaglini*, Series 2, Vol. IX (1902), pp. 184–191.

‡ Cf. first footnote, Art. 30. The quintic with five double lines was erroneously omitted from the classification there referred to. The equation of such a surface is

$$(x_1^2+x_2^2+x_3^2)^2 x_4 = \phi_5(x_1, x_2, x_3),$$

where $\phi_5 = 0$ is the equation of a quintic cone with five double generators which lie on $x_1^2+x_2^2+x_3^2=0$.